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Some New Properties of The Real Quaternion Matrices and Matlab Applications

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Abstract. In this study, firstly, it was shown that the set of real quaternion matrices $M_n(H_{\mathbb{R}})$ is a 4-dimensional module over the real matrix ring $M_n(\mathbb{R})$ and 2-dimensional module over the complex matrix ring $M_n(\mathbb{C})$. Moreover, some new properties of the real quaternion matrices were described. Then, matrix representations of the real quaternion matrices were found easily by Matlab. These matrices were also applied to find the inverse of the real quaternion matrices and inverse matrices were obtained easily with these matrices. In addition, some new properties for matrix representations of the real quaternion matrices were found. Also, the inverse of the 2×2 real quaternion block matrices was obtained by new methods. Finally, a new method to calculate the determinant of the 2×2 real quaternion matrices was found and the determinant of these matrices was calculated easily with Matlab application.

Keywords**:** Quaternions, real quaternions, real quaternion matrices, real matrix representation, determinant, block matrices.

Reel Kuaterniyon Matrislerinin Bazı Yeni Özellikleri ve Matlab Uygulamaları

Özet. Bu çalışmada, ilk olarak, $M_n(H_{\mathbb{R}})$ reel kuaterniyon matrislerin kümesinin $M_n(\mathbb{R})$ reel matris halkası üzerinde 4 boyutlu bir modül olduğu ve $M_n(\mathbb{C})$ kompleks matris halkası üzerinde 2 boyutlu bir modül olduğu gösterilmiştir. Ayrıca, reel kuaterniyon matrislerin bazı yeni özellikleri tanımlanmıştır. Daha sonra, reel kuaterniyon matrislerin matris temsilleri Matlab uygulamaları ile kolayca elde edilmiştir. Bu matrisler reel kuaterniyon matrislerin tersini bulmak için de uygulanmış ve bu matrislerle ters matrisler kolaylıkla elde edilmiştir. Buna ek olarak, reel kuaterniyon matrislerin matris temsilleri için bazı yeni özellikler bulunmuştur. Ayrıca, 2×2 tipindeki reel kuaterniyon blok matrislerin tersi yeni yöntemlerle elde edilmiştir. Son olarak, 2×2 tipindeki reel kuaterniyon matrislerin determinantını hesaplamak için yeni bir yöntem bulunmuş ve Matlab uygulaması ile bu matrislerin determinantı kolayca hesaplanmıştır.

Anahtar Kelimeler**:** Kuaterniyonlar, reel kuaterniyonlar, reel kuaterniyon matrisler, reel matris temsili, determinant, blok matrisler.

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1. INTRODUCTION

The set of quaternions can be represented as

$$H_{\mathbb{R}} = \left\{ q = q_0 + iq_1 + jq_2 + kq_3 \;\middle|\; q_0, q_1, q_2, q_3 \in \mathbb{R} \right\} \tag{1}$$

where

$$i^2 = j^2 = k^2 = -1, \; i\,j = -j\,i = k, \quad j\,k = -k\,j = i, \quad k\,i = -i\,k = j,$$

which given by Hamilton [1], in 1843. From these rules one can see that multiplication of quaternions is not commutative.

Several authors worked on algebraic properties of quaternion matrices [2-8]. In 1997, Zhang [8] gave a brief survey on quaternions and matrices of quaternions. In his study, properties such as addition, multiplication, conjugate, transpose, conjugate transpose, inverse were examined and adjoint of a quaternion matrix was defined and eigenvalues, determinants of quaternion matrices were discussed. Moreover, properties such as equality, rank, inverse, transpose, conjugate transpose and determinant of quaternion matrices were investigated in [9]. The eigenvalues and properties of quaternion matrices were investigated in [10, 11].

Calculating the inverse of real and complex matrices are great importance. In [12], the Study determinant and q- determinant were used. Moreover, they saw that it is very difficult to generalize inverse method of adjoint matrix to quaternion matrices. The details can be found in [13]. Aslasken investigated many different definitions of determinant (Cayley determinant, Study determinant, Dieudonne determinant and Moore determinant) [13]. Gelfand et al. [14] others investigated Moore determinants of Hermitian quaternion matrices and Quasideterminants, Study determinants of quaternion matrices. Moreover, Lewis mentioned relation between the Bagazgoitia's identity [15] and the Dieudonné determinant [16].

Jiang and Wei [17] defined the real representation of the quaternion matrix and gave their properties. Then, they studied the solution of the quaternion matrix equation by means of real representation. Song and others used real representation method for solving Yakubovich-j-conjugate quaternion matrix equation in [18]. Two types of universal factorization equalities for real quaternions and matrices of real quaternions were presented in [19] and real representation of the quaternion matrix was used in this study. In [20], determinants based on real matrix representations of quaternion matrices and linear matrix equations with quaternion coefficients are studied.

Lin and Wang [21] completed a 2×2 block matrix of real quaternions with a partially specified inverse. Also in [22], the general partitioned linear representation form of matrix quaternions are obtained. Localization theorems are discussed for the left and right eigenvalues of block quaternion matrices in [23]. In [24], some sufficient conditions for two, three and four quaternion matrices are block independent in the least squares inverse, the minimum norm inverse and the 1,3,4-inverse are derived respectively.

2. REAL QUATERNIONS

A set of real quaternions is denoted by

$$H_{\mathbb{R}} = \{ q = a + bi + c\,j + d\,k \mid a,b,c,d \in \mathbb{R}, i, j, k \notin \mathbb{R} \}$$

where the basis elements i, j, k satisfy the following multiplication rules [25]:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

A real quaternion may be defined as a pair (S_q, V_q), where $S_q = a \in \mathbb{R}$ is scalar part and $V_q = bi + cj + dk \in \mathbb{R}^3$ is the vector part of q. If $a = 0$, then q is called pure real quaternion. Addition of any real quaternions $q = a + bi + cj + dk$ and $p = a_2 + b_2 i + c_2 j + d_2 k$ is defined as

$$\begin{aligned} q + p &= (a + bi + cj + dk) + (a_2 + b_2 i + c_2 j + d_2 k) \\ &= (a + a_2) + (b + b_2)i + (c + c_2)j + (d + d_2)k \\ &= (S_q + S_p) + (V_q + V_p). \end{aligned}$$

The addition rule preserves the associativity and commutativity properties of addition. The product of scalar $(\mu \in \mathbb{R})$ and a real quaternion are defined as

$$\mu q = (\mu a)1 + (\mu b)i + (\mu c)j + (\mu d)k = (\mu S_q) + (\mu V_q).$$

The real quaternion product of two quaternions $q = a + bi + cj + dk$ and $p = a_2 + b_2 i + c_2 j + d_2 k$ is defined as:

$$\begin{aligned} qp = aa_2 - (bb_2 + cc_2 + dd_2) + a(b_2 i + c_2 j + d_2 k) + a_2(bi + cj + dk) \\ +(cd_2 - dc_2)i + (-bd_2 + db_2)j + (bc_2 - cb_2)k \end{aligned}$$

$$= S_q S_p - \langle V_q, V_p \rangle + S_q V_p + S_p V_q + V_q \wedge V_p.$$

The conjugate of a real quaternion is denoted by $\bar{q}$ and norm of a real quaternion is denoted by $\|q\|$ as follows:

$$\bar{q} = a - (bi + cj + dk) = S_q - V_q.$$

$$\|q\| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

If $\|q\| = 1$, then q is called unit real quaternion. The inverse of the real quaternion q is

$$q^{-1} = \frac{\bar{q}}{\|q\|^2}, \quad \text{if } \|q\| \neq 0.$$

The set $H_{\mathbb{R}}$ is a 4-dimensional vector space on $\mathbb{R}$ and its basis is the set $\{1, i, j, k\}$ [8, 26].

Theorem 2.1. Let $p, q \in H_{\mathbb{R}}$ and $\mu, \eta \in \mathbb{R}$. The conjugate, norm and inverse of real quaternions satisfy the following properties [8];

(i) $\bar{\bar{q}} = q$,

(ii) $\overline{pq} = \bar{q}\,\bar{p}$,

(iii) $\|qp\| = \|q\|\|p\|$,

(iv) $\left\|q^{-1}\right\|=\dfrac{1}{\|q\|}$.

2.1. Real Matrix Representations of Real Quaternions

Let $q=a+bi+cj+dk\in H_{\mathbb{R}}$ be a real quaternion. The left linear map $L_q:H_{\mathbb{R}}\to H_{\mathbb{R}}$ is defined by $L_q(p)=pq$ for all $p\in H_{\mathbb{R}}$. Then the left real matrix representation of real quaternion q is

$$L_q=\begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}.$$

Furthermore $det(L_q)=\|q\|^4$. Here $det(L_q)$ is usual determinant of L_q. The right linear map $R_q:H_{\mathbb{R}}\to H_{\mathbb{R}}$ is defined by $R_q(p)=qp$ for all $p\in H_{\mathbb{R}}$. Then the right real matrix representation of real quaternion q is

$$R_q=\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}.$$

Furthermore $det(R_q)=\|q\|^4$ [26]. Here $det(R_q)$ is usual determinant of R_q.

3. COMPLEX BLOCK QUATERNIONS

The determinant of a 2×2 complex block matrix can be calculated by

$$det\begin{pmatrix} A & B \\ C & D \end{pmatrix}=det(A-BD^{-1}C)det(D). \tag{2}$$

where $A,B,C,D\in M_n(\mathbb{C})$ and D is invertible [27, 28]. If D^{-1} does not exist, then the determinant of a 2×2 block matrix can be calculated by

$$det\begin{pmatrix} A & B \\ C & D \end{pmatrix}=det(D-CA^{-1}B)det(A). \tag{3}$$

where $A,B,C,D\in M_n(\mathbb{C})$ and A is invertible [27, 29]. If neither inverse exists, then generalized inverses must be used [30-32].

The inverse of a 2×2 complex block matrix can be calculated by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}=\begin{pmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}. \tag{4}$$

where $A,B,C,D \in M_n(\mathbb{C})$. If A and the 2×2 block matrix are nonsingular. Then, the Schur complement $D-CA^{-1}B$ is nonsingular, too [29, 33]. This formula is called the Banachiewicz inversion formula for the inverse of a nonsingular matrix [34].

The inverse of a 2×2 complex block matrix can be calculated by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}+D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1} \end{pmatrix}. \tag{5}$$

where $A,B,C,D \in M_n(\mathbb{C})$ [27, 29, 35]. If D and the 2×2 block matrix are nonsingular. Then, the Schur complement $A-BD^{-1}C$ is nonsingular, too [29, 33].

If A,D and the 2×2 block matrix are nonsingular. Then, the Schur complements $A-BD^{-1}C$ and $D-CA^{-1}B$ nonsingular, too. The inverse of a 2×2 complex block matrix can be calculated by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}.$$

where $A,B,C,D \in M_n(\mathbb{C})$ [36, 37].

4. REAL QUATERNION MATRICES

The set of real quaternion matrices can be defined as

$$M_{m\times n}(H_{\mathbb{R}}) = \{\tilde{A} = A + Bi + Cj + Dk \mid A,B,C,D \in M_{m\times n}(\mathbb{R})\}$$

where $A=(a_{rs}), B=(b_{rs}), C=(c_{rs}), D=(d_{rs})$ and

$$i^2 = j^2 = k^2 = -1,\ i\,j = -j\,i = k,\quad j\,k = -k\,j = i,\quad k\,i = -i\,k = j.$$

If $m=n$, then the set of real quaternion matrices is denoted by $M_n(H_{\mathbb{R}})$ [16-19].

Let $\tilde{A} = A\tilde{1} + B\tilde{I} + C\tilde{J} + D\tilde{K}$ be a quaternion matrix. We will define the right linear map $\Re_{\tilde{A}}$ as $\Re_{\tilde{A}} : M_n(H_{\mathbb{R}}) \to M_n(H_{\mathbb{R}})$ such that $\Re_{\tilde{A}}(\tilde{B}) = \tilde{A}\tilde{B}$. Using this operator and the basis $\{\tilde{1},\tilde{I},\tilde{J},\tilde{K}\}$ of the module $M_n(H_{\mathbb{R}})$, we can write

$$\begin{aligned}
\Re_{\tilde{A}}(\tilde{1}) &= \tilde{A}\tilde{1} = A\tilde{1} + B\tilde{I} + C\tilde{J} + D\tilde{K}, \\
\Re_{\tilde{A}}(\tilde{I}) &= \tilde{A}\tilde{I} = -B\tilde{1} + A\tilde{I} + D\tilde{J} - C\tilde{K}, \\
\Re_{\tilde{A}}(\tilde{J}) &= \tilde{A}\tilde{J} = -C\tilde{1} - D\tilde{I} + A\tilde{J} + B\tilde{K}, \\
\Re_{\tilde{A}}(\tilde{K}) &= \tilde{A}\tilde{K} = -D\tilde{1} + C\tilde{I} - B\tilde{J} + A\tilde{K}.
\end{aligned}$$

Then, the following right real matrix representation can be found as

$$\Re_{\tilde{A}}=\begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix}_{4n\times 4n} \in S_{4n}(\mathbb{R}) \tag{6}$$

where $S_{4n}(\mathbb{R})\subset M_{4n}(\mathbb{R})$ [17-19, 38, 39].

Example 4.1. The real matrix representations of $\tilde{1},\tilde{I},\tilde{J},\tilde{K}$ are

$$\Re_{\tilde{1}}=\begin{pmatrix} I_n & 0_n & 0_n & 0_n \\ 0_n & I_n & 0_n & 0_n \\ 0_n & 0_n & I_n & 0_n \\ 0_n & 0_n & 0_n & I_n \end{pmatrix}_{4n\times 4n}, \Re_{\tilde{I}}=\begin{pmatrix} 0_n & -I_n & 0_n & 0_n \\ I_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & -I_n \\ 0_n & 0_n & I_n & 0_n \end{pmatrix}_{4n\times 4n},$$

$$\Re_{\tilde{J}}=\begin{pmatrix} 0_n & 0_n & -I_n & 0_n \\ 0_n & 0_n & 0_n & I_n \\ I_n & 0_n & 0_n & 0_n \\ 0_n & -I_n & 0_n & 0_n \end{pmatrix}_{4n\times 4n}, \Re_{\tilde{K}}=\begin{pmatrix} 0_n & 0_n & 0_n & -I_n \\ 0_n & 0_n & -I_n & 0_n \\ 0_n & I_n & 0_n & 0_n \\ I_n & 0_n & 0_n & 0_n \end{pmatrix}_{4n\times 4n}$$

where $\tilde{1},\tilde{I},\tilde{J},\tilde{K}\in M_n(H_{\mathbb{R}}),\Re_{\tilde{1}},\Re_{\tilde{I}},\Re_{\tilde{J}},\Re_{\tilde{K}}\in S_{4n}(\mathbb{R})$ [38]. Furthermore, these real representation matrices satisfy [38]:

$$\begin{gathered}\Re_{\tilde{1}}^2=I_{4n},\Re_{\tilde{I}}^2=\Re_{\tilde{J}}^2=\Re_{\tilde{K}}^2=-I_{4n},\\ \Re_{\tilde{I}}\Re_{\tilde{J}}=-\Re_{\tilde{J}}\Re_{\tilde{I}}=\Re_{\tilde{K}},\\ \Re_{\tilde{J}}\Re_{\tilde{K}}=-\Re_{\tilde{K}}\Re_{\tilde{J}}=\Re_{\tilde{I}},\\ \Re_{\tilde{K}}\Re_{\tilde{I}}=-\Re_{\tilde{I}}\Re_{\tilde{K}}=\Re_{\tilde{J}}.\end{gathered}$$

Then, left real matrix representation can be found in the same way as follows:

$$\mathcal{L}_{\tilde{A}}=\begin{pmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{pmatrix}_{4n\times 4n} \in S_{4n}(\mathbb{R}) \tag{7}$$

where $S_{4n}(\mathbb{R})\subset M_{4n}(\mathbb{R})$ [16].

Example 4.2. The real matrix representations of $\tilde{1},\tilde{I},\tilde{J},\tilde{K}$ are

$$\mathcal{L}_{\tilde{1}}=\begin{pmatrix} I_n & 0_n & 0_n & 0_n \\ 0_n & I_n & 0_n & 0_n \\ 0_n & 0_n & I_n & 0_n \\ 0_n & 0_n & 0_n & I_n \end{pmatrix}_{4n\times 4n}, \mathcal{L}_{\tilde{I}}=\begin{pmatrix} 0_n & I_n & 0_n & 0_n \\ -I_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & -I_n \\ 0_n & 0_n & I_n & 0_n \end{pmatrix}_{4n\times 4n},$$

$$\mathcal{L}_{\tilde{J}}=\begin{pmatrix} 0_n & 0_n & I_n & 0_n \\ 0_n & 0_n & 0_n & I_n \\ -I_n & 0_n & 0_n & 0_n \\ 0_n & -I_n & 0_n & 0_n \end{pmatrix}_{4n\times 4n}, \mathcal{L}_{\tilde{K}}=\begin{pmatrix} 0_n & 0_n & 0_n & I_n \\ 0_n & 0_n & -I_n & 0_n \\ 0_n & I_n & 0_n & 0_n \\ -I_n & 0_n & 0_n & 0_n \end{pmatrix}_{4n\times 4n}$$

where $\tilde{1},\tilde{I},\tilde{J},\tilde{K}\in M_n(H_{\mathbb{R}}),\mathcal{L}_{\tilde{1}},\mathcal{L}_{\tilde{I}},\mathcal{L}_{\tilde{J}},\mathcal{L}_{\tilde{K}}\in S_{4n}(\mathbb{R})$.

Furthermore, these real representation matrices satisfy [38]:

$$\begin{aligned} \mathcal{L}_{\tilde{1}}^2 = I_{4n}, \mathcal{L}_{\tilde{I}}^2 = \mathcal{L}_{\tilde{J}}^2 = \mathcal{L}_{\tilde{K}}^2 = -I_{4n}, \\ \mathcal{L}_{\tilde{I}}\mathcal{L}_{\tilde{J}} = -\mathcal{L}_{\tilde{J}}\mathcal{L}_{\tilde{I}} = \mathcal{L}_{\tilde{K}}, \\ \mathcal{L}_{\tilde{J}}\mathcal{L}_{\tilde{K}} = -\mathcal{L}_{\tilde{K}}\mathcal{L}_{\tilde{J}} = \mathcal{L}_{\tilde{I}}, \\ \mathcal{L}_{\tilde{K}}\mathcal{L}_{\tilde{I}} = -\mathcal{L}_{\tilde{I}}\mathcal{L}_{\tilde{K}} = \mathcal{L}_{\tilde{J}}. \end{aligned}$$

Corollary 4.1. $S_{4n}(\mathbb{R})$ is a special subset of $M_{4n}(\mathbb{R})$.

4.1. Determinant of 2×2 Real Quaternion Matrices

In practice, the determinant of a 2×2 real quaternion matrix is defined by

$$det\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix} = \tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}\tilde{a}_{21} \tag{8}$$

In the above definition the so-called rule “multiplication from above to down below” rule is used [9].

5. SOME NEW PROPERTIES OF REAL QUATERNION MATRICES

In this Section, we will investigate some new properties of quaternion matrices and their real matrix representations. After that we will give some relations between quaternion matrices and their real matrix representations. In addition, matlab applications on this subject will be done.

Definition 5.1. For $\tilde{A}=(\tilde{a}_{rs})=A+Bi+Cj+Dk\in M_{m\times n}(H_{\mathbb{R}})$,

$\tilde{B}=(\tilde{b}_{rs})=A_2+B_2 i+C_2 j+D_2 k\in M_{m\times n}(H_{\mathbb{R}})$, the ordinary matrix addition is defined by

$$\tilde{A}+\tilde{B}=(\tilde{a}_{rs}+\tilde{b}_{rs})\in M_{m\times n}(H_{\mathbb{R}}),$$

or

$$\tilde{A}+\tilde{B}=(A+A_2)+(B+B_2)i+(C+C_2)j+(D+D_2)k.$$

Definition 5.2. For $\tilde{A}=(\tilde{a}_{rs})=A+Bi+Cj+Dk\in M_{m\times n}(H_{\mathbb{R}})$ and

$\tilde{B}=(\tilde{b}_{so})=A_2+B_2i+C_2j+D_2k\in M_{n\times p}(H_{\mathbb{R}})$, the ordinary matrix multiplication is defined by

$$\tilde{A}\tilde{B}=(\sum_{s=1}^{n}\tilde{a}_{rs}\tilde{b}_{so})\in M_{m\times p}(H_{\mathbb{R}}),$$

or

$$\begin{aligned}\tilde{A}\tilde{B}=&AA_2-(BB_2+CC_2+DD_2)+(AB_2+BA_2+CD_2-DC_2)i\\&+(AC_2+CA_2-BD_2+DB_2)j+(AD_2+DA_2+BC_2-CB_2)k.\end{aligned}$$

Corollary 5.1. $\tilde{A}\tilde{B}\neq\hat{B}\hat{A}$ (in general), for suitable real quaternion matrices $\tilde{A}$ and $\tilde{B}$.

5.1. Module $M_n(H_{\mathbb{R}})$ Structure Over the Ring $M_n(\mathbb{C})$

Definition 5.3. For $Q=(q_{tr})\in M_n(\mathbb{C})$ and $\tilde{A}=(\tilde{a}_{rs})=A+Bi+Cj+Dk\in M_n(H_{\mathbb{R}})$, the left multiplication of a real quaternion matrix and a real matrix is defined as

$$Q\tilde{A}=(\sum_{r=1}^{n}q_{tr}\tilde{a}_{rs})\in M_n(H_{\mathbb{R}})$$

or

$$Q\tilde{A}=QA+QBi+QCj+QDk. \tag{9}$$

The right multiplication can be defined in the same way.

Lemma 5.1. The left multiplication has the following properties:

for $\tilde{A},\tilde{B}\in M_n(H_{\mathbb{R}}), Q_1,Q_2\in M_n(\mathbb{C})$,

(i) $(Q_1+Q_2)\tilde{A}=Q_1\tilde{A}+Q_2\tilde{A}$,

(ii) $(Q_1Q_2)\tilde{A}=Q_1(Q_2\tilde{A})$,

(iii) $Q_1(\tilde{A}+\tilde{B})=Q_1\tilde{A}+Q_1\tilde{B}$,

(iv) $I_n(\tilde{A})=\tilde{A}$,

(v) $(Q_1\tilde{A})\tilde{B}=Q_1(\tilde{A}\tilde{B})$,

(vi) $(\tilde{A}Q_1)\tilde{B}=\tilde{A}(Q_1\tilde{B})$.

Proof. (ii), (iii), (iv), (v) and (vi) can be easily shown. Now we will prove (i): Let

$\tilde{A}=A+Bi+Cj+Dk\in M_n(H_{\mathbb{R}})$ and $Q_1,Q_2\in M_n(\mathbb{C})$. From (9) we get

$$
\begin{aligned}
(Q_1+Q_2)\tilde{A} &= (Q_1+Q_2)A+(Q_1+Q_2)Bi+(Q_1+Q_2)Cj+(Q_1+Q_2)Dk \\
&= (Q_1A+Q_2A)+(Q_1B+Q_2B)i+(Q_1C+Q_2C)j+(Q_1D+Q_2D)k \\
&= (Q_1A+Q_1Bi+Q_1Cj+Q_1Dk)+(Q_2A+Q_2Bi+Q_2Cj+Q_2Dk) \\
&= Q_1\tilde{A}+Q_2\tilde{A}.
\end{aligned}
$$

The properties of the right multiplication can be shown in the same way. With the properties above, the following theorems can be given without the proof.

Theorem 5.1. $M_n(H_{\mathbb{R}})$ is a 4-dimensional module left (right) module over $M_n(\mathbb{R})$.

For all $\tilde{A}=A+Bi+Cj+Dk\in M_n(H_{\mathbb{R}})$ we can write

$$\tilde{A}=A\tilde{1}+B\tilde{I}+C\tilde{J}+D\tilde{K}$$

where $\tilde{1}=I_n,\tilde{I}=iI_n,\tilde{J}=jI_n,\tilde{K}=kI_n$. So, the span of $M_n(H_{\mathbb{R}})$ module over the ring $M_n(\mathbb{R})$ is

$$S_1=\{\tilde{1},\tilde{I},\tilde{J},\tilde{K}\}$$

where

$$\tilde{1}=\begin{pmatrix}1&0&\dots&0\\0&1&\dots&0\\\dots&\dots&\ddots&\vdots\\0&0&\dots&1\end{pmatrix},\tilde{I}=\begin{pmatrix}i&0&\dots&0\\0&i&\dots&0\\\dots&\dots&\ddots&\vdots\\0&0&\dots&i\end{pmatrix},\tilde{J}=\begin{pmatrix}j&0&\dots&0\\0&j&\dots&0\\\dots&\dots&\ddots&\vdots\\0&0&\dots&j\end{pmatrix},\tilde{K}=\begin{pmatrix}k&0&\dots&0\\0&k&\dots&0\\\dots&\dots&\ddots&\vdots\\0&0&\dots&k\end{pmatrix}\in M_n(H_{\mathbb{R}})$$

and

$$\tilde{1}^2=\tilde{I}_n,\tilde{I}^2=\tilde{J}^2=\tilde{K}^2=-\tilde{I}_n,\tilde{I}\tilde{J}=-\tilde{J}\tilde{I}=\tilde{K},\tilde{J}\tilde{K}=-\tilde{K}\tilde{J}=\tilde{I},\tilde{K}\tilde{I}=-\tilde{I}\tilde{K}=\tilde{J}.$$

Theorem 5.2. $M_n(H_{\mathbb{R}})$ is a 2-dimensional module module over $M_n(\mathbb{C})$.

For all $\tilde{A}=A+Bi+Cj+Dk\in M_n(H_{\mathbb{R}})$ we can write

$$\tilde{A}=Z_1+Z_2\tilde{J}$$

where $Z_1=A\tilde{1}+B\tilde{I},Z_2=C\tilde{1}+D\tilde{I}\in M_n(\mathbb{C})$ and $\tilde{1}=I_n,\tilde{I}=iI_n,\tilde{J}=jI_n,\tilde{K}=kI_n$. So, the span of $M_n(H_{\mathbb{R}})$ module over the ring $M_n(\mathbb{C})$ is

$$S_2=\{\tilde{1},\tilde{J}\}$$

where

$$\tilde{1}^2=\tilde{I}_n,\tilde{I}^2=\tilde{J}^2=\tilde{K}^2=-\tilde{I}_n,\tilde{I}\tilde{J}=-\tilde{J}\tilde{I}=\tilde{K},\tilde{J}\tilde{K}=-\tilde{K}\tilde{J}=\tilde{I},\tilde{K}\tilde{I}=-\tilde{I}\tilde{K}=\tilde{J}.$$

Definition 5.4. Let $\tilde{A} = A + Bi + Cj + Dk \in M_n(H_{\mathbb{R}})$ and $Q \in M_n(\mathbb{R})$ where $A,B,C,D \in M_n(\mathbb{R})$. We define the Kronecker product of a $n \times n$ real quaternion matrix with $4n \times 4n$ real matrix as follows:

$$Q \otimes \Re_{\tilde{A}} = Q \otimes \begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix} = \begin{pmatrix} QA & -QB & -QC & -QD \\ QB & QA & -QD & QC \\ QC & QD & QA & -QB \\ QD & -QC & QB & QA \end{pmatrix} = \Re_{Q\tilde{A}}. \tag{10}$$

Let $\tilde{A}, \tilde{B} \in M_n(H_{\mathbb{R}})$, $Q \in M_n(\mathbb{R})$ and $\Re_{\tilde{A}}, \Re_{\tilde{B}}, \Re_Q \in S_{4n}(\mathbb{R})$. Then the following properties are satisfied;

(i) $Q \otimes \Re_{\tilde{A}} = \Re_{Q\tilde{A}} = \Re_Q \Re_{\tilde{A}}$,

(ii) $Q \otimes (\Re_{\tilde{A}} + \Re_{\tilde{B}}) = Q \otimes \Re_{\tilde{A}} + Q \otimes \Re_{\tilde{B}}$,

(iii) $(Q_1 + Q_2) \otimes \Re_{\tilde{A}} = Q_1 \otimes \Re_{\tilde{A}} + Q_2 \otimes \Re_{\tilde{A}}$, for $Q_1, Q_2 \in M_n(\mathbb{R})$,

(iv) $(Q_1 Q_2) \otimes \Re_{\tilde{A}} = Q_1 \otimes (Q_2 \otimes \Re_{\tilde{A}})$, for $Q_1, Q_2 \in M_n(\mathbb{R})$.

Note 5.1. The real matrix representation of quaternion matrix obtained as a result of this external operation is equal to Kronecker product of $Q \in M_n(\mathbb{R})$ and $\Re_{\tilde{A}}$.

Now, we will investigate some new properties of the real matrix representation of a quaternion matrix.

Example 5.1. Let $\tilde{A} = \begin{pmatrix} 1+2i-4j+5k & 2-i+2j+k \\ 2+5i+j-2k & 3+4i-3j+k \end{pmatrix}$ be a real quaternion matrix.

Then the right real matrix representation of $\tilde{A}$ is

$$\Re_{\tilde{A}} = \begin{pmatrix} 1 & 2 & -2 & 1 & 4 & -2 & -5 & -1 \\ 2 & 3 & -5 & -4 & -1 & 3 & 2 & -1 \\ 2 & -1 & 1 & 2 & -5 & -1 & -4 & 2 \\ 5 & 4 & 2 & 3 & 2 & -1 & 1 & -3 \\ -4 & 2 & 5 & 1 & 1 & 2 & -2 & 1 \\ 1 & -3 & -2 & 1 & 2 & 3 & -5 & -4 \\ 5 & 1 & 4 & -2 & 2 & -1 & 1 & 2 \\ -2 & 1 & -1 & 3 & 5 & 4 & 2 & 3 \end{pmatrix}.$$

and the left real matrix representation of $\tilde{A}$ is

$$\mathcal{L}_{\tilde{A}} = \begin{pmatrix} 1 & 2 & 2 & -1 & -4 & 2 & 5 & 1 \\ 2 & 3 & 5 & 4 & 1 & -3 & -2 & 1 \\ -2 & 1 & 1 & 2 & -5 & -1 & -4 & 2 \\ -5 & -4 & 2 & 3 & 2 & -1 & 1 & -3 \\ 4 & -2 & 5 & 1 & 1 & 2 & -2 & 1 \\ -1 & 3 & -2 & 1 & 2 & 3 & -5 & -4 \\ -5 & -1 & 4 & -2 & 2 & -1 & 1 & 2 \\ 2 & -1 & -1 & 3 & 5 & 4 & 2 & 3 \end{pmatrix}.$$

We can find the real representation of real quaternion matrix in Example 5.1 with Matlab. The Matlab command, followed by the output, is shown in the box below.

```
>> m=2; A=[1 2; 2 3]; B=[2 -1; 5 4]; C=[-4 2; 1 -3]; D=[5 1; -2 1];
Re1=vertcat(A, B, C, D); Re2=vertcat(-B, A, D, -C);
Re3=vertcat(-C, -D, A, B); Re4=vertcat(-D, C, -B, A);
Re=horzcat(Re1, Re2, Re3, Re4)
Re =
 1  2 -2  1  4 -2 -5 -1
 2  3 -5 -4 -1  3  2 -1
 2 -1  1  2 -5 -1 -4  2
 5  4  2  3  2  -1  1 -3
-4  2  5  1  1  2  -2  1
 1 -3 -2  1  2  3  -5 -4
 5  1  4 -2  2 -1   1  2
-2  1 -1  3  5  4   2  3
Le1=vertcat(A, -B, -C, -D); Le2=vertcat(B, A, D, -C);
Le3=vertcat(C, -D, A, B); Le4=vertcat(D, C, -B, A);
Le=horzcat(Le1, Le2, Le3, Le4)
Le =
 1  2  2 -1 -4  2  5  1
 2  3  5  4  1  -3 -2 1
-2  1  1  2 -5 -1 -4  2
-5 -4  2  3  2 -1  1 -3
 4 -2  5  1  1  2  -2  1
-1  3 -2  1  2  3 -5 -4
-5 -1  4 -2  2 -1  1  2
 2 -1 -1  3  5  4  2   3
```

Theorem 5.3. Let $\tilde{A}, \tilde{B} \in M_n(H_{\mathbb{R}})$. Then the followings are satisfied;

(i) $\Re_{\tilde{I}_n} = I_{4n}$,

(ii) Let $\tilde{A} \in M_n(H_{\mathbb{R}})$. Then $\tilde{A}$ is Hermitian, anti-Hermitian, unitary or normal if and only if $\Re_{\tilde{A}}$ is symmetric, anti-symmetric, orthogonal or normal, respectively,

(iii) Let $\tilde{A} \in M_n(H_{\mathbb{R}})$ be an invertible matrix. If the inverse of the $4n \times 4n$ real matrix is

$$(\Re_{\tilde{A}})^{-1} = \begin{pmatrix} A_2 & -B_2 & -C_2 & -D_2 \\ B_2 & A_2 & -D_2 & C_2 \\ C_2 & D_2 & A_2 & -B_2 \\ D_2 & -C_2 & B_2 & A_2 \end{pmatrix} = \Re_{\tilde{A}^{-1}}$$

where $A_2, B_2, C_2, D_2 \in M_n(\mathbb{R})$ then

$$\tilde{A}^{-1} = A_2 + B_2 i + C_2 j + D_2 k,$$

(iv) Let $\tilde{A} \in M_n(H_{\mathbb{R}})$ be an invertible matrix. If the inverse of the $4n \times 4n$ real matrix is

$$(\mathcal{L}_{\tilde{A}})^{-1} = \begin{pmatrix} A_2 & B_2 & C_2 & D_2 \\ -B_2 & A_2 & -D_2 & C_2 \\ -C_2 & D_2 & A_2 & -B_2 \\ -D_2 & -C_2 & B_2 & A_2 \end{pmatrix} = \mathcal{L}_{\tilde{A}^{-1}}$$

where $A_2, B_2, C_2, D_2 \in M_n(\mathbb{R})$ then

$$\tilde{A}^{-1} = A_2 + B_2 i + C_2 j + D_2 k .$$

Proof. (i) and (iv) can be easily shown. Now we will prove one condition of (ii):

Let $\tilde{A} \in M_n(H_{\mathbb{R}})$. Firstly, $\tilde{A}$ is Hermitian matrix. Then we get

$$A = A^T, B = -B^T, C = -C^T, D = -D^T. \tag{11}$$

and

$$(\Re_{\tilde{A}})^T = \begin{pmatrix} A^T & B^T & C^T & D^T \\ -B^T & A^T & D^T & -C^T \\ -C^T & -D^T & A^T & B^T \\ -D^T & C^T & -B^T & A^T \end{pmatrix}. \tag{12}$$

If we use (11) in (12), we get

$$(\Re_{\tilde{A}})^T = \Re_{\tilde{A}} .$$

Thus we find, if $\tilde{A}$ is Hermitian matrix, then $\Re_{\tilde{A}}$ is symmetric matrix. Secondly, if $\Re_{\tilde{A}}$ is symmetric matrix, we get (11). By using (11) in $\tilde{A}$, we obtain

$$\tilde{A}^* = -\tilde{A} .$$

So we find, if $\Re_{\tilde{A}}$ is symmetric matrix, then $\tilde{A}$ is Hermitian matrix. Consequently, $\tilde{A}$ is Hermitian if and only if $\Re_{\tilde{A}}$ is symmetric.

The other conditions can be done in the same way. Then, we will prove (iii):

If $\tilde{A}$ is invertible then $\tilde{A}\tilde{A}^{-1} = \tilde{A}^{-1}\tilde{A} = I_n$. By property (i), we get

$$I_{4n} = \Re_{\tilde{I}_n} = \Re_{\tilde{A}\tilde{A}^{-1}} = \Re_{\tilde{A}}\Re_{\tilde{A}^{-1}}.$$

Thus, we obtain

$$(\Re_{\tilde{A}})^{-1} = \Re_{\tilde{A}^{-1}}.$$

Hence, we may find inverse of $\tilde{A}$ by using Theorem 5.3 / (iii), (iv). Following example will show us how to find the inverse of a dual quaternion matrix by using inverse of its real matrix representation.

Example 5.2. Let $\tilde{A} = \begin{pmatrix} i & j \\ 0 & k \end{pmatrix} \in M_2(H_{\mathbb{R}})$. The right real matrix representation of $\tilde{A}$ is

$$\Re_{\tilde{A}} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here $det(\Re_{\tilde{A}}) = 1 \neq 0$. Hence, $\tilde{A}$ is invertible.

The inverse of $(\Re_{\tilde{A}})^{-1}$ is found by using Matlab as follows:

$$(\Re_{\tilde{A}})^{-1} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By using the inverse of $\Re_{\tilde{A}}$ we find

$$\tilde{A}^{-1} = \begin{pmatrix} -i & 1 \\ 0 & -k \end{pmatrix}.$$

Moreover, the left real matrix representation of $\tilde{A}$ is

$$\mathcal{L}_{\tilde{A}}=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here $det(\mathcal{L}_{\tilde{A}})=1\neq 0$. Hence, $\tilde{A}$ is invertible.

The inverse of $(\mathcal{L}_{\tilde{A}})^{-1}$ is found by using Matlab as follows:

$$(\mathcal{L}_{\tilde{A}})^{-1}=\begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By using the inverse of $\mathcal{L}_{\tilde{A}}$ we find

$$\tilde{A}^{-1}=\begin{pmatrix} -i & 1 \\ 0 & -k \end{pmatrix}.$$

Corollary 5.2. Each real quaternion matrix can be factorized. The number of elementary row (column) operations for a real matrix representation of a quaternion matrix is four times than the number of elementary row (column) operations for this quaternion matrix.

6. THE INVERSE OF 2×2 REAL QUATERNION MATRICES

We will find the inverse of 2×2 real quaternion block matrices.

Theorem 6.1. The inverse of a 2×2 real quaternion block matrix can be calculated by

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}^{-1}=\begin{pmatrix} (\tilde{A}-\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1} & -(\tilde{A}-\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1} \\ -\tilde{D}^{-1}\tilde{C}(\tilde{A}-\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1} & \tilde{D}^{-1}+\tilde{D}^{-1}\tilde{C}(\tilde{A}-\tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1} \end{pmatrix}. \tag{13}$$

where $\tilde{A},\tilde{B},\tilde{C},\tilde{D}\in M_n(H_{\mathbb{R}})$ if $\tilde{D}$ is invertible.

Proof.

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}.$$

where $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in M_n(H_{\mathbb{R}})$, $x, y, c, d \in H_{\mathbb{R}}$. Namely,

$$\begin{aligned} \tilde{A}x + \tilde{B}y &= c \\ \tilde{C}x + \tilde{D}y &= d \end{aligned}$$

If $\tilde{D}$ is invertible, we solve for y getting

$$y = \tilde{D}^{-1}(d - \tilde{C}x)$$

and using y in the first equation, we get

$$(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})x = c - \tilde{B}\tilde{D}^{-1}d.$$

If the quaternion matrix $(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}$ is invertible, then we can get

$$\begin{aligned} x &= (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}(c - \tilde{B}\tilde{D}^{-1}d) \\ y &= \tilde{D}^{-1}(d - \tilde{C}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}(c - \tilde{B}\tilde{D}^{-1}d)). \end{aligned}$$

Then, we get the inverse of $\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$ as

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}^{-1} = \begin{pmatrix} (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1} & -(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1} \\ -\tilde{D}^{-1}\tilde{C}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1} & \tilde{D}^{-1} + \tilde{D}^{-1}\tilde{C}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1}\tilde{B}\tilde{D}^{-1} \end{pmatrix}. \tag{14}$$

We can write (14) as

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{I}_n & \tilde{0}_n \\ -\tilde{D}^{-1}\tilde{C} & \tilde{I}_n \end{pmatrix}\begin{pmatrix} (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1} & \tilde{0}_n \\ \tilde{0}_n & \tilde{D}^{-1} \end{pmatrix}\begin{pmatrix} \tilde{I}_n & -\tilde{B}\tilde{D}^{-1} \\ \tilde{0}_n & \tilde{I}_n \end{pmatrix}$$

and we obtain

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} \tilde{I}_n & \tilde{B}\tilde{D}^{-1} \\ \tilde{0}_n & \tilde{I}_n \end{pmatrix}\begin{pmatrix} (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C}) & \tilde{0}_n \\ \tilde{0}_n & \tilde{D} \end{pmatrix}\begin{pmatrix} \tilde{I}_n & \tilde{0}_n \\ \tilde{D}^{-1}\tilde{C} & \tilde{I}_n \end{pmatrix}$$

The above expression shows that only the inverse of $\tilde{D}$ is needed.

Theorem 6.2. The inverse of a 2×2 real quaternion block matrix can be calculated by

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{A}^{-1} + \tilde{A}^{-1}\tilde{B}(\tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B})^{-1}\tilde{C}\tilde{A}^{-1} & -\tilde{A}^{-1}\tilde{B}(\tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B})^{-1} \\ -(\tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B})^{-1}\tilde{C}\tilde{A}^{-1} & (\tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B})^{-1} \end{pmatrix}. \tag{15}$$

where $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in M_n(H_{\mathbb{R}})$ if $\tilde{A}$ is invertible.

Example 6.1. Let $\tilde{Q} = \begin{pmatrix} i & j & 0 & k \\ 0 & k & j & 0 \\ 1 & 0 & i & j \\ 0 & k & 0 & k \end{pmatrix} \in M_4(H_{\mathbb{R}})$ be a real quaternion matrix.

We can write $\tilde{Q}$ as 2×2 real quaternion block matrix.

$$\tilde{Q} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$$

where $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in M_2(H_{\mathbb{R}})$. $\tilde{D}$ is invertible. Then by using Theorem 6.1, we get

$$\tilde{Q}^{-1} = \begin{pmatrix} \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j + \frac{1}{2}k & \frac{1}{2} - \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k & \frac{1}{2} - \frac{1}{2}i - \frac{1}{2}j - \frac{1}{2}k & i \\ -\frac{1}{2}i - \frac{1}{2}j & \frac{1}{2} - \frac{1}{2}k & -\frac{1}{2} - \frac{1}{2}k & 0 \\ \frac{1}{2} - \frac{1}{2}k & \frac{1}{2}i - \frac{1}{2}j & -\frac{1}{2}i + \frac{1}{2}j & 0 \\ \frac{1}{2}i + \frac{1}{2}j & -\frac{1}{2} + \frac{1}{2}k & \frac{1}{2} + \frac{1}{2}k & -k \end{pmatrix}.$$

7. THE DETERMINANT OF 2×2 REAL QUATERNION MATRICES

We now discuss the determinant of 2×2 real quaternion matrices. We will find (8) by a different method. The determinant of a 2×2 real quaternion matrix $\tilde{A}$ is a real quaternion number and denoted by $det(\tilde{A})$.

Theorem 7.1. Let $\tilde{A} = A + Bi + Cj + Dk \in M_2(H_{\mathbb{R}})$. Then,

$$\begin{aligned} det(\tilde{A}) &= det(A) - det(B) - det(C) - det(D) + (\sum_{s=1}^{2} \{det([A\,|\,B]_s) + det([C\,|\,D]_s^*)\})i \\ &+ (\sum_{s=1}^{2} \{det([A\,|\,C]_s) + det([D\,|\,B]_s^*)\})j + (\sum_{s=1}^{2} \{det([A\,|\,D]_s) + det([B\,|\,C]_s^*)\})k. \end{aligned} \tag{16}$$

Proof. Let $\hat{A} = A + Bi + Cj + Dk \in M_2(H_{\mathbb{R}})$. By $det([A\,|\,B]_s), det([A\,|\,C]_s), det([A\,|\,D]_s)$, $det([C\,|\,D]_s^*), det([D\,|\,B]_s^*), det([B\,|\,C]_s^*)$ and usual determinant of A, B, C, D, we can obtain (16), easily.

Note 7.1. The matrices $[A|B]_s, [A|C]_s$ and $[A|D]_s$ mean changing $s.$ column components of the matrices B,C and D instead of $s.$ column components of matrix A. The matrices $[C|D]_s^*$ mean changing $s.$ column components of the matrices D instead of $s.$ column components of matrix C . The matrices $[D|B]_s^*$ mean changing $s.$ column components of the matrices B instead of $s.$ column components of matrix D. The matrices $[B|C]_s^*$ mean changing $s.$ column components of the matrices C instead of $s.$ column components of matrix B.

$*$ means product $s.$ component of the second column with minus one.

Example 7.1. Let $\tilde{A} = \begin{pmatrix} 2+i+2j+k & -1-i+2j+k \\ 3+2i-j+k & 1+2i+j+2k \end{pmatrix} \in M_2(H_{\mathbb{R}})$. Then we get the determinant of $\tilde{A}$ as

$$det(\tilde{A}) = -4+10i-6j+3k.$$

We can find determinant of real quaternion matrix on Example 7.1 by (16) with Matlab. The Matlab command, followed by the output, is shown in the box below.

```
>> A=[2 -1;3 1]; B=[1 -1;2 2]; C=[2 2;-1 1]; D=[1 1;1 2]; Aoriginal=A; Boriginal=B;
Coriginal=C; Doriginal=D; X1=det(A)-det(B)-det(C)-det(D); X2=0; X3=0; X4=0;
for r = 1:2
B(:,r) = A(:,r);
D(:,r) = C(:,r);
D(2,r) = -D(2,r);
X2 = X2+det(B)+det(D);
B=Boriginal;
D=Doriginal;
end
for r = 1:2
C(:,r) = A(:,r);
B(:,r) =D(:,r);
B(2,r)= -B(2,r);
X3 = X3+det(C)+det(B);
B=Boriginal;
C=Coriginal;
end
for r = 1:2
D(:,r) = A(:,r);
C(:,r) = B(:,r);
C(2,r)= -C(2,r);
X4 = X4+det(C)+det(D);
C=Coriginal;
D=Doriginal;
end
disp(['Det= 'num2str(X1) ' + 'num2str(X2) ' i 'num2str(X3) ' j +'num2str(X4) ' k ' ])
Det= -4 + 10i - 6j + 3k
```

Note 7.2. It is difficult to directly calculate the determinant of the real quaternion matrix, the determinant of the 2×2 real quaternion matrices can be easily calculated by this method with Matlab.

REFERENCES

[1]. Hamilton W.R., Elements of Quaternions, London: Longmans, Green & Company, 1866.

[2]. Brenner J.L., Matrices of Quaternions, Pacific Journal of Mathematics, 1 (1951) 329-335.

[3]. Erdoğdu M. and Özdemir M., On Complex Split Quaternion Matrices, Advances in Applied Clifford Algebras, 23-3 (2013) 625-638.

[4]. Erdoğdu M. and Özdemir M., Split Quaternion Matrix Representation of Dual Split Quaternions and Their Matrices, Advances in Applied Clifford Algebras, 25-4 (2015) 787-798.

[5]. Kösal H. H. and Tosun, M., Commutative Quaternion Matrices, Advances in Applied Clifford Algebras, 24-3 (2014) 769-779.

[6]. Wiegmann N.A., Some Theorems on Matrices with Real Quaternion Elements, Canadian Journal of Mathematics, 7 (1955) 191-201.

[7]. Wolf L.A., Similarity of Matrices in which the Elements are Real Quaternions, Bulletin of the American Mathematical Society, 42-10 (1936) 737-743.

[8]. Zhang F., Quaternions and Matrices of Quaternions, Linear Algebra and Its Applications, 251 (1997) 21-57.

[9]. Morais J.P., Georgiev S. and Sprößig W., Real Quaternionic Calculus Handbook, Birkhäuser, Springer Basel, 2014.

[10]. Huang L. and So W., On Left Eigenvalues of a Quaternionic Matrix, Linear Algebra and Its Applications, 323 (2001) 105-116.

[11]. Zhang F., Geršgorin Type Theorems for Quaternionic Matrices, Linear Algebra and Its Applications, 424 (2007) 139-153.

[12]. Cohen N. and De Leo S., The Quaternionic Determinant, Electronic Journal of Linear Algebra, 7 (2000) 100-111.

[13]. Aslaksen H., Quaternionic Determinants, The Mathematical Intelligencer, 18-3 (1996) 57-65.

[14]. Gelfand I., Retakh V. and Wilson R.L., Quaternionic Quasideterminants and Determinants, Translations of the American Mathematical Society-Series 2, 210 (2003) 111-124.

[15]. Bagazgoitia A., A Determinantal Identity for Quaternions, In Proceedings of 1983 Conference on Algebra Lineal y Aplicaciones, Vitoria-Gasteiz, Spain, (1984) 127-132.

[16]. Lewis D., A Determinantal Identity for Skewfields, Linear algebra and its applications, 71 (1985) 213-217.

[17]. Jiang T. S. and Wei M. S., On a Solution of the Quaternion Matrix Equation $X - A\tilde{X}B = C$ and Its Application, Acta Mathematica Sinica, 21-3 (2005) 483-490.

[18]. Song C., Feng J.-e, Wang X. and Zhao J., A Real Representation Method for Solving Yakubovich-j-Conjugate Quaternion Matrix Equation, Abstract and Applied Analysis, Hindawi, (2014).

[19]. Tian Y., Universal Factorization Equalities for Quaternion Matrices and Their Applications, Mathematical Journal of Okayama University, 41 (1999) 45-62.

[20]. Rodman L., Topics in Quaternion Linear Algebra, Princeton: Princeton University Press, 2014.

[21]. Lin Y. and Wang Q.-W., Completing a Block Matrix of Real Quaternions with a Partial Specified Inverse, Journal of Applied Mathematics, (2013).

[22]. Al-Zhour Z., Some New Linear Representations of Matrix Quaternions with Some Applications, Journal of King Saud University-Science, (2017).

[23]. Ahmad S.S. and Ali I., Bounds for Eigenvalues of Matrix Polynomials over Quaternion Division Algebra, Advances in Applied Clifford Algebras, 26-4 (2016) 1095-1125.

[24]. Song G. and Zhou Y., Block Independence in Various Generalized Inverses of Partitioned Quaternion Matrices, Iranian Journal of Science and Technology, Transactions A: Science, (2018) 1-10.
[25]. Hamilton W.R., Lectures on Quaternions, Dublin: Hodges and Smith, 1853.
[26]. Ward J., Quaternions and Cayley Numbers: Algebra and Applications, Mathematics and Its Applications, Dordrecht: Kluwer, 1997.
[27]. Powell P.D., Calculating Determinants of Block Matrices, arXiv:1112.4379 (2011).
[28]. Silvester J.R., Determinants of Block Matrices, The Mathematical Gazette, 84 (2000) 460-467.
[29]. Tian Y. and Takane Y., More on Generalized Inverses of Partitioned Matrices with Banachiewicz-Schur forms, Linear Algebra and its Applications, 430 (2009) 1641-1655.
[30]. Meyer Jr C.D., Generalized Inverses and Ranks of Block Matrices, SIAM Journal on Applied Mathematics, 25-4 (1973) 597-602.
[31]. Moors E., On the Reciprocal of the General Algebraic Matrix, Bull. Amer. Math. Soc., 26 (1920) 394-395.
[32]. Penrose R., A Generalized Inverse for Matrices, Mathematical Proceedings of the Cambridge Philosophical Society, Cambridge University Press, 51 (1955) 406-413.
[33]. Zhang F., The Schur Complement and Its Applications, Springer Science & Business Media, 2006.
[34]. Banachiewicz T., Zur Berechnung der Determinanten, wie auch der Inversen und zur darauf basierten Auflosung der Systeme linearer Gleichungen, Acta Astronom. Ser. C, 3 (1937) 41-67.
[35]. Frazer R.A., Duncan W.J. and Collar A. R., Elementary Matrices and Some Applications to Dynamics and Differential Equations, Cambridge: Cambridge University Press, 1938.
[36]. Gallier J., The Schur Complement and Symmetric Positive Semidefinite (and definite) Matrices, Penn Engineering, (2010).
[37]. Horn R.A. and Johnson C.R., Matrix Analysis, Cambridge: Cambridge University Press, 1990.
[38]. De Leo S., Scolarici G. and Solombrino L., Quaternionic Eigenvalue Problem, Journal of Mathematical Physics, 43-11 (2002) 5815-5829.
[39]. Jiang T. and Ling S., On a Solution of the Quaternion Matrix Equation $A\tilde{X} - XB = C$ and Its Applications, Advances in Applied Clifford Algebras, 23-3 (2013) 689-699.