Selfadjointness and Positiveness of the Differential Operators Generated by New Type Sturm-Liouville Problems

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Abstract. It is purpose of this paper to investigate Sturm-Liouville equation $-u''(x) + q(x)u(x) = \lambda u(x)$ on many-interval $[\alpha, \gamma_1^-) \cup (\gamma_1^+, \gamma_2^-) \cup (\gamma_2^+, \beta]$ with the eigenvalue parameter appearing linearly in the boundary conditions and with two supplementary transmission conditions. The classical Sturmian theory did not cover such type of many-interval boundary value transmission problems. For the classical Sturm-Liouville problems it is guaranteed that the problem is self-adjoint with compact resolvent, the spectrum is disctrete and consist of eigenvalues and the corresponding eigenfunctions form an orthogonal basis in the well-known Hilbert space $L_2[\alpha, \beta]$. But the boundary-value-transmission problems are not self-adjoint and the system of eigenfunctions did not form a basis in the classical Hilbert space $L_2[\alpha, \beta]$ in general. Taking in view this fact we suggest a new approach for self-adjoint realization of such type transmission problems. Moreover, we define some new Hilbert spaces to establish positiveness of corresponding operator-pencil. At first we define a concept of generalized eigenfunctions for this kind of spectral problems. In particular it is shown that if the potential $q(x)$ is continuous then the generalized eigenfunctions satisfies the considered problem is the classical sense. Then we introduce to the consideration some compact operators such a way that the considered boundary-value-transmission problem can be reduced to the appropriate operator-pencil equation. Finally, we prove that this operator-pencil is self-adjoint and positive definite for sufficiently large negative values of the eigenparameter. It is important to note that the obtained results extends classical results associated with regular Sturm-Liouville problems.

Keywords: Boundary value problems, eigenfunctions, boundary and transmission conditions, positive operators.

Yeni Tipten Sturm-Liouville Problemlerinin Ürettiği Diferansiyel Operatörlerin Kendine Eşlenikliği ve Pozitivliği

Özet. Bu makalenin amacı $[\alpha, \gamma_1^-) \cup (\gamma_1^+, \gamma_2^-) \cup (\gamma_2^+, \beta]$ çok-aralığında tanımlı olan, özdeğer parametresini doğrusal olarak sınır şartlarında bulunduran ve iki tane ek geçiş şartı içeren $-u''(x) + q(x)u(x) = \lambda u(x)$ Sturm-Liouville problemini araştırmaktır. Klasik Sturm-Liouville teorisi bu tipten çok-aralıklı sınır-değer-geçiş problemlerini kapsamaktadır. Klasik Sturm-Liouville problemleri için kendine-eşleniklik, rezolventin kompaktlığı, spektrumun diskretliği ve uygun özfonksiyonların iyi bilinen $L_2[\alpha, \beta]$ Hilbert uzayında ortogonal baz oluşturma özelliği sağlanmaktadır. Genellikle sınır-değer-geçiş problemleri kendine-eşlenik değildir ve özfonksiyonlar sistemi klasik $L_2[\alpha, \beta]$ Hilbert uzayında baz oluşturmuyor. Bunu dikkate alarak, bu tipten geçiş problemlerinin kendine-eşlenik biçimde sonuçlanabilmesi için yeni bir yaklaşım önermişiz.

Bunun dışında uygun operatör-demetinin pozitifliğini gösterebilmek için bazı yeni Hilbert uzayları tanımladık. İlk olarak bu türden spektral problemlerin genelleştirilmiş özfonksiyonları kavramını tanımladık. Özel olarak gösterdik ki, eğer $q(x)$ potansiyeli sürekli ise, o halde genelleşmiş özfonksiyonlar incelediğimiz problemi klasik anlamda da sağlıyor. Daha sonra bazı kompakt operatörleri öyle tanımladık ki araştırılan sınır-değer-geçiş problemlerini uygun operatör demetine dönüştürmek mümkün olsun. Son olarak özdeğer parametresinin mutlak değerce yeteri kadar büyük negatif değerleri için bu operatör demetinin kendine eşlenik ve pozitif olduğunu ispat ettik. Elde edilen sonuçların düzgün Sturm-Liouville problemlerinin sağladığı klasik sonuçları genelleştirmesi önem arz etmektedir.

Anahtar Kelimeler: Sınır değer problemleri, özfonksiyonlar, sınır ve geçiş şartları, pozitif operatörler.

1. INTRODUCTION

In the present work we consider a Sturm-Liouville equation

$$-u''(x) + q(x)u(x) = \lambda u(x) \tag{1}$$

on $[\alpha, \gamma_1^-) \cup (\gamma_1^+, \gamma_2^-) \cup (\gamma_2^+, \beta]$, together with eigendependent boundary conditions at the end-points $x = \alpha$ and $x = \beta$, given by

$$\kappa_1 u(\alpha) - \kappa_3 u'(\alpha) - \lambda \kappa_2 u'(\alpha) = 0 \tag{2}$$
$$u'(\beta) = 0 \tag{3}$$

and with transmission conditions at two interior points $x = \gamma_1$ and $x = \gamma_2$, given by

$$u(\gamma_1^+) - \tau_1 u(\gamma_1^-) = 0 \tag{4}$$

$$u'(\gamma_1^+) - \frac{1}{\tau_1} u'(\gamma_1^-) - u(\gamma_1^-) = 0 \tag{5}$$

$$u(\gamma_2^+) - \tau_2 u(\gamma_2^-) = 0 \tag{6}$$

$$u'(\gamma_2^+) - \frac{1}{\tau_2} u'(\gamma_2^-) - u(\gamma_2^-) = 0 \tag{7}$$

where $q(x)$ is a real valued function which is continuous in $\mho_1 = (\alpha, \gamma_1^-)$, $\mho_2 = (\gamma_1^+, \gamma_2^-)$ $\mho_3 = (\gamma_2^+, \beta)$ and has finite limits $q(\alpha + 0), q(\gamma_1^\pm \pm 0), q(\gamma_2^\pm \pm 0), q(\beta - 0), \lambda$ is a complex spectral parameter, κ_i, τ_j ($i = 1, 2, 3$ and $j = 1, 2$) are real numbers and $\rho = \kappa_1 \kappa_2 > 0$, $\tau_j > 0$.

Some properties of the Sturm-Liouville Problem (1)-(3) has been studied by many authors (see, e.g., [1-4]). The main goal of those papers is the analysis of the spectrum and justification of the eigenfunction expansion. The considerations of [2, 4] are based on the operator-theoretic formulation of the Sturm-Liouville problem (1)-(3). The discrete version of this problem was dealt with by Harmsen and Li [5], Atkinson [6]. The boundary value problem with the parameter appearing nonlinearly in the boundary conditions is also important and has a variety of applications. The continuous version of this problem was dealt with by Binding et al. [7-8], Greenberg and Babuska [9]. Oscillation and comparison results have been obtained in [7]. The completeness of the eigenfunctions and eigenfunction expansions in various function spaces for the Sturm-Liouville problems with spectral parameter in the boundary conditions have been considered in [10-16]. Problems with various singularities have been analyzed in [17-19, 20-30].

In different areas of applied mathematics and physics many problems arise in the form of boundary value problems involving transmission conditions at the interior singular points. Such problems are called boundary-value-transmission problems. This kind of problems arise in various problems of mathematics and physics as well as in applications. For example, some boundary value problems with transmission conditions arise in heat and mass transfer problems [31], in vibrating string problems [32], in diffraction problems [33] and etc. The basis properties and eigenfunction expansions in various function spaces of the eigenfunction of the Sturm-Liouville problems with spectral parameter in the boundary conditions have been considered in [10-13, 31, 34]. The concept of generalized solutions in a Hilbert space (see, [34]) allows the eigenvalue problem to be reduced to an operator-pencil equation. In suitable Hilbert space Belinskiy and Dauer in [10, 11] have considered the generalized eigenfunctions of a regular Sturm-Liouville problem on a finite interval with the eigenvalue parameter appearing linearly in the boundary conditions.

It is the aim of this paper is to reduce the boundary-value-transmission problem (1)-(7) to an operator pencil equation

$$L(\lambda)\psi = 0, \quad L(\lambda) := \sum_{i=0}^{n} \lambda^i A_i$$

in a corresponding Hilbert space. We introduce to the consideration some compact operators such a way that the considered boundary-value problem can be reduced to the appropriate operator-pencil equation and prove that this operator-pencil is self-adjoint and positive definite for sufficiently large negative values of the eigenparameter.

2. SOME AUXILIARY FACTS

We use the standard notations for the Hilbert spaces $\bigoplus_{i=1}^{3} L_2(\mho_i), \bigoplus_{i=1}^{3} H^1(\mho_i)$. The inner products in those spaces are given by

$$\langle u, \vartheta \rangle_0 = \int_{\alpha}^{\gamma_1^-} u(x)\overline{\vartheta}(x)dx + \int_{\gamma_1^+}^{\gamma_2^-} u(x)\overline{\vartheta}(x)dx + \int_{\gamma_2^+}^{\beta} u(x)\overline{\vartheta}(x)dx$$

and

$$\langle u, \vartheta \rangle_1 = \int_{\alpha}^{\gamma_1^-} \left\{ u'(x)\overline{\vartheta}'(x) + u(x)\overline{\vartheta}(x) \right\} dx$$

$$+ \int_{\gamma_1^+}^{\gamma_2^-} \left\{ u'(x)\overline{\vartheta}'(x) + u(x)\overline{\vartheta}(x) \right\} dx$$

$$+ \int_{\gamma_2^+}^{\beta} \left\{ u'(x)\overline{\vartheta}'(x) + u(x)\overline{\vartheta}(x) \right\} dx$$

with the corresponding norms are

$$\|u\|_0 = \sqrt{\langle u, u \rangle_0}$$

and

$$\|u\|_1 = \sqrt{\langle u, u \rangle_0} \tag{8}$$

respectively. According to the embedding theorems for the Sobolev spaces ([35]) the functions in $\bigoplus_{i=1}^{3} H^1(\mho_i)$ are continuous on $\mho_i$, but their (generalized) derivatives can only be assumed to be elements of $\bigoplus_{i=1}^{3} L_2(\mho_i)$.

In below we shall use the following inequalities which can be derived from the well-known embedding theorems for Sopbolev spaces (see, for example ([34])). For any $u \in \bigoplus_{i=1}^{3} H^1(\mho_i)$, the following inequalities hold

$$\left| u(x_j) \right|^2 \le d_j \left\| u' \right\|_0^2 + \frac{2}{d_j} \left\| u \right\|_0^2, \tag{9}$$

where $j = 1, 2, 3, 4, x_1 = \alpha, x_2 = \gamma_1^{\pm} \pm 0, x_3 = \gamma_2^{\pm} \pm 0, x_4 = \beta$ and $d_j (j = 1, 2, 3, 4)$ are any positive real numbers are small enough. Moreover, for any $\xi \in \mho = \mho_1 \cup \mho_2 \cup \mho_3$ the inequality

$$\left| u(\xi) \right| \le C(\xi) \left\| u \right\|_1 \tag{10}$$

holds, where the constant $C(\xi)$ is independent of the function u, i.e. is dependent only of ξ.

Remark 1 Note, the normal derivative of a function $u \in \bigoplus_{i=1}^{3} H^1(\mho_i)$, might not be defined as an element of $\bigoplus_{i=1}^{3} L_2(\mho_i)$.

Remark 2 Without loss of generality the function $q(x)$ can be assumed to be positive on $\mho$. In fact, after the shift of the spectral parameter $\lambda \to \lambda - h$, where $h = \max q(x)$ for $x \in \mho$, the function $q(x)$ can be asumed to be positive eveywhere on $\mho$.

3. THE CONCEPT OF GENERALIZED EIGENFUNCTION

Let us introduce to the consideration a new Hilbert space Ξ which is the main Hilbert space considered in the remainder of this article.

Definition 1 *The inner product in the Hilbert space* $\Xi = \bigoplus_{i=1}^{3} H^1(\mho_i) \oplus \mathbb{C}$ *is defined by*

$$\langle U, V \rangle_{\Xi} = \langle u(x), \vartheta(x) \rangle_{\bigoplus_{i=1}^{3} H^1(\mho_i)} + u_1 \overline{\vartheta_1} \tag{11}$$

for $U, V \in \Xi$, *where* $u(x), \vartheta(x) \in \bigoplus_{i=1}^{3} H^1(\mho_i)$ *and* $u_1, \vartheta_1 \in \mathbb{C}$.

For further investigation we shall introduce a new inner product on the same linear space $\bigoplus_{i=1}^{3} H^1(\mho_i)$ as follows:

$$\langle u, \vartheta \rangle_{\bigoplus_{i=1}^{3} H_q^1(\mho_i)} = \langle u, q\vartheta \rangle_0 + \langle u', \vartheta' \rangle_0 .$$

Since $q(x)$ is bounded, positively defined and measurable function, there exist constants $m>0$ and $M>0$ such that

$$m\|u\|_1 < \|u\|_{\bigoplus_{i=1}^{3} H^1(\mho_i)} < M\|u\|_1$$

for all $u(x) \in \bigoplus_{i=1}^{3} H^1(\mho_i)$.

Consequently the inner product (11) generates a norm that is equivalent to the standard norm produced by (8) and therefore the inner product space $\bigoplus_{i=1}^{3} H^1(\mho_i)$ is also Hilbert space.

Now we shall define a concept of generalized solution which is fundamental to this work. The definition of a generalized solutions of the Sturm-Liouville problem (1)-(7) follows by the known procedure (see [34], [36]). Multiplying equation (1) by a conjugate to an arbitrary function $\vartheta(x) \in \bigoplus_{i=1}^{3} H^1(\mho_i)$ and integrate by parts over the intervals $\mho_i (i = 1, 2, 3)$ and applying the boundary and transmission conditions (2)-(7), we can reduce it to the integral form

$$\int_{\mho_1} \left\{ u'(x)\overline{\vartheta'}(x) + q(x)u(x)\overline{\vartheta}(x) \right\} dx + \int_{\mho_2} \left\{ u'(x)\overline{\vartheta'}(x) + q(x)u(x)\overline{\vartheta}(x) \right\} dx$$
$$+ \int_{\mho_3} \left\{ u'(x)\overline{\vartheta'}(x) + q(x)u(x)\overline{\vartheta}(x) \right\} dx + \tau_1 u(\gamma_1^-)\overline{\vartheta}(\gamma_1^-) + \tau_2 u(\gamma_2^-)\overline{\vartheta}(\gamma_2^-) + u'(\alpha)\overline{\vartheta}(\alpha) \quad (12)$$
$$= \lambda \left\{ \int_{\mho_1} u(x)\overline{\vartheta}(x) dx + \int_{\mho_2} u(x)\overline{\vartheta}(x) dx + \int_{\mho_3} u(x)\overline{\vartheta}(x) dx \right\}.$$

According to the structure of the space $\bigoplus_{i=1}^{3} H^1(\mho_i)$ the values of derivatives at a point should be excluded from the idntity (12).

Denoting $\omega := \kappa_2 u'(\alpha)$ and taking in view the boundary condition (2), we have $u'(\alpha) = \dfrac{\omega}{\kappa_2}$. Then the boundary condition (2) can be written in the following form: $\kappa_1 u(\alpha) - \kappa_3 u'(\alpha) = \lambda\omega$.

Taking in view the last relation it is easy to see that the first boundary condition (2) takes the form

$$\frac{u(\alpha)}{\kappa_2} - \frac{\kappa_3}{\kappa_2} \frac{\omega}{\rho} = \lambda \frac{\omega}{\rho}. \tag{13}$$

Putting $u'(\alpha) = \dfrac{\omega}{\kappa_2}$ in the integral identity (17), we obtain

$$\langle u, \vartheta \rangle_{\underset{i=1}{\overset{3}{\oplus}} H^1(\mho_i)} + \tau_1 u(\gamma_1^-)\overline{\vartheta}(\gamma_1^-) + \tau_2 u(\gamma_2^-)\overline{\vartheta}(\gamma_2^-) + \frac{\omega}{\kappa_2}\overline{\vartheta}(\alpha) = \lambda \langle u, \vartheta \rangle_0 . \tag{14}$$

Thus the integro-differential equation (12) is transformed into the system of equalities (13)-(14) all terms of which are defined for the $u, \vartheta \in \overset{3}{\underset{i=1}{\oplus}} H^1(\mho_i)$.

Now we are ready to introduce a concept of generalized solution for the problem under consideration.

Definition 2 *The two-component element* $(u(x), \omega)$ *of the Hilbert space* Ξ *is said to be a generalized solution of the Sturm-Liouville problem (1)-(7), if this element satisfy the equations (13) and (14) for any* $\vartheta \in \overset{3}{\underset{i=1}{\oplus}} H^1(\mho_i)$.

The concept of the generalized solution is based on the following result.

Lemma 1 *Let the function* u *is twice continuously differentiable and the function* q *continuously on* $\mho$ *. Then the generalized solution of the Sturm-Liouville problem satisfies equations (1)-(7) in the classical sense.*

Consequently the concept of generalized solution is an extension of a classical solution.

4. OPERATOR-PENCIL REALIZATION OF THE PROBLEM

The reduction of identities (13)-(14) to an operator-pencil equation is based on the following result.

Lemma 2 *For the bilinear functionals*

$$\ell_1(u, \vartheta) := \tau_1 u(\gamma_1^-)\overline{\vartheta}(\gamma_1^-) + \tau_2 u(\gamma_2^-)\overline{\vartheta}(\gamma_2^-) , \tag{15}$$

$$\ell_2(u, \vartheta) := \langle u, \vartheta \rangle_0 , \tag{16}$$

and

$$\ell_3(\omega, \vartheta) := \frac{\omega}{\kappa_2}\overline{\vartheta}(\alpha) \tag{17}$$

there are such bounded linear operators T_1, T_2 *and* T_3 *respectively such that*

$$\ell_k(u, \vartheta) = \langle T_k u, \vartheta \rangle_{\underset{i=1}{\overset{3}{\oplus}} H^1(\mho_i)} \; for \, k = 1, 2$$

and

$$\ell_3(\omega, \vartheta) = \langle T_3 \omega, \vartheta \rangle_{\underset{i=1}{\overset{3}{\oplus}} H^1(\mho_i)} \; for \, k = 3. \tag{18}$$

Moreover the operators $T_1, T_2 : \overset{3}{\underset{i=1}{\oplus}} H^1(\mho_i) \to \overset{3}{\underset{i=1}{\oplus}} H^1(\mho_i)$ *and* $T_3 : \mathbb{C} \to \overset{3}{\underset{i=1}{\oplus}} H^1(\mho_i)$ *are compact, the operators* T_1 *and* T_2 *are selfadjoint and positive.*

Proof. The functionals $\ell_k \ (k=1,2,3)$ allows the following obvious estimates:

$$\left|\ell_1(u,\vartheta)\right| \le C_1 \left\{\left|\tau_1\right| \ \left|u(\gamma_1^-)\right|\left|\overline{\vartheta}(\gamma_1^-)\right| + \left|\tau_2\right| \ \left|u(\gamma_2^-)\right|\left|\overline{\vartheta}(\gamma_2^-)\right|\right\},$$

$$\left|\ell_2(u,\vartheta)\right| \le C_2 \left\|u\right\|\left\|\vartheta\right\|,$$

and

$$\left|\ell_3(\omega,\vartheta)\right| \le C_3 \left|\omega\right|\left|\vartheta(\alpha)\right|.$$

Here and below $C_k \ (k=1,2,...)$ denote different positive constant whose exact values are not important for the proof.

The inequality (9)-(10) imply

$$\left\|u\right\| \le C_4 \left\|u\right\|_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)}$$

and

$$\left|u(d)\right| \le C_5 \left\|u\right\|_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)} \ for \ any \ d \in \mho_i.$$

Hence, the functionals $\ell_k \ (k=1,2,3)$ alllows the following estimate:

$$\left|\ell_1(u,\vartheta)\right| \le C_6 \left\|u\right\|_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)} \left\|\vartheta\right\|_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)},$$

$$\left|\ell_2(u,\vartheta)\right| \le C_7 \left\|u\right\|_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)} \left\|\vartheta\right\|_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)},$$

$$\left|\ell_3(\omega,\vartheta)\right| \le C_8 \left|\omega\right|\left\|\vartheta\right\|_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)}.$$

Thefore, $\ell_k \ (k=1,2,3)$ are linear functionals in $\vartheta \in \underset{i=1}{\overset{3}{\oplus}}H^1(\mho_i)$ for any given $u \in \underset{i=1}{\overset{3}{\oplus}}H^1(\mho_i), k=1,2,$ and $\omega \in \mathbb{C}, k=3,$ respectively. The Riesz representation theorem shows that the representations (15)-(17) are valid for some bounded operators $T_k \ (k=1,2,3)$. The selfadjointness and positiveness of the operators T_1 and T_2 are obvious. The compactness of T_2 is well-known fact (see, [34]). The proof of the compactness of T_1 can be found by using the same arguments, as in [10, 11] and [13].

Now the compactness of the operator T_3 will be proven. It is easy to verify that the adjoint operator of T_3 is defined on whole $\underset{i=1}{\overset{3}{\oplus}}H^1(\mho_i)$ with equality $T_3^* u = \dfrac{1}{\kappa_2}u(\alpha)$. From this representation it follows that , the operator T_3^* from $\underset{i=1}{\overset{3}{\oplus}}H^1(\mho_i)$ to $\mathbb{C}$ are bounded, i.e

$$\left|T_3^* u\right| \ \le \ C_9 \left\|u\right\|_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)}.$$

Consequently the operator T_3^* is bounded linear operator with finite dimensional range and therefore is compact.

Then by virtue of well-known theorem of Functional Analysis (see, for example [37]) the operator T_3 is also compact. The proof is complete.

Lemma 3 *The generalized eigenfunctions of the Sturm-Liouville problem (1)-(7) satisfy the following operator polynomial equation in* Ξ

$$L(\lambda)\Phi = 0, \quad L(\lambda) = L_1 - \lambda L_2,$$

$$L_1 = \begin{pmatrix} I + T_1 & T_3 \\ T_3^* & -\dfrac{\kappa_3}{\kappa_2 \rho}I \end{pmatrix}, \quad L_2 = \begin{pmatrix} T_2 & 0 \\ 0 & \dfrac{1}{\rho}I \end{pmatrix}. \tag{19}$$

Here I *is the identity operator and the space* Ξ *is described in Definition1 and* $\Phi = \begin{pmatrix} u(x) & \omega \end{pmatrix}$.

Proof. By using Lemma 2 the identities (13)-(14) can be rewritten as follows

$$\langle u, \vartheta \rangle_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)} + \langle T_1 u, \vartheta \rangle_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)} + \langle T_3\omega, \vartheta \rangle_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)} = \lambda \langle T_2 u, \vartheta \rangle_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)},$$

$$T_3^* u - \frac{\kappa_3}{\kappa_2}\frac{\omega}{\rho} = \lambda \frac{\omega}{\rho}.$$

The arbitrariness of $\vartheta \in \underset{i=1}{\overset{3}{\oplus}}H^1(\mho_i)$ implies

$$u + T_1 u + T_3\omega = \lambda T_2 u$$

and hence, the proof is complete.

5. SELFADJOINTNESS AND POSITIVENESS OF THE CORRESPONDING OPERATOR-PENCIL

Theorem 1 *The operators* L_1 *and* L_2 *are self-adjoint in the Hilbert space* Ξ .

Proof. Let $\Phi_1 = \begin{pmatrix} u(.), \omega \end{pmatrix}$ and $\Phi_2 = \begin{pmatrix} v(.), \omega' \end{pmatrix}$ be any two element of Ξ . Then making use of the representation $T_3^* u = \dfrac{1}{\kappa_2}u(\alpha)$ from (15)-(18) and (19) we have

$$\langle L_1\Phi_1, \Phi_2 \rangle_\Xi = \langle u + T_1 u + T_3\omega, v \rangle_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)} + \left(T_3^* u - \frac{\kappa_3}{\kappa_2}\frac{\omega}{\rho} \right)\overline{\omega'} \tag{20}$$

$$\langle \Phi_1, L_1\Phi_2 \rangle_\Xi = \overline{\langle L_1\Phi_2, \Phi_1 \rangle_\Xi} = \overline{\langle v + T_1 v + T_3\omega', u \rangle}_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)} + \overline{\left(T_3^* v - \frac{\kappa_3}{\kappa_2}\frac{\omega'}{\rho} \right)\overline{\omega}}. \tag{21}$$

Substracting (21) from (20) and realizing that

$$\langle T_i\omega, v \rangle_{\underset{i=1}{\overset{3}{\oplus}}H_q^1(\mho_i)} = \omega\overline{\left(T_i^* v \right)} \quad \text{and} \quad T_i^* u\,\overline{\omega'} = \langle u, T_i\omega' \rangle_\Xi$$

by definition of the adjoint operator, we see that

$$\left\langle L_1\Phi_1 , \Phi_2 \right\rangle_\Xi - \left\langle \Phi_1 , L_1\Phi_2 \right\rangle_\Xi = 0$$

so the operator L_1 is self-adjoint in the Hilbert space Ξ. Similarly we have

$$\left\langle L_2\Phi_1 , \Phi_2 \right\rangle_\Xi = \left\langle T_2 u , v \right\rangle_{\underset{i=1}{\overset{3}{\oplus}} H_q^1(\mho_i)} + \frac{\omega\,\overline{\omega'}}{\rho^2},$$

$$\left\langle \Phi_1 , L_2\Phi_2 \right\rangle_\Xi = \overline{\left\langle L_2\Phi_2 , \Phi_1 \right\rangle_\Xi} = \overline{\left\langle T_2 v , u \right\rangle}_{\underset{i=1}{\overset{3}{\oplus}} H_q^1(\mho_i)} + \overline{\frac{\omega'\overline{\omega}}{\rho^2}} = \left\langle T_2 u , v \right\rangle_{\underset{i=1}{\overset{3}{\oplus}} H_q^1(\mho_i)} + \frac{\omega\,\overline{\omega'}}{\rho^2}.$$

Consequently,

$$\left\langle L_2\Phi_1 , \Phi_2 \right\rangle_\Xi - \left\langle \Phi_1 , L_2\Phi_2 \right\rangle_\Xi = 0$$

so the operator L_2 is self-adjoint in the Hilbert space Ξ. The operators L_1 and L_2 are also compact in the Hilbert space Ξ, which completes the proof.

Corollary 1 *The operator* $L\left(-\lambda\right) = L_1 + \lambda L_2$ *is self-adjoint for each real* λ.

Remark 3 *Let* $\lambda \in \mathbb{C}$ *be any eigenvalue with the generalized eigenfunction* $\tilde{u}\left(x,\lambda\right) := \left(u\left(x,\lambda\right), \omega \right)$. *Then the operator polynomial equation* $L\left(-\lambda\right)\tilde{u}\left(.,\lambda\right) = 0$ *is hold in the Hilbert space* Ξ.

Lemma 4 *The inequality*

$$2\left| \mathrm{Re}\left(\int_\mho u(x)\,\overline{\vartheta}(x)dx \right) \right| \leq \|u\|^2 + \|\vartheta\|^2 \tag{22}$$

is hold for any $u, \vartheta \in \overset{3}{\underset{i=1}{\oplus}} L_2(\mho_i)$.

Proof. The inequality (22) is a direct consequence of the polar identity.

Now, making use the Lemma 2, Theorem 1 and inequality (22) we have the following important result.

Theorem 2 There exists $\mu > 0$ such that for each real $\lambda_0 > \mu$, the operators $L\left(-\lambda_0\right)$ is positive defined in the Hilbert space Ξ, that is for real positive λ_0 large enough, there exists a constant $C = C(\lambda_0) > 0$ depending only of parameter λ_0, such that

$$\left\langle L\left(-\lambda_0\right)\Phi , \Phi \right\rangle_\Xi \geq C(\lambda_0) \|\Phi\|_\Xi^2$$

for all $\Phi \in \Xi$.

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