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(k, s)-Riemann-Liouville fractional integral and applications

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Abstract

In this paper, we introduce a new approach on fractional integration, which generalizes the Riemann-Liouville fractional integral. We prove some properties for this new approach. We also establish some new integral inequalities using this new fractional integration.

Keywords: Riemann-Liouville fractional integrals, synchronous function, Chebyshev inequality, Hölder inequality.

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1. Introduction

Fractional calculus and its widely application have recently been paid more and more attentions. For more recent development on fractional calculus, we refer the reader to [7, 12, 15, 16, 19]. There are several known forms of the fractional integrals of which two have been studied extensively for their applications [5, 10, 11, 14, 21]. The first is the Riemann-Liouville fractional integral of $\alpha \geq 0$ for a continuous function f on $[a, b]$ which is defined by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha \geq 0, \; a < x \leq b.$$

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This integral is motivated by the well known Cauchy formula:

$$\int_a^x dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_{n-1}} f(t_n) dt_n = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) dt, n \in \mathbb{N}^*.$$

The second is the Hadamard fractional integral introduced by Hadamard [9]. It is given by:

$$J_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad \alpha > 0, \ x > a.$$

The Hadamard integral is based on the generalization of the integral

$$\int_a^x \frac{dt_1}{t_1} \int_a^{t_1} \frac{dt_2}{t_2} ... \int_a^{t_{n-1}} \frac{f(t_n)}{t_n} dt_n = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{n-1} f(t) \frac{dt}{t}$$

for $n \in \mathbb{N}^*$.

In [10], Katugampola gave a new fractional integration which generalizes both the Riemann-Liouville and Hadamard fractional integrals into a single form. This generalization is based on the observation that, for $n \in \mathbb{N}^*$,

$$\int_a^x t_1^s dt_1 \int_a^{t_1} t_2^s dt_2 ... \int_a^{t_{n-1}} t_n^s f(t_n) dt_n = \frac{(s+1)^{1-n}}{\Gamma(\alpha)} \int_a^x \left(x^{s+1} - t^{s+1}\right)^{n-1} t^s f(t) dt,$$

which gives the following fractional version

$${}^s J_a^{\alpha} f(x) = \frac{(s+1)^{1-n}}{\Gamma(\alpha)} \int_a^x \left(x^{s+1} - t^{s+1}\right)^{\alpha-1} t^s f(t) dt,$$

where α and $s \neq -1$ are real numbers.

Recently, in [6], Diaz and Pariguan have defined new functions called k-gamma and k-beta functions and the Pochhammer k-symbol that is respectively generalization of the classical gamma and beta functions and the classical Pochhammer symbol:

$$\Gamma_k(x) = \lim_{n \to \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \ k > 0,$$

where $(x)_{n,k}$ is the Pochhammer k-symbol for factorial function. It has been shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, explicitly given by

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt, x > 0.$$

Clearly, $\Gamma(x) = \lim_{k \to 1} \Gamma_k(x), \ \Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma(\frac{x}{k})$ and $\Gamma_k(x+k) = x \Gamma_k(x)$. Furthermore, k-beta function is defined as follows

$$B_k(x,y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt,$$

so that $B_k(x,y) = \frac{1}{k} B(\frac{x}{k}, \frac{y}{k})$ and $B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$.

Later, under the above definitions, in [13], Mubeen and Habibullah have introduced the k-fractional integral of the Riemann-Liouville type as follows:

$${}_k J^{\alpha} f(x) = \frac{1}{k \Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad \alpha > 0, \ x > 0.$$

Note that when $k \to 1$, then it reduces to the classical Riemann-liouville fractional integral.

2. (k,s)-Riemann-Liouville fractional integral

In this section, we present the (k,s) fractional integration which generalizes all of the above Riemann-Liouville fractional integrals as follows:

2.1. Definition. Let f be a continuous function on on a the finite real interval $[a,b]$. Then (k,s)-Riemann-Liouville fractional integral of f of order $\alpha>0$ is defined by:

$$ {}_k^sJ_a^\alpha f(x) := \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s f(t)dt, \quad x\in[a,b], \tag{2.1}$$

where $k>0, s\in\mathbb{R}\backslash\{-1\}$.

In the following theorem, we prove that the (k,s) fractional integral is well defined:

2.2. Theorem. *Let $f\in L_1[a,b], s\in\mathbb{R}\backslash\{-1\}$ and $k>0$. Then ${}_k^sJ_a^\alpha f(x)$ exists for any $x\in[a,b], \alpha>0$.*

Proof. Let $\Delta := [a,b]\times[a,b]$ and $P:\Delta\to\mathbb{R}$; $P(x,t)=\left[\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}t^s\right]$. It clear to see that $P=P_+ + P_-$, where

$$P_+(x,t) := \begin{cases} \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}t^s & ,a\le t\le x\le b \\ 0 & ,a\le x\le t\le b \end{cases}$$

and

$$P_-(x,t) := \begin{cases} \left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s & ,a\le t\le x\le b \\ 0 & ,a\le x\le t\le b. \end{cases}$$

Since P is measurable on Δ, then we can write

$$\int_a^b P(x,t)dt = \int_a^x P(x,t)dt = \int_a^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}t^s dt = \frac{k}{\alpha}\left(x^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}}.$$

By using the repeated integral, we obtain

$$\begin{aligned}\int_a^b\left(\int_a^b P(x,t)\left|f(x)\right|dt\right)dx &= \int_a^b \left|f(x)\right|\left(\int_a^b P(x,t)dt\right)dx \\ &= \frac{k}{\alpha}\int_a^b \left(x^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}}\left|f(x)\right|dx \\ &\le \frac{k}{\alpha}\left(b^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}}\int_a^b\left|f(x)\right|dx.\end{aligned}$$

That is

$$\begin{aligned}\int_a^b\left(\int_a^b P(x,t)\left|f(x)\right|dt\right)dx &= \int_a^b \left|f(x)\right|\left(\int_a^b P(x,t)dt\right)dx \\ &\le \frac{k}{\alpha}\left(b^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}}\left\|f(x)\right\|_{L_1[a,b]}<\infty.\end{aligned}$$

Therefore, the function $Q:\Delta\to\mathbb{R}$; $Q(x,t):=P(x,t)f(x)$ is integrable over Δ by Tonelli's theorem. Hence, by Fubini's theorem $\int_a^b P(x,t)f(x)dx$ is an integrable function on $[a,b]$, as a function of $t\in[a,b]$. That is, ${}_k^sJ_a^\alpha f(x)$ exists. □

Now, we prove the commutativity and the semigroup properties of the (k,s)-Riemann-Liouville fractional integral. We have:

2.3. Theorem. *Let f be continuous on $[a,b], k>0$ and $s\in\mathbb{R}\backslash\{-1\}$. Then,*

$$ {}_k^sJ_a^{\alpha}\left[\ {}_k^sJ_a^{\beta}f(x)\right] = \ {}_k^sJ_a^{\alpha+\beta}f(x) = \ {}_k^sJ_a^{\beta}\left[\ {}_k^sJ_a^{\alpha}f(x)\right], $$

for all $\alpha>0,\beta>0,x\in[a,b]$.

Proof. Thanks to Definition 1 and by Dirichlet's formula, we have

$$ {}_k^sJ_a^{\alpha}\left[\ {}_k^sJ_a^{\beta}f(x)\right] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}t^s\ {}_k^sJ_a^{\beta}f(t)dt $$

$$ = \tfrac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}t^s\left[\tfrac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s f(\tau)d\tau\right]dt $$

$$ = \tfrac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}t^s\left[\tfrac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s f(\tau)d\tau\right]dt. $$

That is

(2.2)

$$ {}_k^sJ_a^{\alpha}\left[\ {}_k^sJ_a^{\beta}f(x)\right] = \frac{(s+1)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x\tau^s f(\tau)\left[\int_\tau^x\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}t^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\beta}{k}-1}dt\right]d\tau. $$

Using the change of variable $y=\left(t^{s+1}-\tau^{s+1}\right)/\left(x^{s+1}-\tau^{s+1}\right)$, we can write

(2.3)

$$ \int_\tau^x\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\beta}{k}-1}t^s dt \quad = \frac{\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}}{s+1}\int_0^1(1-y)^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}dy $$

$$ = \frac{\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}}{s+1}\int_0^1(1-y)^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}dy \qquad = \frac{\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}}{s+1}kB_k(\alpha,\beta). $$

According to the k-beta function and by (2.2) and (2.3), we obtain

$$ \begin{aligned} {}_k^sJ_a^{\alpha}\left[\ {}_k^sJ_a^{\beta}f(x)\right] &= \frac{(s+1)^{1-\frac{\alpha+\beta}{k}}}{k\Gamma_k(\alpha+\beta)}\int_a^x\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}\tau^s f(\tau)d\tau \\ &= \ {}_k^sJ_a^{\alpha+\beta}f(x). \end{aligned} $$

This completes the proof of the Theorem 2.3. □

2.4. Theorem. *Let $\alpha,\beta>0,k>0$ and $s\in\mathbb{R}\backslash\{-1\}$. Then, we have*

$$ {}_k^sJ_a^{\alpha}\left[\left(x^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1}\right] = \frac{\Gamma_k(\beta)}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+\beta)}\left(x^{s+1}-a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}, \tag{2.4} $$

where Γ_k denotes the k-gamma function.

Proof. By Definition 1 and using the change of variable $y = \left(x^{s+1} - t^{s+1}\right) / \left(x^{s+1} - a^{s+1}\right); x \in]a, b]$, we get

$$\begin{aligned} {}_{k}^{s}J_{a}^{\alpha}\left[\left(x^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1}\right] &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s \left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1} dt \\ &= \frac{(s+1)^{-\frac{\alpha}{k}}\left(x^{s+1}-a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}}{k\Gamma_k(\alpha)} \int_0^1 (1-y)^{\frac{\alpha}{k}-1} y^{\frac{\beta}{k}-1} dy \\ &= \frac{\left(x^{s+1}-a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha)} B_k(\alpha,\beta). \end{aligned}$$

The case $a = x$ is trivial. The proof of Theorem 2.4 is complete. □

2.5. Remark. (i :) Taking $s = 0, k > 0$ in (2.4), we obtain

$$\text{(2.5)} \qquad {}_{k}J_{a}^{\alpha}\left[(x-a)^{\frac{\beta}{k}-1}\right] = \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} (x-a)^{\frac{\alpha+\beta}{k}-1}.$$

(ii :) The formula (2.4) for $s = 0, k = 1$ becomes

$$J_{a}^{\alpha}\left[(x-a)^{\beta-1}\right] = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\alpha+\beta-1}.$$

2.6. Corollary. *Let $k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then the formula*

$$\text{(2.6)} \qquad {}_{k}^{s}J_{a}^{\alpha}(1) = \frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} \left(x^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}$$

is valid for any $\alpha > 0$.

2.7. Remark. (a :) For $s = 0, k > 0$ in (2.6), we get

$$\text{(2.7)} \qquad {}_{k}J_{a}^{\alpha}(1) = \frac{1}{\Gamma_k(\alpha+k)} (x-a)^{\frac{\alpha}{k}-2}.$$

(b :) For $s = 0, k = 1$ we have

$$J_{a}^{\alpha}(1) = \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha-2}.$$

3. Some new (k, s)-Riemann-Liouville fractional integral inequalities

Chebyshev inequalities can be represented in (k, s)-fractional integral forms as follows:

3.1. Theorem. *Let f and g be two synchronous on $[0, \infty)$. Then for all $t > a \geq 0$, $\alpha > 0$, $\beta > 0$, the following inequalities for (k, s)-fractional integrals hold:*

$$\text{(3.1)} \qquad {}_{k}^{s}J_{a}^{\alpha} fg(t) \geq \frac{1}{J_{a}^{\alpha}(1)}\, {}_{k}^{s}J_{a}^{\alpha} f(t)\, {}_{k}^{s}J_{a}^{\alpha} g(t)$$

$$\text{(3.2)} \qquad {}_{k}^{s}J_{a}^{\alpha} fg(t)\, {}_{k}^{s}J_{a}^{\beta}(1) + {}_{k}^{s}J_{a}^{\beta} fg(t)\, {}_{k}^{s}J_{a}^{\alpha}(1) \geq {}_{k}^{s}J_{a}^{\alpha} f(t)\, {}_{k}^{s}J_{a}^{\beta} g(t) + {}_{k}^{s}J_{a}^{\alpha} g(t)\, {}_{k}^{s}J_{a}^{\beta} f(t).$$

Proof. Since the functions f and g are synchronous on $[0, \infty)$, then for all $x, y \geq 0$, we have

$$(f(x) - f(y))(g(x) - g(y)) \geq 0.$$

Therefore

$$\text{(3.3)} \qquad f(x)g(x) + f(y)g(y) \geq f(x)g(y) + f(y)g(x).$$

Multiplying both sides of (3.3) by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s$, then integrating the resulting inequality with respect to x over (a,t), we obtain

$$\begin{aligned} & \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(x)\,g(x)\,dx \\ & +\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(y)\,g(y)\,dx \\ \geq\ & \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(x)\,g(y)\,dx \\ & +\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(y)\,g(x)\,dx, \end{aligned}$$

i.e.

$$(3.4)\qquad {}_k^sJ_a^\alpha fg(t)+f(y)\,g(y)\ {}_k^sJ_a^\alpha(1)\geq g(y)\ {}_k^sJ_a^\alpha f(t)+f(y)\ {}_k^sJ_a^\alpha g(t).$$

Multiplying both sides of (3.3) by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1}-y^{s+1}\right)^{\frac{\alpha}{k}-1}y^s$, then integrating the resulting inequality with respect to y over (a,t), we obtain

$$\begin{aligned} & {}_k^sJ_a^\alpha fg(t)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\alpha}{k}-1}y^s dy \\ & +\ {}_k^sJ_a^\alpha(1)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\alpha}{k}-1}y^s f(y)\,g(y)\,dy \\ \geq\ & {}_k^sJ_a^\alpha f(t)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\alpha}{k}-1}y^s g(y)dy \\ & +\ {}_k^sJ_a^\alpha g(t)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\alpha}{k}-1}y^s f(y)\,dy, \end{aligned}$$

that is

$${}_k^sJ_a^\alpha fg(t)\geq\frac{1}{J_a^\alpha(1)}\ {}_k^sJ_a^\alpha f(t)\ {}_k^sJ_a^\alpha g(t).$$

The first inequality is thus proved.

Multiplying both sides of (3.3) by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s$, then integrating the resulting inequality with respect to y over (a,t), we obtain

$$\begin{aligned} & {}_k^sJ_a^\alpha fg(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s dy \\ & +\ {}_k^sJ_a^\alpha(1)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,g(y)\,dy \\ \geq\ & {}_k^sJ_a^\alpha f(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s g(y)dy \\ & +\ {}_k^sJ_a^\alpha g(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,dy, \end{aligned}$$

that is

$$ {}_{k}^{s}J_{a}^{\alpha} fg(t)\ {}_{k}^{s}J_{a}^{\beta}(1) + {}_{k}^{s}J_{a}^{\beta} fg(t)\ {}_{k}^{s}J_{a}^{\alpha}(1) \geq {}_{k}^{s}J_{a}^{\alpha} f(t)\ {}_{k}^{s}J_{a}^{\beta} g(t) + {}_{k}^{s}J_{a}^{\alpha} g(t)\ {}_{k}^{s}J_{a}^{\beta} f(t) $$

and the second inequality is proved. The proof is completed. □

3.2. Theorem. *Let f and g be two synchronous on $[0,\infty)$, $h \geq 0$. Then for all $t > a \geq 0$, $\alpha > 0$, $\beta > 0$, the following inequalities hold:*

$$ \begin{aligned} & \frac{1}{(s+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-2}\ {}_{k}^{s}J_{a}^{\alpha} fgh(t) \\ & + \frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}\ {}_{k}^{s}J_{a}^{\beta} fgh(t) \\ \geq\ & {}_{k}^{s}J_{a}^{\alpha} fh(t)\ {}_{k}^{s}J_{a}^{\beta} g(t) + {}_{k}^{s}J_{a}^{\alpha} gh(t)\ {}_{k}^{s}J_{a}^{\beta} f(t) - {}_{k}^{s}J_{a}^{\alpha} h(t)\ {}_{k}^{s}J_{a}^{\beta} fg(t) - {}_{k}^{s}J_{a}^{\alpha} fg(t)\ {}_{k}^{s}J_{a}^{\beta} h(t) \\ & + {}_{k}^{s}J_{a}^{\alpha} f(t)\ {}_{k}^{s}J_{a}^{\beta} gh(t) + {}_{k}^{s}J_{a}^{\alpha} g(t)\ {}_{k}^{s}J_{a}^{\beta} fh(t). \end{aligned} $$

Proof. Since the functions f and g are synchronous on $[0,\infty)$ and $h \geq 0$, then for all $x, y \geq 0$, we have

$$ (f(x) - f(y))(g(x) - g(y))(h(x) + h(y)) \geq 0. $$

Hence,

$$ \begin{aligned} & f(x)g(x)h(x) + f(y)g(y)h(y) \\ \geq\ & f(x)g(y)h(x) + f(y)g(x)h(x) - f(y)g(y)h(x) \\ & - f(x)g(x)h(y) + f(x)g(y)h(y) + f(y)g(x)h(y). \end{aligned} \tag{3.5} $$

Multiplying both sides of (3.5) by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s$, then integrating the resulting inequality with respect to x over (a,t), we obtain

$$\begin{aligned}
& \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(x)\,g(x)\,h(x)\,dx \\
& +f(y)\,g(y)\,h(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s dx \\
\geq\ & g(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(x)\,h(x)\,dx \\
& +f(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s g(x)\,h(x)\,dx \\
& -f(y)\,g(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s h(x)\,dx \\
& -h(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(x)\,g(x)\,dx \\
& +g(y)\,h(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(x)\,dx \\
& +f(y)\,h(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s g(x)\,dx.
\end{aligned}$$

That is

$$\begin{aligned}
(3.6)\qquad & {}_k^sJ_a^\alpha fgh(t)+f(y)\,g(y)\,h(y)\ {}_k^sJ_a^\alpha(1) \\
\geq\ & g(y)\ {}_k^sJ_a^\alpha fh(t)+f(y)\ {}_k^sJ_a^\alpha gh(t)-f(y)\,g(y)\ {}_k^sJ_a^\alpha h(t)-h(y)\ {}_k^sJ_a^\alpha fg(t) \\
& +g(y)\,h(y)\ {}_k^sJ_a^\alpha f(t)+f(y)\,h(y)\ {}_k^sJ_a^\alpha g(t).
\end{aligned}$$

Multiplying both sides of (3.6) by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s$, then integrating the resulting inequality with respect to y over (a,t), we obtain

$$\begin{aligned}
& {}_k^sJ_a^\alpha fgh(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s dy \\
& +{}_k^sJ_a^\alpha(1)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,g(y)\,h(y)dy \\
\geq\quad & {}_k^sJ_a^\alpha fh(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s g(y)\,dy \\
& +{}_k^sJ_a^\alpha gh(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,dy \\
& -{}_k^sJ_a^\alpha h(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,g(y)\,dy \\
& -{}_k^sJ_a^\alpha fg(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s h(y)dy \\
& +{}_k^sJ_a^\alpha f(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s g(y)\,h(y)\,dy \\
& +{}_k^sJ_a^\alpha g(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,h(y)\,dy.
\end{aligned}$$

Consequently,

$$\begin{aligned}
{}_k^sJ_a^\alpha fgh(t){}_k^sJ_a^\beta(1)+{}_k^s J_a^\alpha(1){}_k^sJ_a^\beta fgh(t) \quad\geq\quad & {}_k^sJ_a^\alpha fh(t)\ {}_k^sJ_a^\beta g(t)+\ {}_k^sJ_a^\alpha gh(t)\ {}_k^sJ_a^\beta f(t) \\
& -\ {}_k^sJ_a^\alpha h(t)\ {}_k^sJ_a^\beta fg(t)-\ {}_k^sJ_a^\alpha fg(t)\ {}_k^sJ_a^\beta h(t) \\
& +\ {}_k^sJ_a^\alpha f(t)\ {}_k^sJ_a^\beta gh(t)+\ {}_k^sJ_a^\alpha g(t)\ {}_k^sJ_a^\beta fh(t).
\end{aligned}$$

The proof is thus complete. □

3.3. Corollary. *Let f and g be two synchronous on $[0,\infty)$, $h\geq 0$. Then for all $t>a\geq 0$, $\alpha>0$, the following inequalities hold:*

$$\begin{aligned}
& \frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}\ {}_k^sJ_a^\alpha fgh(t) \\
\geq\quad & {}_k^sJ_a^\alpha fh(t)\ {}_k^sJ_a^\alpha g(t)+\ {}_k^sJ_a^\alpha gh(t)\ {}_k^sJ_a^\alpha f(t)-\ {}_k^sJ_a^\alpha h(t)\ {}_k^sJ_a^\alpha fg(t).
\end{aligned}$$

3.4. Theorem. *Let f,g and h be three monotonic functions defined on $[0,\infty)$ satisfying the following*

$$(f(x)-f(y))\,(g(x)-g(y))\,(h(x)-h(y))\geq 0$$

for all $x, y \in [a, t]$. Then for all $t > a \geq 0,\ \alpha > 0,\ \beta > 0,$ the following inequalities are valid:

$$\begin{aligned}
&\frac{1}{(s+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-2}\ {}_k^sJ_a^\alpha fgh(t)\\
&-\frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}\ {}_k^sJ_a^\beta fgh(t)\\
\geq\ & {}_k^sJ_a^\alpha fh(t)\ {}_k^sJ_a^\beta g(t) + {}_k^sJ_a^\alpha gh(t)\ {}_k^sJ_a^\beta f(t) - {}_k^sJ_a^\alpha h(t)\ {}_k^sJ_a^\beta fg(t) + {}_k^sJ_a^\alpha fg(t)\ {}_k^sJ_a^\beta h(t)\\
&-{}_k^sJ_a^\alpha f(t)\ {}_k^sJ_a^\beta gh(t) - {}_k^sJ_a^\alpha g(t)\ {}_k^sJ_a^\beta fh(t).
\end{aligned}$$

Proof. We use the same arguments as in the proof of Theorem 3.2. □

3.5. Theorem. *Let f and g be two functions on $[0, \infty)$. Then for all $t > a \geq 0,\ \alpha > 0,\ \beta > 0,$ the following inequalities for (k, s)-fractional integrals hold:*

$$\begin{aligned}
&\frac{1}{(s+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-2}\ {}_k^sJ_a^\alpha f^2(t)\\
&+\frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}\ {}_k^sJ_a^\beta g^2(t)\\
\geq\ & 2\ {}_k^sJ_a^\alpha f(t)\ {}_k^sJ_a^\beta g(t)
\end{aligned} \tag{3.7}$$

$${}_k^sJ_a^\alpha f^2(t)\ {}_k^sJ_a^\beta g^2(t) + {}_k^sJ_a^\beta f^2(t)\ {}_k^sJ_a^\alpha g^2(t) \geq 2\ {}_k^sJ_a^\alpha fg(t)\ {}_k^sJ_a^\beta fg(t). \tag{3.8}$$

Proof. Since

$$(f(x) - g(y))^2 \geq 0,$$

then, we have

$$f^2(x) + g^2(y) \geq 2f(x)g(y). \tag{3.9}$$

Multiplying both sides of (3.9) by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s$ and $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s$, then integrating the resulting inequality with respect to x and y over (a, t) respectively, we obtain (3.7).

On the other hand, since

$$(f(x)g(y) - f(y)g(x))^2 \geq 0,$$

then, with the same arguments as before, we obtain (3.8). □

3.6. Corollary. *Let f and g be two functions on $[0, \infty)$, then for all $t > a \geq 0,\ \alpha > 0,$ the following inequalities are valid:*

$$\begin{aligned}
&\frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}\left[{}_k^sJ_a^\alpha f^2(t) + {}_k^sJ_a^\beta g^2(t)\right]\\
\geq\ & 2\ {}_k^sJ_a^\alpha f(t)\ {}_k^sJ_a^\alpha g(t)
\end{aligned}$$

$${}_k^sJ_a^\alpha f^2(t)\ {}_k^sJ_a^\alpha g^2(t) \geq \left[{}_k^sJ_a^\alpha fg(t)\right]^2.$$

3.7. Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ with:*

$$\bar{f}(x) := \int_a^x t^s f(t)dt, \quad x > a \geq 0,\ s \in \mathbb{R}\backslash\{-1\}.$$

Then, for $\alpha \geq k > 0$ we have:

$$ {}_k^s J_a^\alpha f(x) = \frac{1}{k}\ {}_k^s J_a^{\alpha-k} \bar{f}(x) $$

Proof. By definition of the (k, s)-fractional integral and by using Dirichlet's formula, we have

$$
\begin{aligned}
{}_k^s J_a^\alpha \bar{f}(x) &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s \int_a^t u^s f(u) du dt \\
&= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x u^s f(u) \int_u^x \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s dt du \\
&= \frac{(s+1)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \int_a^x \left(x^{s+1} - u^{s+1}\right)^{\frac{\alpha}{k}} u^s f(u) du \\
&= k\ {}_k^s J_a^{\alpha+k} f(x).
\end{aligned}
$$

This completes the proof of Theorem 3.7. □

We give the generalized Cauchy-Buniakovsky-Schwarz inequality as follows:

3.8. Lemma. *Let $f, g, h : [a, b] \to (0, \infty)$ be three functions $0 \leq a < b$. Then*

(3.10)
$$ \left(\int_a^b g^m(t) h^x(t) f(t) dt\right) \left(\int_a^b g^n(t) h^y(t) f(t) dt\right) \geq \left(\int_a^b g^{\frac{m+n}{2}}(t) h^{\frac{x+y}{2}}(t) f(t) dt\right)^2, $$

where m, n, x, y arbitrary real numbers.

Proof.

$$ \int_a^b \left[\sqrt{g^m(t)h^x(t)f(t)}\sqrt{\int_a^b g^n(t)h^y(t)f(t)dt} - \sqrt{g^n(t)h^y(t)f(t)}\sqrt{\int_a^b g^m(t)h^x(t)f(t)dt}\right]^2 dt \geq 0 $$

$$
\begin{aligned}
&\int_a^b \Bigg[g^m(t)h^x(t)f(t) \int_a^b g^n(t)h^y(t)f(t)dt + g^n(t)h^y(t)f(t) \int_a^b g^m(t)h^x(t)f(t)dt \\
&-2\, g^{\frac{m+n}{2}}(t) h^{\frac{x+y}{2}}(t) f(t) \sqrt{\int_a^b g^m(t)h^x(t)f(t)dt}\sqrt{\int_a^b g^n(t)h^y(t)f(t)dt}\Bigg] dt \\
\geq\ & 0 \\
&2\left(\int_a^b g^m(t)h^x(t)f(t)dt\right)\left(\int_a^b g^n(t)h^y(t)f(t)dt\right) \\
\geq\ & 2\left(\int_a^b g^{\frac{m+n}{2}}(t) h^{\frac{x+y}{2}}(t) f(t) dt\right)\sqrt{\int_a^b g^m(t)h^x(t)f(t)dt}\sqrt{\int_a^b g^n(t)h^y(t)f(t)dt}
\end{aligned}
$$

which gives the desired inequality. □

3.9. Theorem. *Let $f \in L_1[a, b]$. Then*

(3.11)
$$ \left({}_k^s J_a^{m(\frac{\alpha}{k}-1)+1} f^r(x)\right)\left({}_k^s J_a^{n(\frac{\alpha}{k}-1)+1} f^p(x)\right) \geq \left({}_k^s J_a^{\frac{m+n}{2}(\frac{\alpha}{k}-1)+1} f^{\frac{r+p}{2}}(x)\right)^2, $$

for $k, m, n, r, p > 0$ and $\alpha > 1$.

Proof. By taking $g(t)=\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}$, $f(t)=\frac{t^{s}(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_{k}(\alpha)}$ and $h(t)=f(t)$ in (3.10), we obtain

$$\begin{aligned}&\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-t^{s+1}\right)^{m(\frac{\alpha}{k}-1)}t^sf^r(t)dt\right)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-t^{s+1}\right)^{n(\frac{\alpha}{k}-1)}t^sf^p(t)dt\right)\\ \geq\ &\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-t^{s+1}\right)^{\frac{m+n}{2}(\frac{\alpha}{k}-1)}t^sf^{\frac{r+p}{2}}(t)dt\right)^2\end{aligned}$$

which can be written as (3.11). □

3.10. Remark. For $k=1$ in (3.11), we get the following inequalities:

$$\left(J_a^{m(\alpha-1)+1}f^r(x)\right)\left({}_kJ_a^{n(\alpha-1)+1}f^s(x)\right)\geq\left({}_kJ_a^{\frac{m+n}{2}(\alpha-1)+1}f^{\frac{r+s}{2}}(x)\right)^2.$$

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