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Parameter-Dependent Dirac Systems on Time Scales

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Abstract. In this study, we consider two generalized Dirac systems on a time scale and a boundary-value problem with boundary conditions depending on the spectral parameter. We give some sufficient conditions for disconjugacy of the systems and obtain a formula about the number of eigenvalues of the problem.

Keywords: Time scale, Dynamic Equations, Dirac-System, parameter-dependent boundary conditions, disconjugacy.

Zaman Skalaları Üzerinde Parametreye Bağlı Dirac Sistemleri

Özet. Bu çalışmada bir zaman skalası üzerinde iki farklı genelleştirilmiş Dirac sistemi ve parametreye bağlı sınır koşulları ile üretilen bir sınır değer problemi ele alınmıştır. Sistemlerin eşleniksiz (disconjugate) olması için yeterli koşullar ve problemin özdeğerlerinin sayısı ile ilgili bir formül elde edilmiştir.

Anahtar Kelimeler: Zaman Skalası, Dinamik denklemler, Dirac sistemi, parametreye bağlı sınır koşulları, eşleniksizlik.

1. INTRODUCTION

The time scale theory was introduced by Hilger in 1988. He gave a new derivation in order to unify continuous and discrete analysis [1]. From then on this approach has received a lot of attention and has applied quickly to various areas in mathematics. Especially, this theory allows us to study differential and difference equations in the same subject. Because, a result obtained for a dynamic equation given in an arbitrary time scale is also valid for differential and difference equations.

The studies about spectral theory on time scales have focused on Sturm–Liouville equation. Sturm-Liouville theory on time scales was studied first by Erbe and Hilger [2] in 1993. Some important results on the properties of eigenvalues and eigenfunctions of the classical Sturm-Liouville problem on time scales were given in various publications (see e.g. [3-18] and the references therein).

The basic terminology of time scales theory such as Δ-derivation, Δ-integration; the operators σ, ρ and μ; the set of rd-continuous functions C_{rd} and the set of rd-continuously delta-differentiable functions $C_{rd}^1(\mathbb{T})$ can be found in [19].

Additionally, we need to recall some other notations.

Let $p(t)$ be a rd-continuous function satisfying the condition $1+\mu(t)p(t)\neq 0$ for each t in the time scale $\mathbb{T}$. The exponential function $e_p(t,t_0)$ and the trigonometric functions $\sin_p(t,t_0)$ and $\cos_p(t,t_0)$ are defined on $\mathbb{T}$ as follows:

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$$e_p(t,t_0) = \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right),$$

$$\sin_p(t,t_0) = \frac{e_{ip}(t,t_0) - e_{-ip}(t,t_0)}{2i},$$

$$\cos_p(t,t_0) = \frac{e_{ip}(t,t_0) + e_{-ip}(t,t_0)}{2}$$

where $\xi_{\mu(\tau)}(p(\tau)) = \begin{cases} \frac{1}{\mu(\tau)} Log(1+\mu(\tau)p(\tau)), & \mu(\tau) \neq 0 \\ p(\tau), & \mu(\tau) = 0 \end{cases}.$

It is proved in [19] that these functions satisfy the following relations (for further informations about these functions see also [19])

$i)$ $e_p^{\Delta}(t,t_0) = p(t)e_p(t,t_0)$, $e_p(t_0,t_0) = 1$;

$ii)$ $\sin_p^{\Delta}(t,t_0) = p(t)\cos_p(t,t_0)$, $\sin_p(t_0,t_0) = 0$;

$iii)$ $\cos_p^{\Delta}(t,t_0) = -p(t)\sin_p(t,t_0)$, $\cos_p(t_0,t_0) = 1$.

For a fixed $\lambda \in \mathbb{R}$, a scalar function $y(t,\lambda)$ is said to have a zero at $t_0 \in \mathbb{T}$ if $y(t_0,\lambda) = 0$, and it has a node on $(t_0, \sigma(t_0))$ if $y(t_0,\lambda)y^{\sigma}(t_0,\lambda) < 0$. A generalized zero of y is then defined as a zero or a node [20].

A first order linear system on a time scale $\mathbb{T}$ can be given as follows

$$Y^{\Delta}(t) = A(t)Y(t) + f(t) \tag{1}$$

where $A(t)$ is an $n \times n$ −matrix-valued rd-continuous function on $\mathbb{T}$ and $f: \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. If $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^k$, then we say that the system (1) is regressive, where I is nxn identity matrix. A function $Y: \mathbb{T} \to \mathbb{R}^n$ is called a solution of (1) if Y is Δ-differentiable on $\mathbb{T}$ and satisfies (1). The system (1) is called as disconjugate provided there is no nontrivial real solution with one (or more) generalized zeros in $\mathbb{T}^{\kappa}$ [20].

Theorem 1([19]) *Let $t_0 \in \mathbb{T}^{\kappa}$ and $Y_0 \in \mathbb{R}^n$. If (1) is a regressive system, then initial value problem*

$$Y^{\Delta}(t) = A(t)Y(t) + f(t)$$

$$Y(t_0) = Y_0$$

has a unique solution.

Two types Dirac systems can be given on a time scale $\mathbb{T}$ as follows

$$BY^{\Delta}(t) + \Omega(t)Y(t) = \lambda Y(t) \tag{2}$$

$$BY^{\Delta}(t) + \Omega(t)Y^{\sigma}(t) = \lambda Y^{\sigma}(t) \tag{3}$$

where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ is unknown vector-valued function, $Y^{\sigma}(t) = Y(\sigma(t))$ and $\Omega(t)$ is a matrix with dimension 2x2 defined on $\mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then the classical Dirac system

$$BY'(t) + \Omega(t)Y(t) = \lambda Y(t) \tag{4}$$

is obtained from each of (2) and (3).

Although, the literature about the spectral problems for Sturm-Liouville equation on time scales is vast; there are only a few studies about Dirac-type dynamic equation systems. It can be referred [21] and [22] for example to boundary-value problems generated by the Dirac system on a time scale.

In the present paper, we consider two generalized Dirac systems:

$$BY^{\Delta}(t) + \Omega(t,\lambda)Y(t) = \lambda^m Y(t), \quad t \in \mathbb{T}^{\kappa} \tag{5}$$

$$BY^{\Delta}(t) + \Omega(t,\lambda)Y^{\sigma}(t) = \lambda^m Y^{\sigma}(t), \quad t \in \mathbb{T}^{\kappa} \tag{6}$$

where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, $\Omega(t,\lambda) = \sum_{j=0}^{m-1} \lambda^j Q_j(t)$ such that $Q_j(t) = \begin{pmatrix} p_j(t) & 0 \\ 0 & q_j(t) \end{pmatrix}$ are continuous on $\mathbb{T}$ for $j = \overline{0, m-1}$.

Equations (5) and (6) can be written as follows:

$$y_1^{\Delta}(t) = [q(t,\lambda) - \lambda^m]y_2(t), \quad y_2^{\Delta}(t) = [\lambda^m - p(t,\lambda)]y_1(t)$$

$$y_1^{\Delta}(t) = [q(t,\lambda) - \lambda^m]y_2^{\sigma}(t), \quad y_2^{\Delta}(t) = [\lambda^m - p(t,\lambda)]y_1^{\sigma}(t)$$

where $q(t,\lambda) = \sum_{j=0}^{m-1} \lambda^j q_j(t)$ and $p(t,\lambda) = \sum_{j=0}^{m-1} \lambda^j p_j(t)$.

2. DISCONJUGACY

Denote $h_1(t,\lambda) := q(t,\lambda) - \lambda^m$ and $h_2(t,\lambda) := p(t,\lambda) - \lambda^m$.

It can be proved that the systems (5) and (6) are regressive for each fixed λ in $H := \{\lambda \in \mathbb{C} : 1 + \mu^2(t)h_1(t,\lambda)h_2(t,\lambda) \neq 0 \text{ for all } t \in \mathbb{T}^{\kappa}\}$.

Lemma 1 *Let* $Y(t,\lambda_0) = \begin{pmatrix} y_1(t,\lambda_0) \\ y_2(t,\lambda_0) \end{pmatrix}$ *be a non-trivial solution of (5) for a fixed* $\lambda_0 \in \mathbb{R}$.

i) If $y_1(t,\lambda_0)$ *has a node on* $(t_0, \sigma(t_0))$, *then* $h_1(t_0,\lambda_0)$ *and* $(y_1y_2)(t_0,\lambda_0)$ *are of opposite signs.*

ii) If $y_2(t,\lambda_0)$ has a node on $(t_0,\sigma(t_0))$, then $h_2(t_0,\lambda_0)$ and $(y_1y_2)(t_0,\lambda_0)$ have the same sign.

Proof. It is clear that the following relations hold for all t and λ.

$$(y_1y_1^\sigma)(t,\lambda)=[y_1(t,\lambda)]^2+\mu(t)h_1(t,\lambda)(y_1y_2)(t,\lambda)$$

$$(y_2y_2^\sigma)(t,\lambda)=[y_2(t,\lambda)]^2-\mu(t)h_2(t,\lambda)(y_1y_2)(t,\lambda)$$

Since $\lambda_0\in\mathbb{R}$, both assertions are true.

Lemma 2 *Let $Y(t,\lambda_0)=\begin{pmatrix}y_1(t,\lambda_0)\\ y_2(t,\lambda_0)\end{pmatrix}$ be a non-trivial solution of (6) for a fixed $\lambda_0\in\mathbb{R}$.*

i) If $y_1(t,\lambda_0)$ has a node on $(t_0,\sigma(t_0))$, then $h_1(t_0,\lambda_0)$ and $(y_1y_2^\sigma)(t_0,\lambda_0)$ are of opposite signs.

ii) If $y_2(t,\lambda_0)$ has a node on $(t_0,\sigma(t_0))$, then $h_2(t_0,\lambda_0)$ and $(y_1^\sigma y_2)(t_0,\lambda_0)$ have the same sign.

Proof. Similar to previous lemma.

Theorem 2 *Let $Y(t,\lambda_0)=\begin{pmatrix}y_1(t,\lambda_0)\\ y_2(t,\lambda_0)\end{pmatrix}$ be a non-trivial solution of (5) or (6) for a fixed $\lambda_0\in\mathbb{R}$. If $det(\Omega(t_0,\lambda_0)-\lambda^m I)>0$ for a $t_0\in\mathbb{T}^\kappa$, then $y_1(t,\lambda_0)$ and $y_2(t,\lambda_0)$ can not have a node on $(t_0,\sigma(t_0))$ at the same time.*

Proof. If we assume conversely that $y_1(t,\lambda_0)$ and $y_2(t,\lambda_0)$ have a node on $(t_0,\sigma(t_0))$, then we have a contradiction from Lemma 1 and Lemma 2. The proof is clear.

Corollary 1 *Systems (5) and (6) are disconjugate for each fixed $\lambda\in\mathbb{R}$ which satisfies $det(\Omega(t,\lambda)-\lambda^m I)>0$ on $\mathbb{T}^\kappa$.*

3. A BOUNDARY-VALUE PROBLEM

Now, let us consider the following boundary conditions.

$$U(y):=a(\lambda)y_1(\alpha)-b(\lambda)y_2(\alpha)=0 \quad (7)$$

$$V(y):=c(\lambda)y_1(\beta)-d(\lambda)y_2(\beta)=0 \quad (8)$$

where $\alpha=\inf\mathbb{T}$, $\beta=\sup\mathbb{T}$, $\alpha\neq\beta$; $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ and $d(\lambda)$ are real polynomials whose leading coefficients are a_n, b_n, c_{n_c} and d_{n_d}, respectively. We assume $n=\deg a(\lambda)=\deg b(\lambda)$, (it may be $n_c=\deg c(\lambda)\neq\deg d(\lambda)=n_d$).

We denote boundary value problem (5), (7), (8) by L.

Let $\varphi(t,\lambda)=\begin{pmatrix}\varphi_1(t,\lambda)\\ \varphi_2(t,\lambda)\end{pmatrix}$ be the solution of (5) under the initial conditions $\varphi_1(\alpha,\lambda)=b(\lambda)$, $\varphi_2(\alpha,\lambda)=a(\lambda)$ for a fixed $\lambda\in H$. Existence and uniqueness of $\varphi(t,\lambda)$ follow from Theorem 1.

The following integral equations are valid for each fixed λ in H such that $1+\mu^2(t)\lambda^{2m} \neq 0$ for all $t \in \mathbb{T}^\kappa$.

$$\begin{aligned}\varphi_1(t,\lambda) = & -a(\lambda)\sin_{\lambda^m}(t,\alpha) + b(\lambda)\cos_{\lambda^m}(t,\alpha) \\ & + \int_\alpha^t \frac{1}{e_{\mu\lambda^{2m}}(\sigma(\tau),\alpha)} \sin_{\lambda^m}(\sigma(\tau),\alpha)\cos_{\lambda^m}(t,\alpha)p(\tau)\varphi_1(\tau,\lambda)\Delta\tau \\ & - \int_\alpha^t \frac{1}{e_{\mu\lambda^{2m}}(\sigma(\tau),\alpha)} \cos_{\lambda^m}(\sigma(\tau),\alpha)\sin_{\lambda^m}(t,\alpha)p(\tau)\varphi_1(\tau,\lambda)\Delta\tau \\ & - \int_\alpha^t \frac{1}{e_{\mu\lambda^{2m}}(\sigma(\tau),\alpha)} \sin_{\lambda^m}(\sigma(\tau),\alpha)\sin_{\lambda^m}(t,\alpha)q(\tau)\varphi_2(\tau,\lambda)\Delta\tau \\ & - \int_\alpha^t \frac{1}{e_{\mu\lambda^{2m}}(\sigma(\tau),\alpha)} \cos_{\lambda^m}(\sigma(\tau),\alpha)\cos_{\lambda^m}(t,\alpha)q(\tau)\varphi_2(\tau,\lambda)\Delta\tau\end{aligned}$$

$$\begin{aligned}\varphi_2(t,\lambda) = & \, a(\lambda)\cos_{\lambda^m}(t,\alpha) + b(\lambda)\sin_{\lambda^m}(t,\alpha) \\ & + \int_\alpha^t \frac{1}{e_{\mu\lambda^{2m}}(\sigma(\tau),\alpha)} \sin_{\lambda^m}(\sigma(\tau),\alpha)\sin_{\lambda^m}(t,\alpha)p(\tau)\varphi_1(\tau,\lambda)\Delta\tau \\ & + \int_\alpha^t \frac{1}{e_{\mu\lambda^{2m}}(\sigma(\tau),\alpha)} \cos_{\lambda^m}(\sigma(\tau),\alpha)\cos_{\lambda^m}(t,\alpha)p(\tau)\varphi_1(\tau,\lambda)\Delta\tau \\ & - \int_\alpha^t \frac{1}{e_{\mu\lambda^{2m}}(\sigma(\tau),\alpha)} \cos_{\lambda^m}(\sigma(\tau),\alpha)\sin_{\lambda^m}(t,\alpha)q(\tau)\varphi_2(\tau,\lambda)\Delta\tau \\ & + \int_\alpha^t \frac{1}{e_{\mu\lambda^{2m}}(\sigma(\tau),\alpha)} \sin_{\lambda^m}(\sigma(\tau),\alpha)\cos_{\lambda^m}(t,\alpha)q(\tau)\varphi_2(\tau,\lambda)\Delta\tau.\end{aligned}$$

It is obvious that the zeros of the function

$$\Delta(\lambda) := c(\lambda)\varphi_1(\beta,\lambda) - d(\lambda)\varphi_2(\beta,\lambda) \tag{9}$$

coincide with the eigenvalues of the problem L.

The next theorem gives the number of eigenvalues of the problem L on a finite time scale.

Theorem 3 *Let $\mathbb{T}$ be a finite time scale, the number of elements of $\mathbb{T}$ be denoted by s and $r := max\{degc(\lambda), degd(\lambda)\}$. Then the problem L has at most $\chi = n + r + (s-1)m$ many eigenvalues in H.*

Proof. Since all points of $\mathbb{T}$ are isolated we can write it as

$$\mathbb{T} = \{\alpha, \sigma(\alpha), \sigma^2(\alpha), \dots, \sigma^{s-2}(\alpha), \sigma^{s-1}(\alpha) = \beta\}$$

where $\sigma^j = \sigma^{j-1} \circ \sigma$, for $j \geq 2$. It can be calculated from (5) that

$$\begin{cases}\varphi_1^\sigma(t) = \varphi_1(t) + \mu(t)[q(t,\lambda) - \lambda^m]\varphi_2(t) \\ \varphi_2^\sigma(t) = \varphi_2(t) + \mu(t)[\lambda^m - p(t,\lambda)]\varphi_1(t)\end{cases}, t \in \mathbb{T}^k$$

Therefore, one can obtain the following equalities.

$$\begin{cases} \varphi_1(\alpha) = b_n\lambda^n + [\lambda^{n-1}], \\ \varphi_2(\alpha) = a_n\lambda^n + [\lambda^{n-1}], \\ \varphi_1^\sigma(\alpha) = -a_n\mu(\alpha)\lambda^{n+m} + [\lambda^{n+m-1}], \\ \varphi_2^\sigma(\alpha) = b_n\mu(\alpha)\lambda^{n+m} + [\lambda^{n+m-1}], \\ \varphi_1^{\sigma^K}(\alpha) = A_K\lambda^{n+Km} + [\lambda^{n+Km-1}], \text{ for } K \geq 2 \\ \varphi_2^{\sigma^K}(\alpha) = B_K\lambda^{n+Km} + [\lambda^{n+Km-1}], \text{ for } K \geq 2 \end{cases} \tag{10}$$

where

$$A_K = \begin{cases} (-1)^{\frac{K}{2}} b_n \prod_{j=0}^{K-1} \mu^{\sigma^j}(\alpha), & \text{if } K \text{ is even} \\ (-1)^{\frac{K+1}{2}} a_n \prod_{j=0}^{K-1} \mu^{\sigma^j}(\alpha), & \text{if } K \text{ is odd} \end{cases},$$

$$B_K = \begin{cases} (-1)^{\frac{K}{2}} a_n \prod_{j=0}^{K-1} \mu^{\sigma^j}(\alpha), & \text{if } K \text{ is even} \\ (-1)^{\frac{K-1}{2}} b_n \prod_{j=0}^{K-1} \mu^{\sigma^j}(\alpha), & \text{if } K \text{ is odd} \end{cases}$$

and $[\lambda^j]$ denotes a polynomial with degree j. It is obvious from (9) that

$$\Delta(\lambda) = c(\lambda)\varphi_1(\sigma^s(\alpha),\lambda) - d(\lambda)\varphi_2(\sigma^s(\alpha),\lambda). \tag{11}$$

Thus, $\Delta(\lambda)$ has $n + r + (s-1)$ many roots on the complex plane. However some of them may not be belong in H. Hence the proof is clear.

Example 1 *Let us consider the following boundary value problem*

on $\mathbb{T} = \{1,2,\ldots,s\}$*:*

$$BY^\Delta(t) + \left[\sum_{j=0}^{m-1} \lambda^j Q_j(t)\right]Y(t) = \lambda^m Y(t), \quad 1 \leq t \leq s-1$$

$$(\lambda + a_0)y_1(1) - (\lambda + b_0)y_2(1) = 0$$

$$(\lambda + c_0)y_1(s) - (\lambda^2 + d_1\lambda + d_0)y_2(s) = 0$$

According to Theorem 3, this boundary value problem has at most $(s-1)m + 3$ *many eigenvalues.*

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