



On a Relation Between Twistors and Killing Spinors

Özgür AÇIK

Ankara University, Faculty of Sciences, Department of Physics, Ankara, TURKEY

Received: 19.06.2018; Accepted: 13.11.2018

<http://dx.doi.org/10.17776/cs.j.434630>

Abstract. Inspiring from the consequence of constructing conformal Killing-Yano forms out of Killing-Yano forms and closed conformal Killing-Yano forms, this work includes a method for building up twistors from Killing spinors which can be analogously interpreted as the quantum electrodynamical pair annihilation process in background gravitational fields. The former consequence is easily verified if one introduces a new defining differential equation for (possibly inhomogeneous) Killing-Yano forms which is free from auxiliary vector fields, as is done in this text. From this point of view a neat relation between the symmetry operators of massive and massless Dirac equation is also introduced. Some other physical interpretations are also included.

Keywords: Clifford Algebras and Spinors, Twistors, Killing-Yano forms, Killing spinors.

Twistörler ve Killing Spinörleri Arasındaki Bir İlişki Üzerine

Özet. Killing-Yano formları ve kapalı konformal Killing-Yano formlarından, konformal Killing-Yano formları oluşturabilme sonucundan esinlenmek suretiyle, bu çalışma Killing spinörlerinden twistörler inşa etmek için bir yöntem içermektedir ki bu benzeşik olarak arka-zemin kütle-çekimsel alanlarda kuantum elektrodinamiksel çift yokoluşu olarak yorumlanabilmektedir. Eğer bu makalede yapıldığı gibi (homojen olmaması muhtemel) Killing-Yano formları için yardımcı vektör alanlarından muhafazalı yeni tanımlayıcı bir diferensiyel denklem tanıtılırsa ilk sonuç kolayca doğrulanabilir. Bu bakış açısıyla kütleli ve kütsüz Dirac denkleminin simetri işlemcileri arasında muntazam bir ilişki tanıtılabilmektedir. Bazı diğer fiziksel yorumlar da içermektedir.

Anahtar Kelimeler: Clifford cebirleri ve spinörler, Twistörler, Killing-Yano formları, Killing spinörleri.

1. INTRODUCTION

Killing spinors are fruitful objects that give rise to many differential equations that appear in mathematical physics. Up to now bosonic equations were constructed from the real bilinear covariants of Killing spinors which include Duffin-Kemmer-Petiau and Maxwell-like equations in gravitational fields [1], but relation to possible fermionic field equations were absent. Connectedly the construction of twistors from Killing spinors is to be seen by us as a first step in this direction, that may be tied [2] to the generation of higher spin fields by known methods such as spin raising/lowering of non-gravitational fields in curved spacetimes [3-5]. We show that, in analogy

with the quantum electrodynamical pair annihilation process, Killing spinors correspond to the electron-positron pair and twistors correspond to the pure radiation field. The mathematics literature include the classification of pseudo-Riemannian manifolds admitting twistors [6,7] and the classification of pseudo-Riemannian manifolds admitting Killing spinors [8]; so our method will form a bridge between these classifications.

We prefer to use coordinate-free differential geometry for the sake of notational elegance and brevity and also for clarity in direct geometrical or physical interpretations. The usage of componentwise tensor calculus is more common in physics community; closing this gap could only be

possible by transforming one language to the other. We tried to do this translation at least for a set of equations that we feel important and any reader can use this example as a dictionary in between; this is done in Appendix C. Pre-metric notions are handled through the usage of Cartan calculus on manifolds [9, 10], the participation of a metric tensor field into the set of spacetime structures [37] forces us to use the full power of Clifford bundle formulation of physics and geometry. This latter choice is precious for admitting a local equivalence between spinor fields and the sections of minimal left ideal bundle of the Clifford bundle. A good account of this tradition can be found in [11, 12]. Throughout the text we assume a Riemannian spacetime [38] i.e. a smooth manifold endowed with an indefinite metric g and its Levi-Civita connection ∇ . The organisation of the paper is as follows. In section II an explicit method for constructing conformal Killing-Yano (CKY) forms from Killing-Yano (KY) forms and closed conformal Killing-Yano (CCKY) forms is given; before that a new form for the definitive equation for KY forms is built up. Then the first order symmetry operators for massive and massless Dirac equations are reconsidered and are related in an elegant way in the view of the above setup. An intuitional question, which leads to the core of this paper, is asked at the end of section II. The positive answer of this main question is given in terms of a proposition and its proof. The analysis is bifurcated into fermionic and bosonic sectors both of which contain geometric identities that are derived in the corresponding subsections; the fermionic sector is detailed by considerations on charge conjugation, time reversal, helicity and their relation to Killing reversal due to the preceding main physical interpretation. After the conclusion of our paper at section IV, two appendices are given at the end. Appendix A is based on the calculation of the adjoints of the symmetry operators associated to KY and CCKY forms and ends up with a comment on symmetry algebra. Appendix B contains the derivation of the coordinate expressions of our primitive set of equations which is important for the understanding of general reader who are used to the notation of tensor components.

2. BUILDING UP CONFORMAL KILLING-YANO FORMS

A. A Different Perspective for Defining Killing-Yano Forms

The important roles played by KY forms and CKY forms are various. They define the hidden symmetries of the ambient spacetime generalising to higher degree forms the ones induced by Killing and Conformal Killing vector fields, and they respectively can be used for setting up first order symmetry operators for massive and massless Dirac equations in curved spacetimes [14, 15]. These symmetry operators reduce to the Lie derivative on spinor fields for the lowest possible degree, so if carefully worked out they may define Lie multi-flows of spinor fields in a classical spacetime. Achieving this shall also extend the concept of Lie multi-flows to form fields and multi-vector fields. As a consequence, the analysis of higher rank infinitesimal symmetries will heavily simplify [16, 17]. On the other side, via Noether's theorem, they induce conserved quantities associated to geodesics of (point) particles [18] and more generally to world-immersions of p-branes [19-21]; where there is much work to do for the later case [22]. Also the definition of some charges for higher dimensional black holes rely on the existence of these symmetry generating totally anti-symmetric tensor fields [23].

Now we want to redefine KY forms from a different perspective that is more general. The usual definition of a KY-form of degree p is the solution of

$$\nabla_X \omega = \frac{1}{p+1} i_X d\omega. \quad (1)$$

Exterior derivative d and co-derivative $d^\dagger := *^{-1} d * \eta$ can be given by the Riemannian relations $d = e^a \wedge \nabla_{X_a}$, $d^\dagger = -i_{X^a} \nabla_{X_a}$ and i_X is the internal contraction with respect to the vector field X . If T is a mixed tensor field of covariant degree r and contravariant degree s , its covariant differential ∇T which has one contravariant degree more than T is defined as

$$(\nabla T)(X, Y_1, \dots, Y_r, \beta^1, \dots, \beta^s) = (\nabla_X T)(Y_1, \dots, Y_r, \beta^1, \dots, \beta^s) \quad (2)$$

in terms of vector fields X, Y_i and co-vector fields β^j . Also remembering the identity

$$(i_X \phi)(Y_1, \dots, Y_{p-1}) = p\phi(X, Y_1, \dots, Y_{p-1}) \quad (3)$$

for any p-form ϕ , we can define KY-forms in a different manner. If we write (1) as

$$(\nabla_X \omega)(Y_1, \dots, Y_p) = \left(\frac{i_X d\omega}{p+1} \right)(Y_1, \dots, Y_p)$$

then by using (2) and (3) for the left and right hand sides respectively we obtain

$$(\nabla \omega)(X, Y_1, \dots, Y_p) = (d\omega)(X, Y_1, \dots, Y_p) \quad (4)$$

which is valid for any $X, Y_i \in \Gamma TM$. As a result the new defining equation for a KY form is

$$(\nabla - d)\omega = 0 \quad (5)$$

and is not necessarily homogeneous. An integrability condition equivalent to Poincaré's lemma is $\nabla^2 \omega = 0$. Remembering the definition of the alternating idempotent map; i.e.

$$\begin{aligned} Alt: \Gamma T^r(M) &\rightarrow \Gamma T\Lambda^r(M) \\ G &\mapsto Alt(G) \end{aligned}$$

such that

$$Alt(G)(Y_1, \dots, Y_r) = \frac{1}{r!} \sum_{\sigma \in \mathcal{S}(r)} sign(\sigma) G(Y_{\sigma(1)}, \dots, Y_{\sigma(r)})$$

it is possible to write (5) as

$$((1 - Alt)\nabla)\omega = 0.$$

B. Adding Closed Conformal Killing-Yano forms to Killing-Yano forms

In this section our main objects will be CKY forms and we will give a trivial method for their construction. CKY form equation is

$$\nabla_X \rho = \frac{1}{p+1} i_X d\rho - \frac{1}{n-p+1} \tilde{X} \wedge d^\dagger \rho \quad (6)$$

where an inhomogeneous generalisation was given in [1], \tilde{X} is the g -dual of the vector field X . We want to emphasize the duality between the space of Killing-Yano forms and the space of CCKY forms, that is if $\hat{\omega}$ is a CCKY p-form then its Hodge dual $*\hat{\omega}$ is a KY (n - p)-form and vice-versa. Although the usual definition of a CCKY p-form $\hat{\omega}$ is given by

$$\nabla_X \hat{\omega} = -\frac{1}{n-p+1} \tilde{X} \wedge d^\dagger \hat{\omega} \quad (7)$$

it is trivial from (6) that this equation can also be written as

$$(\nabla^\dagger - d^\dagger)\hat{\omega} = 0 \quad (8)$$

where $\nabla^\dagger = *^{-1} \nabla * \eta$. η is the main automorphism of the tensor algebra. In a Riemannian spacetime, the ρ in (6) satisfies (7)/(1) if it is closed/co-closed; so it is trivial that if ω is a KY p-form and $\widehat{\omega}$ is a CCKY p-form then we can built up a CKY p-form simply as

$$\rho = \omega + \widehat{\omega}.$$

Proof: Covariant derivative of ρ with respect to a vector field X is

$$\nabla_X \rho = \nabla_X \omega + \nabla_X \widehat{\omega},$$

using (1) for the first term and (9) for the second term at the right hand side gives

$$\nabla_X \rho = \frac{1}{p+1} i_X d\omega - \frac{1}{n-p+1} \tilde{X} \wedge d^\dagger \widehat{\omega}$$

Adding $\widehat{\omega}$ to ω and ω to $\widehat{\omega}$ changes nothing at the right hand side, that is because ω is co-closed and $\widehat{\omega}$ is closed

$$\nabla_X \rho = \frac{1}{p+1} i_X d(\omega + \widehat{\omega}) - \frac{1}{n-p+1} \tilde{X} \wedge d^\dagger(\widehat{\omega} + \omega)$$

and the result (8) holds.

It is also possible to construct inhomogeneous self dual or anti-self dual CKY forms from a KY form or equivalently from a CCKY form. The sign is determined only by the degree of the unit Yano form [39] because of the action of Hodge square on homogeneous forms; that is

$$**\alpha = (-1)^{p(n-p)} \frac{\det \mathbf{g}}{|\det \mathbf{g}|} \alpha ; \quad \alpha \in \Gamma \Lambda^p(M)$$

here \mathbf{g} is the chart matrix of the metric tensor g . If $\alpha \in \{\omega, \widehat{\omega}\}$ is a unit Yano form and if $**\alpha = \alpha$ then $\alpha + *\alpha$ is a self dual inhomogeneous CKY form; but if $**\alpha = -\alpha$ then $i\alpha + *\alpha$ is an anti-self dual inhomogeneous CKY form.

Here a physical interpretation is inevitable according to the results of [1]; there the generalised Dirac currents (real homogeneous bilinears) of Killing spinors were satisfying a kind of higher degree Maxwell equations, so superposing the field associated to the co-existent Killing reversed spinors with the former field, a null higher-degree electromagnetic field for homogeneous CKY forms could be obtained. Passing to the inhomogeneous domain by linearity will turn out this null higher-degree electromagnetic field into a self dual or an anti-self dual one. The definition of $i = \sqrt{-1}$ is ambiguous here but since it exists at least if the Hodge map defines a complex structure on $\Gamma \Lambda^p(M)$ which is always possible for some p [24].

C. Symmetry considerations

As mentioned before, KY forms were used for the generation of first order symmetries of the massive Dirac equation $D\psi = m\psi$

$$L_{(\omega)} = \omega^a \nabla_{X_a} + \frac{p}{2(p+1)} d\omega \tag{9}$$

namely $DL_{(\omega)}\psi = L_{(\omega)}D\psi = mL_{(\omega)}\psi$; $\omega^a := i_{X^a}\omega$ and Clifford product is shown by the juxtaposition of the factors, but we put a dot when left Clifford acting on spinor fields. CKY forms corresponds similarly to massless Dirac equation $D\psi = 0$ as

$$L_{(\rho)} = \rho^a \nabla_{X_a} + \frac{p}{2(p+1)} d\rho - \frac{n-p}{2(n-p+1)} d^\dagger \rho \quad (10)$$

i.e. $DL_{(\omega)}\psi = 0$. Since $L_{(\rho)} = L_{(\omega+\hat{\omega})} = L_{(\omega)} + L_{(\hat{\omega})}$ then we can deduce that the first order operator generated by a CCKY form $\hat{\omega}$ is

$$L_{(\hat{\omega})} = \hat{\omega}^a \nabla_{X_a} - \frac{n-p}{2(n-p+1)} d^\dagger \hat{\omega}. \quad (11)$$

The question here is, which equation admits this as its first order symmetry operator? We do not know the answer to this question; but we should note that this operator does not reduce to the usual Lie derivative on spinor fields for degree one case, as opposed to $L_{(\omega)}$. The definition of $L_{(\rho)}$ for massless Dirac equation necessitates the existence of an operator R [40] such that

$$DL_{(\rho)} = RD \quad (12)$$

i.e. $L_{(\rho)}$, R -commutes with D . Although ρ , ω and $\hat{\omega}$ are all homogeneous and of the same degree for our purposes, for general considerations they are taken as inhomogeneous forms which can be separated to their Clifford even and Clifford odd parts if needed. So, if Φ is one of the above Yano forms that is inhomogeneous, its even and odd parts respectively $\Phi_{odd} = \sum_{p \text{ odd}} \Phi_p$ and $\Phi_{even} = \sum_{p \text{ even}} \Phi_p$ where Φ_p is the degree p part of Φ obtained by applying

the p -form projector \wp_p on Φ [41]. For these general inhomogeneous investigations, the condition (14) transforms into

$$[D, L_{(\rho)}]_{GCC} = RD \quad (13)$$

$[\cdot, \cdot]_{GCC}$ is the Graded Clifford Commutator and the symmetry condition may be termed as graded R -commuting. Note that the odd and even parts of the first order symmetry operators satisfy

$$L'_{(\Phi)}^{odd} = L'_{(\Phi \text{ even})}, L'_{(\Phi)}^{even} = L'_{(\Phi \text{ odd})}; L' \in \{L, \hat{L}\}.$$

A known result: In any dimensions with arbitrary signature and curvature the below results hold [14, 15, 25].

- The first order symmetry operator of an odd KY form Clifford commutes with the Dirac operator: $L_{(\omega \text{ odd})}D = DL_{(\omega \text{ odd})}$.

- The first order symmetry operator of an even CCKY form Clifford commutes with the Dirac operator: $\hat{L}_{(\hat{\omega} \text{ even})}D = D\hat{L}_{(\hat{\omega} \text{ even})}$.

- The first order operator of an even KY form Clifford anti-commutes with the Dirac operator: $L_{(\omega \text{ even})}D = -DL_{(\omega \text{ even})}$.

- The first order operator of an odd CCKY form Clifford anti-commutes with the Dirac operator: $\hat{L}_{(\hat{\omega} \text{ odd})}D = -D\hat{L}_{(\hat{\omega} \text{ odd})}$.

This result will be important for our homogeneous analysis, especially when the analog relations in the spinor sector will be derived. Here in the anti-commuting cases the first order operators are no more symmetry operators, but they in some sense have a physical meaning. In this case, if ψ is a solution of the massive Dirac equation then $L\psi$ will be a solution of the Dirac equation with negative mass representing a hypothetical spinning particle. We will comment on this issue at Section III, but as every one knows that the quantum vacuum is full of these particles as part of pairs with energies of different signs permitted by the Heisenberg's quantum uncertainty principle.

D. Physical Interpretation and the Main Question

In a recent work [1] we worked out some properties of bilinears generated by twistors and Killing spinors. The Killing spinor case, accompanied by a data set, was more sophisticated and rich. All possible outcomes obtainable from the Killing spinor bilinears were determined by the restrictive reality conditions imposed on them for physical reasons. As a reward we uncovered both kinematical and dynamical equations satisfied by the corresponding generalized Dirac currents of Killing spinor. The *primitive set* of generating equations were seen to be

$$\nabla_{X_a}(\psi\bar{\psi})_p = 2\lambda e^a \wedge (\psi\bar{\psi})_{p-1} \tag{14}$$

$$\nabla_{X_a}(\psi\bar{\psi})_{p_*} = 2\lambda i_{X_a}(\psi\bar{\psi})_{p_*+1} \tag{15}$$

giving rise to the *principal set*

$$d(\psi\bar{\psi})_p = 0 \quad , \quad d^\dagger(\psi\bar{\psi})_p = -2\lambda(n - p + 1)(\psi\bar{\psi})_{p-1} \tag{16}$$

$$d(\psi\bar{\psi})_{p_*} = 2\lambda(p_* + 1)(\psi\bar{\psi})_{p_*+1} \quad , \quad d^\dagger(\psi\bar{\psi})_{p_*} = 0. \tag{17}$$

Here p_* means that it has a different parity than p , i.e. $p_* + p$ is always odd. These equations imply that the homogeneous realified parts of Killing spinor bilinears represent Duffin-Kemmer-Petiau (DKP) fields or Maxwell-like fields on the dynamical side, where on the kinematical side when the degree p part of the bilinear satisfies KY equation the degree p_* part satisfies CCKY equation and vice-versa. The kinematical results could then be applied by using the machinery of the previous subsection. Namely for a Killing spinor ψ , we know that for some p , $(\psi\bar{\psi})_p$ is a KY form which requires that $(\psi\bar{\psi})_{p_*}$ is a CCKY form then we also know that $(\psi\bar{\psi})_p + (\psi\bar{\psi})_{p_*}$ is a CKY form which is generally associated to a twistor spinor's bilinear. Now the remarkable question arises!

The main question: *By using the generalized currents of Killing spinors one can deduce directly the generalized currents of twistors. Is there a way for generating twistors from Killing spinors?*

Before passing to the next section in search for the answer, we want to repeat a previous interpretation of the equations (14) and (15). With a slight change of understanding, the primitive equations could be reinterpreted as follows: The propagation of a brane in spacetime is triggered by the creation of a brane with one lower dimension and because of the unstable motion of the higher dimensional brane it is annihilated and this gives rise to the propagation of the lower dimensional (stable) brane. Of course in the context of General Relativity this process should be observed in a locally inertial frame $\{X_a\}$ co-moving with the stable brane to which it is adapted in a such way that $g(X_0, X_0) + \sum_{i=1,2,3} g(X_i, X_i) < 0$ and this corresponds to the physical (semi-classical) motion of this system (for details see the insight in [1]). Some classical references for the details of the motion of extendons in General Relativity are [26-30]. The selection of the local frame is important for the preservation of local causality which temporally orders the equations in the primitive set in accordance with the above scenario; otherwise the mathematical

simultaneity of the equations will be deceptive. The temporal parameter should be taken as the local proper time measured by the locally inertial (time-like) observer instead of the local coordinate time. If one remembers that $e^a \wedge$ is equal to $i_{X_a}^\dagger := *^{-1} i_{X_a} * \eta$ (see App. A of [1]), the primitive set could be rewritten as

$$i_{X_a}^\dagger(\psi\bar{\psi})_{p-1} = (2\lambda)^{-1}\nabla_{X_a}(\psi\bar{\psi})_p$$

$$i_{X_a}(\psi\bar{\psi})_{p_*+1} = (2\lambda)^{-1}\nabla_{X_a}(\psi\bar{\psi})_{p_*}$$

and these contain more physical intuition. Our scenario could also be thought as equivalent to Dirac's [31] where he models the electron as an extended elementary particle and the muon corresponds to the first excited state of the electron; a similar and more accurate construction may be found in [32]. From this point of view, our model may be seen as a gas of one level $(p-2)$ -branes (i.e. the $(p-2; p-1)$ -brane couple), if n is the dimension of spacetime then p ranges from $2, 4, \dots, n+2$ in odd dimensions and from $2, 4, \dots, n$ in even dimensions. This set up requires one level branes because i_{X_a} and $i_{X_a}^\dagger$ are both nilpotent of index two.

3. BUILDING UP TWISTORS FROM KILLING SPINORS

A. The Fermionic Sector

The answer to the main question is positive. If ψ is a Killing spinor

$$\nabla_X \psi = \lambda \tilde{X} \cdot \psi \quad (18)$$

then we define the Killing reversal ψ^ζ of ψ by

$$\nabla_X \psi^\zeta = -\lambda \tilde{X} \cdot \psi^\zeta \quad (19)$$

that was defined in [22] technically, but is being known and used for example in [33]. The generation of twistors from Killing spinors is given by the following proposition.

Proposition: To every Killing spinor pair ψ, ψ^ζ there corresponds a twistor pair Ψ^\pm, Ψ^\mp such that

$$\Psi^\pm = \psi \pm \psi^\zeta$$

where ψ^ζ is the Killing reversal of ψ ; and trivially the Killing reversals of the induced twistors are $\Psi^\pm = \pm \Psi^\pm$.

Proof:

$$\nabla_{X_a} \Psi^\pm = \nabla_{X_a}(\psi \pm \psi^\zeta) = \nabla_{X_a} \psi \pm \nabla_{X_a} \psi^\zeta = \lambda e_a \cdot \psi \mp \lambda e_a \cdot \psi^\zeta = \lambda e_a \cdot \Psi^\mp,$$

and if we Clifford contract both hand sides from left with e_a and use the identity $e^a e_a = n$ then we obtain

$$D\Psi^\pm = n\lambda\Psi^\mp$$

and if we put the last identity into the last equality of the first relation we reach the desired result

$$\nabla_{X_a} \Psi^\pm = \frac{1}{n} e_a \cdot D\Psi^\pm$$

namely the twistor equation.

Physical Interpretation:

The process $\psi + \psi^c \rightarrow \Psi^+ + \Psi^-$ mimics the well known quantum electrodynamical pair annihilation process $e + e^+ \rightarrow \gamma + \gamma$ in many ways. From sections II.C and II.D we know that Killing spinors are related to the massive sector and the twistors are related to the massless sector; so just as the mutual annihilation of an electron and a positron results in a pure electromagnetic field the mutual annihilation of a Killing spinor and its Killing reversal results in a pure radiation field the kind of which should be determined by further considerations. This can partially be achieved by the comparison of the properties of the Killing reversal map with that of charge conjugation (mainly) for the complex case. Another possibly related interpretation could be that the $\{\psi, \psi^c\}$ pair is extracted from the quantum vacuum by the intense gravitational field of a black hole around the horizon which conceptually resembles

the gravitationally sourced thermal radiation of a collapsed body into a black hole, namely the Hawking radiation. So in our model, the outward radiation of a Killing spinor field shall be termed the Hawking-Killing radiation of a black hole. Whereas the evaporation is due to the inward motion of the negative energy reversed Killing spinor field. Also, the tie with Wigner's time reversal may also be checked locally because of the relation between charge conjugation and time reversal; this at a first investigation requires the usage of locally Minkowskian nature of spacetime. Since the resultant photons have different helicities, the produced twistors also should have different helicity states for the consistency of our analog model.

Charge Conjugation Versus Killing reversal:

The similarities and differences between charge conjugation and Killing reversal is given in Table 1. Since the charge conjugation in some sense carries the properties of electromagnetic interactions then by comparison the interaction to which Killing reversal belongs to could be worked out. The behaviour of charge conjugation depends on the dimension

Table 1. Properties of charge conjugation and Killing reversal maps.

Charge Conjugation	Killing Reversal
$(\psi^c)^c = \pm\psi$	$(\psi^c)^c = \psi$
$(\nabla_X \psi)^c = \nabla_X \psi^c$	$(\nabla_X \psi)^c \neq \nabla_X \psi^c$
$(\mathcal{E}_K \psi)^c \neq \mathcal{E}_K \psi^c$	$(\mathcal{E}_K \psi)^c = \mathcal{E}_K \psi^c$

and signature but Killing reversal is independent from them; also while acting on real spinors charge conjugation maps a spinor ψ to $\pm\psi$ on the other hand Killing reversal maps both real and complex spinors in the same manner. Charge conjugation and Killing reversal both preserve the spin degrees of freedom, charge conjugation changes the sign of the electric charge whereas the Killing reversal changes the sign of the inertial mass or more generally energy. This could be seen if one Clifford contracts the Killing spinor equation from the left : $e^a \cdot \nabla_{X_a} \psi = \lambda e^a e_a \cdot \psi$ i.e.

$$D\psi = \lambda n\psi \tag{20}$$

and as a result identifies $n\lambda$ with the inertial mass m . From this point of view the Killing reversal map may be associated to pure gravitational interactions. Here \mathcal{E}_K is the spinorial Lie derivative with respect to a Killing vector field and it sends a Killing spinor to a Killing spinor so it commutes with the Killing reversal map. Similarly the first order operator with respect to a odd KY form $L_{(\omega)}$ is a symmetry operator for Killing spinor equation hence

should commute with the Killing reversal i.e. $(L_{(\omega)}\psi)^S = L_{(\omega)}\psi^S$. Exact coincidences between these operations could be possible when a gravitational problem is reducible to an electromagnetic one in curved spacetime quantum field theories [34].

Comment on the Relation to Wigner Time Reversal:

Discrete orientation-changing finite diffeomorphisms could be worked out locally in a general curved spacetime, so then the only guide will be the local validity of special relativity at a first approximation. But if one intends to analyze the higher order effects of the diffeomorphism flow in a curved spacetime generated by the Wigner time reversal operation, which is closely related to charge conjugation in a flat spacetime, the past-future asymmetry of the gravitational field would avoid a well-defined analysis. Furthermore, the requirement of global hyperbolicity on spacetime could pose a well-behaved "time" analysis and then its relation to other operations such as charge conjugation and Killing reversal shall be determined. In a at spacetime, covariances of Dirac equation (possibly coupled to a Maxwell field) under the above mentioned finite isometric diffeomorphisms are formed by labelling theclater by a parallel element s of the smooth sections of Clifford group bundle $\cup_{m \in M} \Gamma_m(M, \eta)$ where $\Gamma_m(M, \eta)$ is the Clifford group at the point $m \in M$. In the presence of a preferred inner product for spinor fields, the Clifford group bundle should be restricted to the invariance group bundle of the spinor product [11].

Helicity Considerations:

The analogy made before makes one to anticipate that the resultant twistor _elds should correspond to different helicity states. If we define $\Psi := \Psi^+ + \Psi^-$ and if we restrict ourselves to even dimensional Lorentzian spacetimes for physical reasons then we can represent the Killing reversal operation by the left action of the volume form $z := * 1$ on a Killing spinor or on a twistor induced from a Killing spinor. If $z^2 = 1$ the helicity operators are $\frac{1}{2}(1 \pm z)$ and if $z^2 = -1$ then the operators in this case are $\frac{1}{2}(1 \pm iz)$; note that in even dimensions Clifford algebras are central simple and z Clifford anti-commutes with odd forms. When $z^2 = -1$ the center is algebraically isomorphic to \mathbb{C} and hence the helicity operators are well defined. Lets see that $\psi^S = z \cdot \psi$ under our assumptions: If $\nabla_X \psi = \lambda \tilde{X} \cdot \psi$

then $\nabla_X(z \cdot \psi) = z \cdot \nabla_X \psi = \lambda z \tilde{X} \cdot \psi = -\lambda \tilde{X} z \cdot \psi$, so it is legitimate to identify both. Let us take $z^2 = 1$ then

$$\frac{1}{2}(1 \pm z)\Psi = (1 \pm z)\psi = (\psi \pm z \cdot \psi) = (\psi \pm \psi^S) = \Psi^\pm$$

which ends the proof. It is interesting to note here that when the (order reversing) main anti-automorphism ξ of the Clifford algebra corresponds to the adjoint involution of the spinor inner product then the Hodge dual of a spinor is well de_ ned and it maps a spinor that is an element of a Minimal Left Ideal to a dual spinor i.e. an element of the associated Minimal Right Ideal. This follows from the well known identity for Clifford forms namely the

Clifford-Kahler-Hodge duality ; $* \Phi = \Phi^\xi z$ so

$$\nabla_X * \psi = * \nabla_X \psi = \lambda * (\tilde{X} \cdot \psi) = \lambda (\tilde{X} \cdot \psi)^\xi \cdot z = \lambda \bar{\psi} \cdot \tilde{X}^\xi z = \lambda \bar{\psi} \cdot * \tilde{X}$$

that means $* \psi$ is the associated dual spinor sharing the same Killing number with ψ .

Geometric Identities: The action of the Hessian $\nabla^2(X_a, X_b) = \nabla_{X_a} \nabla_{X_b} - \nabla_{\nabla_{X_a} X_b}$ on a Killing spinor ψ is

$$\nabla^2(X_a, X_b)\psi = \lambda^2 e_{ba} \cdot \psi$$

and

$$\mathbf{R}(X_a, X_b)\psi = \lambda^2(e_{ba} - e_{ab}) \cdot \psi = 2\lambda^2(e_b \wedge e_a) \cdot \psi$$

or it can be rewritten as $\mathbf{R}(X_a, X_b)\psi = 2\lambda^2 e_{ba} \cdot \psi$ when $a \neq b$. The trace of the Hessian is defined by $\nabla^2 := \nabla^2(X_a, X^a)$ so $\nabla^2\psi = n\lambda^2\psi$, hence since by Schrödinger-Weitzenböck-Bochner-Lichnerowicz formula for spinors, the square of the Dirac operator is

$$\begin{aligned} D^2\psi &= \nabla^2\psi - \frac{1}{2}\psi \cdot \mathbf{R}(X_a, X_b)e^{ba} \\ &= \nabla^2\psi - \frac{1}{2}\psi \cdot R_{ab}e^{ba} = \nabla^2\psi - \frac{1}{4}\mathcal{R}\psi. \end{aligned} \tag{21}$$

Finally from the geometric constraint $\mathcal{R} = -4\lambda^2 n(n - 1)$ for the existence of Killing spinors we have

$$D^2\psi = \lambda^2 n^2 \psi \tag{22}$$

A trivial result which could be directly obtained from (20).

B. The bosonic sector

The inhomogeneous Clifford forms constructed from the induced twistors decompose into the bilinears of the generator Killing spinors as

$$\Psi^\pm \bar{\Psi}^\pm = \psi \bar{\psi} + \psi^\varsigma \bar{\psi}^\varsigma \pm \psi^\varsigma \bar{\psi} \pm \psi \bar{\psi}^\varsigma$$

and the primitive set of equations corresponding to each of them, and their principal sets together with their determining Clifford algebraic operators are

I
$$\nabla_{X_a}(\psi \bar{\psi})_p = 2\lambda e^a \wedge (\psi \bar{\psi})_{p-1} \quad \{\hat{\lambda}^\pm_p\} \tag{23}$$

$$\nabla_{X_a}(\psi \bar{\psi})_{p_*} = 2\lambda i_{X_a}(\psi \bar{\psi})_{p_*+1}$$

$$d(\psi \bar{\psi})_p = 0, \quad d^\dagger(\psi \bar{\psi})_p = -2\lambda(n - p + 1)(\psi \bar{\psi})_{p-1} \tag{24}$$

$$d(\psi \bar{\psi})_{p_*} = 2\lambda(p_* + 1)(\psi \bar{\psi})_{p_*+1}, \quad d^\dagger(\psi \bar{\psi})_{p_*} = 0$$

II
$$\nabla_{X_a}(\psi^\varsigma \bar{\psi}^\varsigma)_p = -2\lambda e^a \wedge (\psi^\varsigma \bar{\psi}^\varsigma)_{p-1} \quad \{-\hat{\lambda}^\pm_p\} \tag{25}$$

$$\nabla_{X_a}(\psi^\varsigma \bar{\psi}^\varsigma)_{p_*} = -2\lambda i_{X_a}(\psi^\varsigma \bar{\psi}^\varsigma)_{p_*+1}$$

$$d(\psi^\varsigma \bar{\psi}^\varsigma)_p = 0, \quad d^\dagger(\psi^\varsigma \bar{\psi}^\varsigma)_p = 2\lambda(n - p + 1)(\psi^\varsigma \bar{\psi}^\varsigma)_{p-1} \tag{26}$$

$$d(\psi^s \bar{\psi}^s)_{p_*} = -2\lambda(p_* + 1)(\psi^s \bar{\psi}^s)_{p_*+1}, \quad d^\dagger(\psi^s \bar{\psi}^s)_{p_*} = 0$$

$$\text{III} \quad \begin{aligned} \nabla_{X_a}(\psi^s \bar{\psi})_p &= 2\lambda i_{X_a}(\psi^s \bar{\psi})_{p+1} && \{\hat{\lambda}^{\mp}_p\} \\ \nabla_{X_a}(\psi^s \bar{\psi})_{p_*} &= 2\lambda e^a \wedge (\psi^s \bar{\psi})_{p_*-1}; \end{aligned} \quad (27)$$

$$\begin{aligned} d(\psi^s \bar{\psi})_p &= 2\lambda(p+1)(\psi^s \bar{\psi})_{p+1}, \quad d^\dagger(\psi^s \bar{\psi})_p = 0 \\ d(\psi^s \bar{\psi})_{p_*} &= 0, \quad d^\dagger(\psi^s \bar{\psi})_{p_*} = -2\lambda(n-p_*+1)(\psi^s \bar{\psi})_{p_*-1} \end{aligned} \quad (28)$$

$$\text{IV} \quad \begin{aligned} \nabla_{X_a}(\psi \bar{\psi}^s)_p &= -2\lambda i_{X_a}(\psi \bar{\psi}^s)_{p+1} && \{-\hat{\lambda}^{\mp}_p\} \\ \nabla_{X_a}(\psi \bar{\psi}^s)_{p_*} &= -2\lambda e^a \wedge (\psi \bar{\psi}^s)_{p_*-1}; \end{aligned} \quad (29)$$

$$\begin{aligned} d(\psi \bar{\psi}^s)_p &= -2\lambda(p+1)(\psi \bar{\psi}^s)_{p+1}, \quad d^\dagger(\psi \bar{\psi}^s)_p = 0 \\ d(\psi \bar{\psi}^s)_{p_*} &= 0, \quad d^\dagger(\psi \bar{\psi}^s)_{p_*} = 2\lambda(n-p_*+1)(\psi \bar{\psi}^s)_{p_*-1} \end{aligned} \quad (30)$$

here $\hat{\lambda}^{\pm}_p = (\lambda 1 \pm (-1)^p \lambda^j c \mathcal{J})$ with \mathcal{J} the adjoint involution of the spinor inner product and j_c the induced involution on the real algebra of complex numbers. The first set pairs I and II do have the same Yano type and the last set pairs III and IV either, but first and last have different types. In fact the twistor bilinears satisfy CKY equation [1], but an alternative proof follows from calculating their covariant derivatives and using the Yano properties of the Killing spinor bilinears. Another thing is that, if one tries to obtain the primitive or principal sets associated to twistor bilinears from those of Killing spinor bilinears, then he/she should remember that while the former ones are free from the reality conditions the latter ones are not.

Geometric Identities:

(a) From (14) the Hessian of $(\psi \bar{\psi})_p$ is found to be

$$\nabla^2(X_a, X_b)(\psi \bar{\psi})_p = 4\lambda^2 e_b \wedge i_{X_a}(\psi \bar{\psi})_p = 4\lambda^2 i_{X_b}^\dagger i_{X_a}(\psi \bar{\psi})_p \quad (31)$$

and necessarily

$$\mathbf{R}(X_a, X_b)(\psi \bar{\psi})_p = -4\lambda^2 (i_{X_a}^\dagger i_{X_b} - i_{X_b}^\dagger i_{X_a})(\psi \bar{\psi})_p \quad (32)$$

This together with

$$\nabla^2(\psi \bar{\psi})_p = 4\lambda^2 p(\psi \bar{\psi})_p$$

Gives

$$\begin{aligned}
 (d - d^\dagger)^2(\psi\bar{\psi})_p &= \nabla^2(\psi\bar{\psi})_p - \frac{1}{2}\mathbf{R}(X_a, X_b)(\psi\bar{\psi})_p e^{ab} \\
 &= 4\lambda^2 p(\psi\bar{\psi})_p + 2\lambda^2((i_{X_a}^\dagger i_{X_b} - i_{X_b}^\dagger i_{X_a})(\psi\bar{\psi})_p)e^{ab} \\
 &= 4\lambda^2 p(n - p + 1)(\psi\bar{\psi})_p.
 \end{aligned}
 \tag{33}$$

(b) Similar identities for (15) are as follows:

$$\nabla^2(X_a, X_b)(\psi\bar{\psi})_{p_*} = 4\lambda^2(\eta_{ab} - e_a \wedge i_{X_b})(\psi\bar{\psi})_{p_*}
 \tag{34}$$

so

$$\mathbf{R}(X_a, X_b)(\psi\bar{\psi})_{p_*} = -4\lambda^2(i_{X_a}^\dagger i_{X_b} - i_{X_b}^\dagger i_{X_a})(\psi\bar{\psi})_{p_*}
 \tag{35}$$

Unifying the last one with

$$\nabla^2(\psi\bar{\psi})_{p_*} = 4\lambda^2(n - p_*)(\psi\bar{\psi})_{p_*}$$

leaves us with

$$(d - d^\dagger)^2(\psi\bar{\psi})_{p_*} = 4\lambda^2(p_* + 1)(n - p_*)(\psi\bar{\psi})_{p_*}
 \tag{36}$$

As a result (33) and (36) are homogeneous equations in the form of eigenvalue equations of the Laplace-Beltrami operator, and are coupled in accordance with (16) and (17). Note that when $p_* = p - 1$ then the coupled fields do have the same mass.

4. CONCLUSION

The aim of our programme is to work out the many details for putting Killing spinors into the main elements of mathematical physics. First results were derived in [1], and a road map was given in [22]. This latter work contained some open questions, each of which could well be a problem of its own. The present work fills one of these important gaps theoretically and promises a companion paper for the application of its results; the plan of this second part will be explained below. Before that, we want to make a brief report about the new contributions to the literature by this work.

We first gave a new elegant form for defining KY forms and CCKY forms, that is complementing the esthetic inhomogeneous definitive equation for CKY forms which was given before in [1]; so a trivial but effective way for building up a CKY form out of a KY form and a CCKY form was at hand. This simple consequence made us identify clearly the corresponding first order symmetry operators for massive and massless Dirac equations in

curved spacetimes admitting KY and CCKY forms, removing some ambiguities existent in the literature. By the way we showed that one can set up self-dual or anti self-dual massless fields from the resultant CKY forms, this result is in accordance with the analogies given in [1]. Some of the above mentioned operators anti-commuting with Dirac operator reveal the negative energy massive spinning particles, at first sight seen as unphysical or virtual but

the value of this became apparent later on. Again from previous publications it was known that the generalised Dirac currents of Killing spinors were identified with KY and CCKY forms on the kinematical side; so we asked the possibility for generating twistors from Killing spinors and reached the positive answer by using some original technical details. The most critical physical interpretation was that the

generation of a twistor pair by the Killing spinor pair, was identified with the quantum electrodynamical pair annihilation process and also eventually to the Hawking radiation of a black hole. The characteristic mathematical operation associated to this analogy which is termed the Killing reversal was compared and related to the charge conjugation and time-reversal for completeness. Also the investigations in the fermionic sector uncovered the relation of Killing number to inertial mass which is dimension dependent; this relates the dimensionality of space-time to the concept of inertia in some sense. The analysis of the bosonic part added three more types of primitive and principal sets of equations to the preceding ones. A thing not to pass without mentioning is the translation of the coordinate-free form of the primitive set of equations to the coordinate-wise ones, given in Appendix B. Another idea could be to use the reverse procedure for obtaining Killing spinors from twistors in spacetimes admitting the latter ones by the method of trial and error roughly. We also emphasized the fact that our technique shall be of use in classifying spacetimes admitting twistors or Killing spinors.

The planned companion work called Part II [35] will be dealing with many topics. These include the stress tensors of the specific spinor fields and the stress tensors of their generalised Dirac currents, the continuation and improvement of the quantum field theoretical formulation of physical examples given in [22] and the present work, selection of a black hole spacetime admitting a Killing spinor for clarifying the Hawking-Killing radiation process, fixing the values of the inherent degrees associated to the dimensions of the brane immersions including strings and membranes and then reconsidering the dynamical equations such as DKP and Maxwell-like equations in this context. Last but not least we hope to calculate the Killing spinor existing in the plane-wave spacetime and push the button for using our method again. Some results concerning the solutions of massless Dirac and Rarita-Schwinger field in plane-wave spacetimes is gathered in a recent work [36].

Appendix A: Evaluation of $L_{(\omega)}^\dagger$ and $\widehat{L}_{(\widehat{\omega})}^\dagger$ and a comment on the symmetry algebra

Recall that from (11), $L_{(\omega)} = \omega^a \nabla_{X_a} + \frac{p}{2(p+1)} d\omega$ so from metric compatibility of the connection one can write

$$(L_{(\omega)})^\dagger = (\omega^a)^\dagger \nabla_{X_a} + \frac{p}{2(p+1)} (d\omega)^\dagger = *^{-1} i_{X^a} \omega * \eta \nabla_{X_a} + \frac{p}{2(p+1)} *^{-1} d\omega * \eta$$

and also using the identity $*^{-1} i_X \Phi = (*^{-1} \Phi \wedge \widetilde{X})$

$$\begin{aligned} (L_{(\omega)})^\dagger &= \left(((*^{-1} \omega) \wedge e^a) \nabla_{X_a} + \frac{p}{2(p+1)} d^\dagger \eta *^{-1} \omega \right) * \eta \\ &= \left((e^a \wedge \eta *^{-1} \omega) \nabla_{X_a} + \frac{p}{2(p+1)} d^\dagger \eta *^{-1} \omega \right) * \eta. \end{aligned}$$

Finally after using the identity $*^{-1} \omega = (-1)^{p(n-p)} \varepsilon(\mathbf{g}) * \omega$, so

$$(L_{(\omega)})^\dagger = (-1)^{(p+1)(n-p)} \varepsilon(\mathbf{g}) \left((e^a \wedge * \omega) \nabla_{X_a} + \frac{p}{2(p+1)} d^\dagger * \omega \right) * \eta.$$

Let us define $\widehat{L}_{(*\omega)} := (e^a \wedge * \omega) \nabla_{X_a} + \frac{p}{2(p+1)} d^\dagger * \omega$ then

$$(L_{(\omega)})^\dagger = (-1)^{(p+1)(n-p)} \varepsilon(\mathbf{g}) \widehat{L}_{(*\omega)} * \eta,$$

here $\varepsilon(\mathbf{g}) = \frac{\det \mathbf{g}}{|\det \mathbf{g}|}$ where \mathbf{g} is the chart matrix of the metric tensor. The last relation could also be written as

$$*L_{(\omega)} = \widehat{L}_{(*\omega)} = (e^a \wedge * \omega) \nabla_{X_a} + \frac{p}{2(p+1)} d^\dagger * \omega. \tag{A1}$$

Another thing to be done for completeness is to calculate $*\widehat{L}_{(\widehat{\omega})}$ and with a little algebra it is found that

$$*\widehat{L}_{(\widehat{\omega})} = (-1)^{(n-p+1)} \left((e^a \wedge * \widehat{\omega}) \nabla_{X_a} + \frac{n-p}{2(n-p+1)} d * \widehat{\omega} \right). \tag{A2}$$

Dimension dependent closure of the symmetry algebra is based on the set of odd KY forms and even CCKY forms, then the associated first order L_{KY}^{even} 's and L_{CCKY}^{odd} 's form an algebra under Killing-Yano brackets [25]. In the last reference a detailed account of symmetry analysis could be found, but there are some sign ambiguities arising from minor errors.

Appendix B: The coordinate expressions of the primitive set of equations

This appendix is intended to make clear the understanding of the basic equations of our programme, for the general reader who is familiar with the more common notation based on local components of tensor fields. We only translate the primitive set of equations. Let us define $(\psi\bar{\psi})_p := \Omega_p$ for brevity and work in a local chart with coordinate functions $x = (x^\mu)$. Since our equations are general covariant we can write, for example (16) as:

$$\nabla_{\frac{\partial}{\partial x^\mu}} \Omega_p = 2\lambda dx^\mu \wedge \Omega_{p-1}$$

The local expansions of the form fields are

$$\Omega_p = \frac{1}{p!} (\Omega_p)_{\sigma_1 \sigma_2 \dots \sigma_p} dx^{\sigma_1} \wedge dx^{\sigma_2} \wedge \dots \wedge dx^{\sigma_p}$$

and

$$\Omega_{p-1} = \frac{1}{(p-1)!} (\Omega_{p-1})_{\sigma_1 \sigma_2 \dots \sigma_{p-1}} dx^{\sigma_1} \wedge dx^{\sigma_2} \wedge \dots \wedge dx^{\sigma_{p-1}}.$$

so

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^\mu}} \left((\Omega_p)_{\sigma_1 \sigma_2 \dots \sigma_p} dx^{\sigma_1} \wedge dx^{\sigma_2} \wedge \dots \wedge dx^{\sigma_p} \right) \\ = 2\lambda p (\Omega_{p-1})_{\sigma_1 \sigma_2 \dots \sigma_{p-1}} dx^\mu \wedge dx^{\sigma_1} \wedge dx^{\sigma_2} \wedge \dots \wedge dx^{\sigma_p}. \end{aligned}$$

The left hand side is

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^\mu}} \left((\Omega_p)_{\sigma_1 \sigma_2 \dots \sigma_p} dx^{\sigma_1} \wedge dx^{\sigma_2} \wedge \dots \wedge dx^{\sigma_p} \right) &= \left(\partial_\mu (\Omega_p)_{\sigma_1 \sigma_2 \dots \sigma_p} \right) dx^{\sigma_1} \wedge dx^{\sigma_2} \wedge \dots \wedge dx^{\sigma_p} \\ &\quad + (\Omega_p)_{\sigma_1 \sigma_2 \dots \sigma_p} \nabla_{\frac{\partial}{\partial x^\mu}} (dx^{\sigma_1} \wedge dx^{\sigma_2} \wedge \dots \wedge dx^{\sigma_p}) \\ &= (\Omega_p)_{\sigma_1 \sigma_2 \dots \sigma_p, \mu} dx^{\sigma_1} \wedge dx^{\sigma_2} \wedge \dots \wedge dx^{\sigma_p} \end{aligned}$$

$$\begin{aligned}
&= -\omega^{\sigma_i \kappa}(\partial_\mu)(\Omega_p)_{\sigma_1 \sigma_2 \dots \sigma_p} (dx^{\sigma_1} \wedge \dots \wedge dx^{\sigma_{i-1}} \wedge dx^\kappa \wedge dx^{\sigma_{i+1}} \dots \wedge dx^{\sigma_p}) \\
&= \left((\Omega_p)_{\sigma_1 \sigma_2 \dots \sigma_p, \mu} - \sum_i \Gamma_{\mu \sigma_i}^\kappa (\Omega_p)_{\sigma_1 \dots \sigma_{i-1} \kappa \sigma_{i+1} \dots \sigma_p} \right) dx^{\sigma_1} \wedge \dots \wedge dx^{\sigma_p}.
\end{aligned}$$

Conventionally written componentwise as

$$(\Omega_p)_{\sigma_1 \sigma_2 \dots \sigma_p; \mu} = (\Omega_p)_{\sigma_1 \sigma_2 \dots \sigma_p, \mu} - \sum_i \Gamma_{\mu \sigma_i}^\kappa (\Omega_p)_{\sigma_1 \dots \sigma_{i-1} \kappa \sigma_{i+1} \dots \sigma_p}$$

and the right hand side reads

$$2\lambda p (\Omega_{p-1})_{\sigma_1 \sigma_2 \dots \sigma_{p-1}} dx^\mu \wedge dx^{\sigma_1} \wedge \dots \wedge dx^{\sigma_{p-1}} = 2(-1)^{p-1} \lambda p g_{\mu \sigma_p} (\Omega_{p-1})_{\sigma_1 \dots \sigma_{p-1}} dx^{\sigma_1} \wedge \dots \wedge dx^{\sigma_p}$$

and finally

$$(\Omega_p)_{\sigma_1 \sigma_2 \dots \sigma_p; \mu} = 2(-1)^{p-1} \lambda p g_{\mu \sigma_p} (\Omega_{p-1})_{\sigma_1 \dots \sigma_{p-1}}. \quad (B1)$$

Easily (17) becomes in component notation as

$$(\Omega_{p_*})_{\sigma_1 \sigma_2 \dots \sigma_{p_*}; \mu} = 2(-1)^{p_*} \frac{\lambda}{(p_*+1)} (\Omega_{p_*+1})_{\sigma_1 \dots \sigma_{p_*} \mu}. \quad (B2)$$

REFERENCES

- [1]. Açık Ö. And Ertem Ü., Higher degree Dirac currents of twistor and Killing spinors in supergravity theories, *Class. Quantum Grav.* 32, (2015) 175007; Açık Ö. and Ertem Ü., "Generating dynamical bosons from kinematical fermions", *CQG+*, (19 August 2015).
- [2]. Açık Ö., Field equations from Killing spinors, *J. Math. Phys.* 59, (2018) 023501.
- [3]. Penrose R. and Rindler W., *Spinors and Space-time*, Vol.2, Cambridge Univ. Press, 1987.
- [4]. Benn I. M. and Kress J., Differential forms relating twistors to Dirac fields, in: *Differential Geometry and its Applications*, Proceedings of the 10th International Conference DGA 2007, World Scientific Publishing, Singapore, 2008, pp. 573.
- [5]. Charlton P., *The Geometry of Pure Spinors with Applications*, PhD thesis, University of Newcastle 1997.
- [6]. Baum H., Leitner F., The twistor equation in Lorentzian spin geometry, *Math. Z.* 247 795812.14 (2004).
- [7]. Lischewski A., Towards a Classification of pseudo-Riemannian Geometries Admitting Twistor Spinors, arXiv:1303.7246v2.
- [8]. Kath I., Killing spinors on pseudo-Riemannian manifolds, *Habilit., Humboldt-Universität zu Berlin* (1999).
- [9]. Tucker R. W., *Extended Particles and Exterior Calculus*, Rutherford Laboratory, Chilton-Didcot-Oxon, OX11 0QX, RL-76-022 (1976).
- [10]. Burton D. A., A primer on exterior differential calculus, *Theoret. Appl. Mech.*, Vol. 30, No. 2, 85-162, Belgrade (2003).
- [11]. Benn I. M. and Tucker R. W., *An Introduction to Spinors and Geometry with Applications in Physics*, IOP Publishing Ltd, Bristol, 1987.
- [12]. Tucker R. W., A Clifford calculus for physical field theories, in J. S. R. Chisholm and A. K. Common (Eds.), *Clifford Algebras and Their Applications in Mathematical Physics*, Dordrecht: D. Reidel Publishing Company, 1986.
- [13]. Toretta R., *Relativity and Geometry*, Dover Publications, Inc. New York, 1983.
- [14]. Benn I. M. and Kress J., First-order Dirac symmetry operators", *Class. Quantum Grav.* 21 (2004) 427.

- [15]. Acik O., Ertem U., Onder M. and Vercin A., First-order symmetries of the Dirac equation in a curved background: a unified dynamical symmetry condition, *Class. Quantum Grav.* 26 (2009) 075001.
- [16]. Ertem Ü., Symmetry operators of Killing spinors and superalgebras in AdS5, *J. Math. Phys.* 57, (2016) 042502.
- [17]. Ertem Ü., Twistor spinors and extended conformal superalgebras, arXiv:1605.03361.
- [18]. Hughston L. P., Penrose R., Sommers P. and Walker M., On a quadratic first integral for the charged particle orbits in the charged Kerr solution, *Commun. Math. Phys.* 27 (1972) 303.
- [19]. Kastor D. and Traschen J., Conserved gravitational charges from Yano tensors, *J. High Energy Phys.* JHEP08 (2004) 045.
- [20]. Acik O., Ertem U., Onder M. and Vercin A., Basic gravitational currents and Killing-Yano forms, *Gen. Relativ. Gravit.* 42 (2010) 2543.
- [21]. Açık Ö. and Ertem Ü., Hidden symmetries and Lie algebra structures from geometric and supergravity Killing spinors, *Class. Quantum Grav.* 33, (2016) 165002.
- [22]. Açık Ö., New developments in Killing spinor programme and more motivations for physics, arXiv:1611.04424v2.
- [23]. Krtous P., Kubiznak D., Page D. N. and Frolov V. P., Killing-Yano tensors, rank-2 Killing tensors, and conserved quantities in higher dimensions, *J. High Energy Phys.* JHEP02 (2007) 004.
- [24]. Trautmann A., Complex structures in physics, arXiv:math-ph/9809022v1.
- [25]. Cariglia M., Krtous P. and Kubiznak D., Commuting symmetry operators of the Dirac equation, Killing-Yano and Schouten-Nijenhuis brackets, *Phys. Rev. D* 84, (2011) 024004.
- [26]. Geroch R. and Traschen J., Strings and other distributional sources in general relativity, *Phys. Rev. D* 36, (1987) 1017.
- [27]. Stachel J., Thickening the string I, The string perfect dust, *Phys. Rev. D* 21, (1980) 2171.
- [28]. Stachel J., Thickening the string II, The null-string dust, *Phys. Rev. D* 21, (1980) 2182.
- [29]. Tucker R. W., Motion of Membranes in Spacetime, Conference on Mathematical Relativity, Centre for Mathematics and its Applications, Mathematical Sciences Institute, The Australian National University, Canberra AUS, (1989) 238-243.
- [30]. Hartley D., Tucker R. W., Tuckey P. A. and Dray T., Tensor Distributions on Signature-changing Space-times", *Gen. Relativ. Gravit.*, 32, 3 (2000).
- [31]. Dirac P. A. M., An extensible model of the electron, *Proc. Roy. Soc. of London A*, 268, (1962) 57-67.
- [32]. Önder M. and Tucker R. W., Membrane Interactions and Total Mean Curvature, *Phys. Lett. B* 202(4), (1988) 501-504.
- [33]. Fujii Y. and Yamagishi K., Killing spinors on spheres and hyperbolic manifolds, *J. Math. Phys.* 27 (4) (1986).
- [34]. Fulling S. A., Aspects of Quantum Field Theory in Curved Space-Time, London Mathematical Society Student Texts 17 (1989).
- [35]. Açık Ö., On a relation between twistors and Killing spinors II, (in preparation).
- [36]. Açık Ö., Spin 1/2 and spin 3/2 field solutions in plane wave spacetimes, *Gen. Relativ. Gravit.*, 50 33 (2018).
- [37]. Of course in almost all geometric theories of gravity metric is the fundamental field, especially in General Relativity.
- [38]. Although we have used the term pseudo-Riemannian in the abstract and in the introduction, we go against this usage with the same lines of thought as R. Toretti [13], and instead we prefer to use Riemannian spacetime generally and especially Lorentzian spacetime for manifolds endowed with one time-like direction in a local g-frame.
- [39]. We use the term Yano form to indicate a KY form or a CCKY form, then the term dual Yano form indicates a CCKY form or a KY form respectively.
- [40]. Confusion with the Riemann curvature R tensor should be avoided.
- [41]. The properties associated to these projectors are given at the Appendix A of [1].