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Some Congruences for Sums Involving Harmonic Numbers

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Abstract. In this paper, we establish some congruences involving sums with harmonic numbers and the terms of second-order linear sequences.

Keywords: Congruences, harmonic numbers, generalized second order linear sequences.

Harmonik sayıları içeren toplamlar için bazı denklikler

Özet. Bu makalede harmonik sayıları ve ikinci mertebeden lineer dizilerin terimlerini içeren toplamlar hakkında bazı denklikler gösterilmiştir.

Anahtar Kelimeler: Denklikler, harmonik sayılar, genelleştirilmiş ikinci mertebeden lineer diziler.

1. INTRODUCTION

Define the generalized second order linear sequence $\{W_n\}$ by

$$W_n = rW_{n-1} + W_{n-2},$$

with $W_0 = a, W_1 = b$ for all integers n. Since $\Delta = r^2 + 4 \neq 0$, the roots α and β of the equation $x^2 - rx - 1 = 0$ are distinct.
Also define the sequence $\{X_n\}$ via the terms of sequence $\{W_n\}$ as $X_n = W_{n+1} + W_{n-1}$.
The Binet formulae for the sequences $\{W_n\}$ and $\{X_n\}$ are

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \text{ and } X_n = A\alpha^n + B\beta^n,$$

respectively, where $A = b - a\beta$ and $B = b - a\alpha$. For $a = 0,\ b = 1$, we denote $W_n = U_n$ and so $X_n = V_n$, respectively. When $r = 1$, $U_n = F_n$ (n th Fibonacci number) and $V_n = L_n$ (n th Lucas number).
The harmonic numbers have interesting applications in many fields of mathematics, such as number theory, combinatorics, analysis and computer science. Harmonic numbers are those rational numbers given by for $n \in \mathbb{N} = \{1,2,\ldots\}$

$$H_0 = 0, \qquad H_n = \sum_{i=1}^{n} \frac{1}{i}.$$

For a prime p and an integer a with $p \nmid a$, we write the Fermat quotient $q_p(a) = (a^{p-1} - 1)/p$. For an odd prime p and an integer a, $\left(\frac{a}{p}\right)$ denotes the Legendre symbol defined by

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$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p|a, \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

In [9], J. Wolstenholme discovered that for any prime $p > 3$,

$$H_{p-1} \equiv 0 \ (mod p^2). \tag{1.1}$$

In [7], Z.W. Sun obtained that for a prime $p > 3$,

$$\sum_{i=1}^{p-1} \frac{H_i}{i2^i} \equiv 0 \ (mod p), \quad \sum_{i=1}^{p-1} H_i^2 \equiv 2p - 2 \ (mod p^2),$$

and for a prime $p > 5$,

$$\sum_{i=1}^{p-1} \frac{H_i^2}{i^2} \equiv 0 \ (mod p).$$

In [8], Z.W. Sun and L.L. Zhao established arithmetic properties of harmonic numbers. For example, for any prime $p > 3$,

$$\sum_{i=1}^{p-1} \frac{H_i}{i2^i} \equiv \frac{7}{24} p B_{p-3} \ (mod p^2),$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers.
In [1], A. Granville showed that for any prime $p > 3$,

$$\sum_{i=1}^{p-1} \frac{x^i}{i} \equiv \frac{1 - x^p + (x-1)^p}{p} (mod p), \tag{1.2}$$

where x is variable.
In [2], E. Kılıç et al. derived general formulae for alternating analogues of Melham's sums given by

$$\sum_{i=1}^{n} (-1)^i F_{2i+\delta}^{2m+\varepsilon} \quad and \quad \sum_{i=1}^{n} (-1)^i L_{2i+\delta}^{2m+\varepsilon},$$

where $\delta, \varepsilon \in \{0,1\}$.
In [3], E. Kılıç et al. gave various nonalternating sums, alternating sums, and sums that alternate according to $(-1)^{\binom{n+1}{2}}$ involving the generalized Fibonacci and Lucas numbers. For example, for even m,

$$\sum_{n=i}^{j} U_{mn+c} W_{mn+d} = \frac{U_{m(j-i+1)} X_{m(j+i)+c+d}}{\Delta U_m} - \frac{(-1)^c (j-i+1) X_{d-c}}{\Delta},$$

and for odd m and the same parities i and j,

$$\sum_{n=i}^{j} (-1)^n U_{mn+c} W_{mn+d} = \frac{(-1)^j V_{m(j-i+1)} X_{m(j+i)+c+d}}{\Delta V_m} - \frac{(-1)^c \left((-1)^i + (-1)^j\right) X_{d-c}}{2\Delta},$$

where c, d and m are fix integers and $j > i$.
In [4], S. Koparal and N. Ömür showed that for $x \in \mathbb{Z}_p$,

$$\sum_{i=0}^{(p-3)/2} \frac{x^i}{2i+1} \equiv \frac{1}{2p\sqrt{x}}\left(\left(\sqrt{x}+1\right)^p + \left(\sqrt{x}-1\right)^p - 2\left(\sqrt{x}\right)^p\right) \ (modp), \qquad (1.3)$$

where $\mathbb{Z}_p$ is the set of those rational numbers whose denominator is not divisible by p.
The authors gave new congruences involving central binomial coefficients, and harmonic and Fibonacci numbers, using same combinatorial identities. For example, for an odd prime p,

$$\sum_{i=1}^{(p-1)/2} (-1)^i \binom{2i}{i} H_{i-1} \equiv \frac{2^p}{p}\left(2F_{p+1} - 5^{(p-1)/2} - 1\right) (modp),$$

and for $\left(\frac{5}{p}\right) = 1$,

$$\sum_{i=1}^{(p-1)/2} \binom{2i}{i} \frac{H_{i-1}F_i}{(-4)^i} \equiv \frac{1}{p}\left(F_{2p+1} - F_{p+2}\right) - \frac{2^p}{p} F_{p-1} \ (modp),$$

where the Legendre symbol $\left(\frac{\cdot}{p}\right)$.
In [6], N. Ömür and S. Koparal gave some congruences. For example, for $\left(\frac{D}{p}\right) = 1$,

$$DV_k^3 \sum_{i=1}^{(p-1)/2} i^2 V_{4ki} H_i \equiv U_{2kp}\left(3 - q_p(2)\left(\frac{V_{4k}}{2} + 3\right)\right) - U_{2k}\left(\frac{V_{2k(p-1)}}{2} + 1\right)$$

$$+ \frac{(-1)^k V_{2k}}{2}\left(V_k^p U_{kp} - D^{(p-1)/2} V_{kp}\right) (modp),$$

where $D = V_k^2 + 4(-1)^{k+1}$.

2. ON CONGRUENCES WITH HARMONIC NUMBERS AND THE TERMS OF SEQUENCES $\{W_n\}$ AND $\{X_n\}$

In this section, we consider some congruences related to the terms of sequences $\{W_n\}$ and $\{X_n\}$. For this, firstly, we will give some auxiliary Lemmas:

Lemma 2.1: Let p be an odd prime. Then

$$\sum_{i=0}^{(p-3)/2} \frac{V_{2i+1}W_{2i}}{2i+1} \equiv (ar-b)q_p(2) + \frac{1}{2p}\left(\Delta^{(p-1)/2} X_{p-1} + r^p W_{p-1} - 2W_{2p-1}\right) (modp),$$

and

$$\sum_{i=1}^{(p-1)/2} \frac{U_{2i}X_{2i-1}}{i} \equiv -2(ar-b)q_p(2) + \frac{1}{p}\left(2b - 2ar - \Delta^{(p-1)/2} X_{p-1} + r^p W_{p-1}\right) (modp).$$

Proof. From Binet formulae of the sequences $\{W_n\}$ and $\{X_n\}$, we write

$$\sum_{i=0}^{(p-3)/2}\frac{V_{2i+1}W_{2i}}{2i+1}=\sum_{i=0}^{(p-3)/2}\frac{1}{2i+1}\frac{(b-a\beta)\alpha^{2i}-(b-a\alpha)\beta^{2i}}{\alpha-\beta}\left(\alpha^{2i+1}+\beta^{2i+1}\right)$$

$$=\frac{1}{\alpha-\beta}\sum_{i=0}^{(p-3)/2}\frac{1}{2i+1}\left(b\alpha^{4i+1}+b\beta+a\alpha^{4i}-a\beta^{2}-b\alpha-b\beta^{4i+1}+a\alpha^{2}-a\beta^{4i}\right)$$

$$=(aU_2-bU_1)\sum_{i=0}^{(p-3)/2}\frac{1}{2i+1}+\frac{1}{\alpha-\beta}\left\{b\left(\alpha\sum_{i=0}^{(p-3)/2}\frac{\alpha^{4i}}{2i+1}-\beta\sum_{i=0}^{(p-3)/2}\frac{\beta^{4i}}{2i+1}\right)\right\}$$

$$+\frac{1}{\alpha-\beta}\left\{a\left(\sum_{i=0}^{(p-3)/2}\frac{\alpha^{4i}}{2i+1}-\sum_{i=0}^{(p-3)/2}\frac{\beta^{4i}}{2i+1}\right)\right\}.$$

By (1.3) and $\alpha\beta=-1$, we have

$$\sum_{i=0}^{(p-3)/2}\frac{V_{2i+1}W_{2i}}{2i+1}\equiv(aU_2-bU_1)\frac{2^{p-1}-1}{p}+\frac{1}{\alpha-\beta}\left\{\frac{b-a\beta}{2p\alpha}\left((\alpha^2+1)^p+(\alpha^2-1)^p-2\alpha^{2p}\right)\right.$$

$$\left.-\frac{b-a\alpha}{2p\beta}\left((\beta^2+1)^p+(\beta^2-1)^p-2\beta^{2p}\right)\right\}(mod p)$$

and thus

$$\sum_{i=0}^{(p-3)/2}\frac{V_{2i+1}W_{2i}}{2i+1}\equiv(aU_2-bU_1)q_p(2)+\frac{1}{\alpha-\beta}\left\{\frac{b-a\beta}{2p\alpha}\left(\alpha^p\Delta^{p/2}+\alpha^p r^p-2\alpha^{2p}\right)\right.$$

$$\left.-\frac{b-a\alpha}{2p\beta}\left(-\beta^p\Delta^{p/2}+\beta^p r^p-2\beta^{2p}\right)\right\}$$

$$=(ar-b)q_p(2)+\frac{1}{2p}\left(\Delta^{(p-1)/2}X_{p-1}+r^pW_{p-1}-2W_{2p-1}\right)(mod p),$$

as claimed. Similarly, the other congruence is obtained. ■

Lemma 2.2: Let p be an odd prime. Then

$$\sum_{i=1}^{(p-1)/2}\frac{W_{4i-2}}{i}\equiv\frac{2}{p}(a-br+ar^2)+\frac{r^p}{p}W_{p-2}-\frac{\Delta^{(p-1)/2}}{p}X_{p-2}\ (mod p),$$

and

$$\sum_{i=1}^{(p-1)/2}\frac{X_{4i-2}}{i}\equiv\frac{2}{p}(2b-3ar+br^2-ar^3)+\frac{r^p}{p}X_{p-2}-\frac{\Delta^{(p+1)/2}}{p}W_{p-2}\ (mod p).$$

Proof: In [4], the authors gave that for $x\in\mathbb{Z}_p$,

$$\sum_{i=1}^{(p-1)/2} \frac{x^i}{i} \equiv \frac{2}{p} - \frac{\left(\sqrt{x}+1\right)^p - \left(\sqrt{x}-1\right)^p}{p} \ (modp).$$

Using this congruence, the proof is similar to the proof of Lemma 2.1. ■

Theorem 2.1: Let p be an odd prime. Then

$$\sum_{i=1}^{p-1} H_i W_{2i} \equiv \frac{1}{pr}\left(ar - b + W_{2p-1} - r^p W_{p-1}\right) (modp),$$

and

$$\sum_{i=1}^{p-1} H_i X_{2i} \equiv -\frac{1}{pr}\left(ar^2 - br + 2a - X_{2p-1} + r^p X_{p-1}\right) (modp).$$

Proof: Observed that

$$\sum_{i=1}^{p-1} H_i W_{2i} = \sum_{j=1}^{p-1} \frac{1}{j} \sum_{i=j}^{p-1} W_{2i} = \sum_{j=1}^{p-1} \frac{1}{j}\left(\sum_{i=0}^{p-1} W_{2i} - \sum_{i=0}^{j-1} W_{2i}\right).$$

With the help of the congruence (1.1), we have

$$\sum_{i=1}^{p-1} H_i W_{2i} \equiv -\sum_{j=1}^{p-1} \frac{1}{j} \sum_{i=0}^{j-1} W_{2i}$$

$$= -\left\{W_0 + \frac{1}{2}\sum_{i=0}^{1} W_{2i} + \frac{1}{3}\sum_{i=0}^{2} W_{2i} + \cdots + \frac{1}{p-1}\sum_{i=0}^{p-2} W_{2i}\right\} (modp^2).$$

From the equality [5]

$$\sum_{n=i}^{j} W_{2n} = \begin{cases} \frac{1}{r} V_{j-i+1} W_{i+j} & \text{if } j-i \equiv 0(mod2), \\ \frac{1}{r} U_{j-i+1} X_{i+j} & \text{if } j-i \equiv 1(mod2), \end{cases}$$

we get

$$\sum_{i=1}^{p-1} H_i W_{2i} \equiv -\left\{W_0 + \frac{1}{2r} U_2 X_1 + \frac{1}{3r} V_3 W_2 + \cdots + \frac{1}{(p-2)r} V_{p-2} W_{p-3} + \frac{1}{(p-1)r} U_{p-1} X_{p-2}\right\}$$

$$= -\left\{\frac{1}{r}\sum_{i=1}^{(p-1)/2} \frac{U_{2i} X_{2i-1}}{2i} + \frac{1}{r}\sum_{i=0}^{(p-3)/2} \frac{V_{2i+1} W_{2i}}{2i+1}\right\} (modp^2).$$

By Lemma 2.1, the desired result is obtained. Similarly, for $j - i \equiv 1(mod2)$, considering the equality[5]

$$\sum_{n=i}^{j} X_{2n} = \frac{\Delta}{2} U_{j-i+1} W_{i+j},$$

the other claim is obtained. Thus we have completed the proof of Theorem 2.1. ■

For example, when $a = 0,\ b = r = 1$ in Theorem 2.1, we have

$$\sum_{i=1}^{p-1} H_i F_{2i} \equiv \frac{1}{p}\left(F_{2p-1} - F_{p-1} - 1\right) \ (modp),$$

and

$$\sum_{i=1}^{p-1} H_i L_{2i} \equiv \frac{1}{p}\left(L_{2p-1} - L_{p-1} + 1\right) \ (modp).$$

Theorem 2.2: Let p be an odd prime. Then

$$\sum_{i=1}^{p-1} (-1)^i H_i W_{2i} \equiv \frac{1}{\Delta p}\left(ar^2 + 2a - br + X_{2p-1} - \Delta^{(p+1)/2} W_{p-1}\right) \ (modp),$$

and

$$\sum_{i=1}^{p-1} (-1)^i H_i X_{2i} \equiv \frac{1}{p}\left(b - ar + W_{2p-1} - \Delta^{(p-1)/2} X_{p-1}\right) \ (modp).$$

Proof: Consider that

$$\sum_{i=1}^{p-1} (-1)^i H_i W_{2i} = \sum_{i=1}^{p-1} (-1)^i W_{2i} \sum_{j=1}^{i} \frac{1}{j} = \sum_{j=1}^{p-1} \frac{1}{j} \sum_{i=j}^{p-1} (-1)^i W_{2i}$$

$$= \sum_{j=1}^{p-1} \frac{1}{j}\left(\sum_{i=0}^{p-1} (-1)^i W_{2i} - \sum_{i=0}^{j-1} (-1)^i W_{2i}\right).$$

By (1.1), we write

$$\sum_{i=1}^{p-1} (-1)^i H_i W_{2i} \equiv -\sum_{j=1}^{p-1} \frac{1}{j} \sum_{i=0}^{j-1} (-1)^i W_{2i}$$

$$= -W_0 - \frac{1}{2}\sum_{i=0}^{1} (-1)^i W_{2i} - \frac{1}{3}\sum_{i=0}^{2} (-1)^i W_{2i} - \cdots - \frac{1}{p-1}\sum_{i=0}^{p-2} (-1)^i W_{2i} \ (modp^2).$$

Since the equality [5]

$$\sum_{n=i}^{j} (-1)^n W_{2n} = (-1)^j U_{j-i+1} W_{i+j},$$

we have

$$\sum_{i=1}^{p-1}(-1)^i H_i W_{2i} \equiv -U_1W_0 + \frac{1}{2}U_2W_1 - \frac{1}{3}U_3W_2 + \cdots + \frac{1}{p-1}U_{p-1}W_{p-2}$$
$$= \sum_{i=1}^{p-1}\frac{(-1)^i}{i}U_iW_{i-1}\ (modp^2).$$

With the help of the equality $W_n = aU_{n-1} + bU_n$ and the Binet formula of the sequence $\{U_n\}$, we have

$$\sum_{i=1}^{p-1}(-1)^i H_i W_{2i} \equiv \sum_{i=1}^{p-1}\frac{(-1)^i}{i}U_i(aU_{i-2} + bU_{i-1})$$

$$= a\sum_{i=1}^{p-1}\frac{(-1)^i}{i}U_iU_{i-2} + b\sum_{i=1}^{p-1}\frac{(-1)^i}{i}U_iU_{i-1}$$

$$= \left(\frac{b}{\Delta\alpha} + \frac{a}{\Delta\alpha^2}\right)\sum_{i=1}^{p-1}\frac{(-\alpha^2)^i}{i} + \left(\frac{b}{\Delta\beta} + \frac{a}{\Delta\beta^2}\right)\sum_{i=1}^{p-1}\frac{(-\beta^2)^i}{i} + \frac{b}{\Delta}rH_{p-1} - \frac{a}{\Delta}(r^2+2)H_{p-1}\ (modp^2).$$

Considering (1.1) and (1.2), the proof is complete. Similarly, the other congruence is given. ■

Similar to the proof methods of the aboving Theorems, we have following results without proof.

Theorem 2.3: Let p be an odd prime. For $p \nmid r$,

$$\sum_{i=1}^{p-1} iH_iW_{2i} \equiv \frac{1}{r^2}\left(a - W_{2(p-1)} - \frac{1}{p}\left(W_{2p} - r^pW_p - a\right)\right)\ (modp),$$

and

$$\sum_{i=1}^{p-1} iH_iX_{2i} \equiv \frac{1}{r^2}\left(2b - ar - X_{2(p-1)} - \frac{1}{p}\left(X_{2p} - r^pX_p + ar - 2b\right)\right)\ (modp).$$

Theorem 2.4: Let p be an odd prime. Then

$$\sum_{i=1}^{(p-1)/2} H_iW_{4i} \equiv -\frac{2}{\Delta r}q_p(2)X_{2p}$$

$$-\frac{1}{\Delta pr}\left(2(2b - 3ar + br^2 - ar^3) + r^pX_{p-2} - \Delta^{(p+1)/2}W_{p-2}\right)\ (modp),$$

and

$$\sum_{i=1}^{(p-1)/2} H_iX_{4i} \equiv -\frac{2}{r}q_p(2)W_{2p} - \frac{1}{pr}\left(2(a - br + ar^2) + r^pW_{p-2} - \Delta^{(p-1)/2}X_{p-2}\right)\ (modp).$$

For example, when $a = 0, b = r = 1$ in Theorem 2.4, we obtain

$$\sum_{i=1}^{(p-1)/2} H_i F_{4i} \equiv -\frac{2}{5} q_p(2) L_{2p} - \frac{1}{5p}\left(L_{p-2} - 5^{(p+1)/2} F_{p-2} + 6\right) (mod p),$$

and

$$\sum_{i=1}^{(p-1)/2} H_i L_{4i} \equiv -2\, q_p(2) F_{2p} - \frac{1}{p}\left(F_{p-2} - 5^{(p-1)/2} L_{p-2} - 2\right) (mod p).$$

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