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A Computational Method for the Time-Fractional Navier-Stokes Equation

İnci ÇİLİNGİR SÜNGÜ[1*], Hüseyin DEMİR[2]

[1]*Ondokuz Mayıs University, Education Faculty, Department of Mathematics and Science Education, Samsun TURKEY*

[2]*Ondokuz Mayıs University, Arts and Science Faculty, Department of Mathematics, Samsun TURKEY*

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Abstract. In this study, Navier-Stokes equations with fractional derivate are solved according to time variable. To solve these equations, hybrid generalized differential transformation and finite difference methods are used in various subdomains. The aim of this hybridization is to combine the stability of the difference method and simplicity of the differential transformation method in use. It has been observed that the computational intensity of complex calculations is reduced and also discontinuity due to initial conditions can be overcome when the size increased in the study. The convergence of the time-dependent series solution is ensured by multi-time-stepping method. This study has shown that the hybridization method is effective, reliable and easy to apply for solving such type of equations.

2010 Mathematics Subject Classification: 35Q30, 35R11, 65M06, 65N55.

Keywords: Hybrid Differential Transform/Finite Difference Method, Time-Fractional Navier-Stokes Equations, Numerical solution.

Zaman Değişkeninde Kesirli Türev İçeren Navier-Stokes Denklemlerinin Sayısal Çözümü

Özet. Bu çalışmada zaman değişkenine göre kesirli türev içeren Navier-Stokes denklemleri çözülmüştür. Denklemlerin çözümünde genelleştirilmiş diferansiyel dönüşüm ve sonlu fark metotları beraber farklı alt aralıklara bölünerek çok adımlı olarak kullanılmıştır. Bu melezleme ile sonlu fark metodunun kararlılık özelliği ve diferansiyel dönüşüm metodunun uygulama kolaylığı özelliklerinin birleştirilmesi amaçlanmıştır. Ele alınan örneklerde karmaşık hesaplamaların getirdiği işlem yükünün azaldığı ve çok boyutlu problemlerde ise başlangıç koşulu nedeniyle oluşan süreksizliğin aşılabildiği görülmüştür. Zamana bağlı seri çözümünün yakınsaklığı ise çok zaman adımlı metot kullanılarak sağlanmıştır. Yapılan çalışma melezleme metodunun bu tür denklemlerin çözümünde etkili, güvenilir ve uygulanması kolay olduğunu göstermiştir.

Anahtar Kelimeler: Diferansiyel Dönüşüm/Sonlu Fark Metodu, Kesirli Navier-Stokes Denklemleri, Sayısal Çözüm.

1. INTRODUCTION

In recent years, the problems which contain fractional order derivatives have been modeled in many areas of science such as fluid mechanics, chemistry, control theory, psychology [1]. The first research on approximations of fractional differential equations was made by Padovan [2]. After that the fractional differential equations has become popular among scientist, additionally Adomian Decomposition Method [3-9], Variational Iteration Method [7-10], Homotopy Perturbation Method [11-12], Fractional Difference

* Corresponding author. *Email address: incicilingir@gmail.com*

Method [13], Fractional Adams-Bashforth-Moulton Method [14] and Generalized Differential Transform Method [15] have been used to get numeric and analytic solutions. Using most of these methods, the fractional differential equation has turned into recurrence relation or system of equations. The others have used to obtain a convergent series solution. Most of the flow motion problems, Newtonian or non-Newtonian flow problems in a tube and also many problems in science bring out the fractional derivative in their differential equations. Navier-Stokes equation which has β order time derivation $0<\beta\leq 1$ was solved by El Shahed and Salem by using Laplace transform, Fourier sine transform and finite Hankel transforms [16]. Momani and Odibat, in 2006, used Adomian Decompositon method to get solutions for time-fractional Navier-Stokes equations [4].

The time-fractional Navier-Stokes equation and continuity equation can be written as

$$\frac{\partial^{\beta}\underline{\varphi}}{\partial t^{\beta}}+\left(\underline{\varphi}\nabla\right)\underline{\varphi}=-\frac{1}{\rho}\nabla P+v\nabla^{2}\underline{\varphi}, \tag{1}$$

$$\nabla.\underline{\varphi}=0. \tag{2}$$

Here t is the time, $\underline{\varphi}$ is the velocity vector, P is the pressure, v is the kinematic viscosity and ρ is the density.

Definition1.1. Suppose that n is the smallest integer greater than β and $\beta>0$. Caputo derivative is defined by

$$D_t^{\beta}\varphi(x,t)=\frac{\partial^{\beta}\varphi(x,t)}{\partial t^{\beta}}=\begin{cases}\dfrac{1}{\Gamma(n-\beta)}\displaystyle\int_0^t(t-\tau)^{n-\beta-1}\frac{\partial^{n}\varphi(x,\tau)}{\partial\tau^{n}}d\tau & , \quad n-1<\beta<n\\ \dfrac{\partial^{n}\varphi(x,t)}{\partial t^{n}} & , \quad \beta=n\in\mathbb{N}\end{cases} \tag{3}$$

[17].

2. COUPLING DIFFERENTIAL TRANSFORM AND FINITE DIFFERENCE METHOD

Differential transform method (DTM) and Finite difference method (FDM) are two of well-known methods to solve differential equations. FDM is used to solve differential equations for decades since its results satisfy consistency and stability conditions with high accuracy. DTM is a kind of newly method when compared to FDM. DTM establishes a series form solution of differential equation that is based on Taylor series method. Zhou [18] is the first who mentioned about DTM to get solutions of differential equations. The DTM is reduced both ordinary and partial differential equation to recurrence relation in order to calculate necessary derivatives symbolically. Consequently, DTM provides truncated series form solution or convergent solution for differential equation considered [19-21]. Generalized differential transform method (GDTM) is the method to get solution of fractional differential equations based on DTM [21-29]. On the other hand, it is difficult to use DTM or GDTM for solving high dimensional problems especially which contain discontinuity on initial or boundary conditions. To keep away from this difficulty, we integrate GDTM and FDM. In [30-33], hybrid DTM and FDM was used to obtain solutions in different problems such as the transient thermal stress problem, Burger equations, dynamic response problem and heat conduction problem.

Definition 2.1. Assume that $\varphi(x,t)$ depends on two variables and also $\varphi(x,t)$ is denoted as separable into variables, i.e. $\varphi(x,t)=p(x)q(t)$. Then, $\varphi(x,t)$ can be represented as

$$\varphi(x,t)=\sum_{i=0}^{\infty}P(i)x^{i}\sum_{j=0}^{\infty}Q(j)t^{j}=\sum_{k=0}^{\infty}\Phi_{k}(x)t^{k}. \tag{4}$$

Here, $\Phi_k(x)$ symbolizes the differential transform of $\varphi(x,t)$.

Definition 2.2. Let the function $\varphi(x,t)$ is continuous and differentiable with respect to time and space variables. Then the differential transform of fractional time derivative of $\varphi(x,t)$ is defined as

$$\Phi_{k}^{\beta}(x)=\frac{1}{\Gamma(\beta k+1)}\left[\frac{\partial^{k\beta}}{\partial t^{k\beta}}\varphi(x,t)\right]_{t=0}, \quad for \quad 0<\beta\leq 1\ . \tag{5}$$

Definition 2.3. The inverse of differential transform is defined by

$$\varphi(x,t)=\sum_{k=0}^{\infty}\Phi_{k}^{\beta}(x)t^{k\beta}. \tag{6}$$

The combination of equation (5) and (6) permits us to have

$$\varphi(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\beta k+1)}\left[\frac{\partial^{k\beta}}{\partial t^{k\beta}}\varphi(x,t)\right]_{t=0}t^{k\beta}. \tag{7}$$

In this study, the hybrid GDTM/FDM process is explained with the time-fractional nonlinear partial differential equation:

$$L(\varphi(x,t))+R(\varphi(x,t))+N(\varphi(x,t))=q(x,t) \tag{8}$$

with initial condition

$$\varphi(x,0)=p(x) \tag{9}$$

where $L=\frac{\partial^{\beta}}{\partial t^{\beta}}$, $0<\beta\leq 1$, β is the order of fractional derivative in the Caputo definition. R and N are linear and nonlinear terms with partial derivatives, respectively. Also $q(x,t)$ is a nonhomogeneous function in the equation. We apply firstly GDTM to discretize of fractional order derivative and secondly FDM to discretize of spatial variable. The region in x direction is split into subdomains which has equal width h. Grid points are denoted by $x_i=ih,\ i=0,1,2,\ldots,M$.

After discretization, this equation reduces to following iteration formula

$$\frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)}\Phi_{k+1}^{\beta}(i)=Q_{k}(i)-R_{1}\left[\Phi_{k}^{\beta}(i)\right]-N_{1}\left[\Phi_{k}^{\beta}(i)\right]. \tag{10}$$

From the initial condition it follows that

$$\Phi_0^{\beta}(i) = P(x_i). \tag{11}$$

Here, $Q_k(i)$ is the transformed function of $q(x,t)$. R_1 and N_1 are the operators in FDM.

In the current study we use hybrid method of GDTM and FDM to solve the time-fractional Navier-Stokes equation. The Navier-Stokes equations represent the flow motion in the cavity and in cylindrical coordinate and these equations can be given by

$$\frac{\partial \varphi}{\partial t} = -\frac{\partial P}{\rho \partial z} + v\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right), \tag{12}$$

$$\varphi(r,0) = f(r). \tag{13}$$

When time derivative term is written as fractional derivative model, the governing equation of motion (12) has the form

$$\frac{\partial^{\beta} \varphi}{\partial t^{\beta}} = p + v\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right), \qquad 0 < \beta \le 1, \tag{14}$$

where $p = -\dfrac{\partial P}{\rho \partial z}$.

After taking the differential transform of time derivative term in equation (14), the time evolutionary equation is transformed to an elliptic equation.

After using GDTM and FDM, the following recurrence relation is achieved;

$$\frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)} \Phi_{k+1}^{\beta}(i) = p\delta(k) + v\left(D^2 \Phi_k^{\beta}(i) + \frac{1}{r\Delta r} D\Phi_k^{\beta}(i)\right). \tag{15}$$

Here D and D^2 are the finite difference operators for first and second order derivatives, respectively. δ is the Kronecker Delta function. Therefore, system of algebraic equation evaluated from different values of i is constructed and this diagonally dominant system is solved by the Gauss-Seidel method which is convergent since coefficient matrix remains nonsingular.

From the initial condition, we have

$$\Phi_0^{\beta}(i) = f(i\Delta r). \tag{16}$$

To provide the approximate solution of $\varphi_{\beta}(x_i,t)$, the differential transform coefficients $\Phi_k^{\beta}(i)$ values calculated from equation (15) with substituting the initial condition (16) into (15). Consequently, we have approximate series solution of order m

$$\varphi_{\beta}(x_i,t) = \sum_{k=0}^{m} \Phi_k^{\beta}(i) t^k. \tag{17}$$

Equation (17) is a truncated series solution getting from applying of hybrid DTM and FDM to time fractional Navier-Stokes equation. It is important to show that this truncated series solution is convergent solution. To gain convergent solution three key parameters are used in the solution procedure. They are the mesh size used in FDM, the time step used in DTM and the order of truncated series. Using small meshes and time steps, simulation results can be developed. But, too many meshes and time steps may cause a divergence in approximation with big rounding error. Also the order of power series (m) can be outnumbered to avoid divergence. If these three parameters are compatible with each other, consistency will be satisfied from Lax equivalence theorem as proved by Jang [24]. When the selection of the values of parameters are satisfied the following condition

$$\Delta t < \left(\frac{\varepsilon}{\Phi_k^\beta(i)} \right)^{1/m+1},$$

the error of the solution procedure will not be exceed the tolerance value ε .

In this study, the total length is divided into 30 or 32 meshes; the time step is selected as 0.0005 or 0.001 and 10 terms are used in power series.

3. APPLICATIONS

The nondimensional form of the time fractional Navier-Stokes equation for unsteady viscous fluid flow in three dimensions is $\frac{\partial\varphi}{\partial t} = p + v\nabla^2\varphi$. It can be rewritten in variables of cylindrical polar coordinates (r, θ, z) by

$$\frac{\partial\varphi}{\partial t} = p + v\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} \right). \tag{18}$$

For simplicity φ is supposed to be independent on z , equation (18) reduces to

$$\frac{\partial\varphi}{\partial t} = p + v\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \right) \tag{19}$$

in two dimensional form.

When the fractional time derivative is used instead of time derivative term, the general equation (19) is written in the following form:

$$\frac{\partial^\beta\varphi}{\partial t^\beta} = p + v\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \right), \quad 0 < \beta \le 1. \tag{20}$$

Example 3. 1.

Consider two dimensional problem in cylindrical coordinate with circular symmetry domain. In this case, we have $\frac{\partial^2\varphi}{\partial\theta^2} = 0$ and equation (20) simplifies to next equation for Newtonian fluid

$$\frac{\partial^\beta \varphi}{\partial t^\beta} = p + v\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right). \tag{21}$$

We can take $v = 1$. Then, equation (21) reduces to

$$\frac{\partial^\beta \varphi}{\partial t^\beta} = p + \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}, \qquad 0 < \beta \le 1 \tag{22}$$

and the initial condition has the form

$$\varphi(r,0) = 1 - r^2. \tag{23}$$

To discretize equation (22) with the initial condition (23) we use hybrid method. Firstly, we apply GDTM with respect to the time variable. Secondly, we use FDM for the spatial variables.

Substituting the generalized differential transformation in equation (21),we have

$$\frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)}\Phi_{k+1}^\beta(r) = p\delta(k) + \frac{\partial^2 \Phi_k^\beta(r)}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi_k^\beta(r)}{\partial r}. \tag{24}$$

Then we apply FDM to derivatives according to variable r,

$$\frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)}\Phi_{k+1}^\beta(i) = p\delta(k) + \frac{\Phi_k^\beta(i+1) - 2\Phi_k^\beta(i) + \Phi_k^\beta(i-1)}{(\Delta r)^2} + \frac{1}{i(\Delta r)}\frac{\Phi_k^\beta(i+1) - \Phi_k^\beta(i-1)}{2(\Delta r)} \tag{25}$$

where Δr is the mesh size in direction r and $r = i(\Delta r)$. For $k = 0$, we get the value of $\Phi_0^\beta(i)$ from the discretization of initial condition (23) as

$$\Phi_0^\beta(i) = 1 - i^2(\Delta r)^2. \tag{26}$$

Substituting $k = 0$ in equation (25), gives that

$$\Phi_1^\beta(i) = (p-4)\frac{1}{\Gamma(\beta+1)}. \tag{27}$$

For $k = 1$, $\Phi_2^\beta(i) = 0$. Substituting this value in equation (25) subsequently,

$$\begin{aligned} &\Phi_3^\beta(i) = 0 \\ &\Phi_4^\beta(i) = 0 \\ &\quad\vdots \end{aligned} \tag{28}$$

Consequently the solution is defined by

$$\begin{aligned}\varphi(r,t) &= \sum_{k=0}^{\infty}\Phi_k^{\beta}(i)t^{\beta k} = \Phi_0^{\beta}(i)+\Phi_1^{\beta}(i)t^{\beta}+\dots \\ &= 1-r^2+\{p-4\}\frac{1}{\Gamma(\beta+1)}\end{aligned} \tag{29}$$

which is the same solution given by [4]. Figure 1a and 1b show the simulation results for the time fractional equation (22) with the initial condition (23) when $p=1;\ \beta=1$ and $\beta=0.5$, respectively.

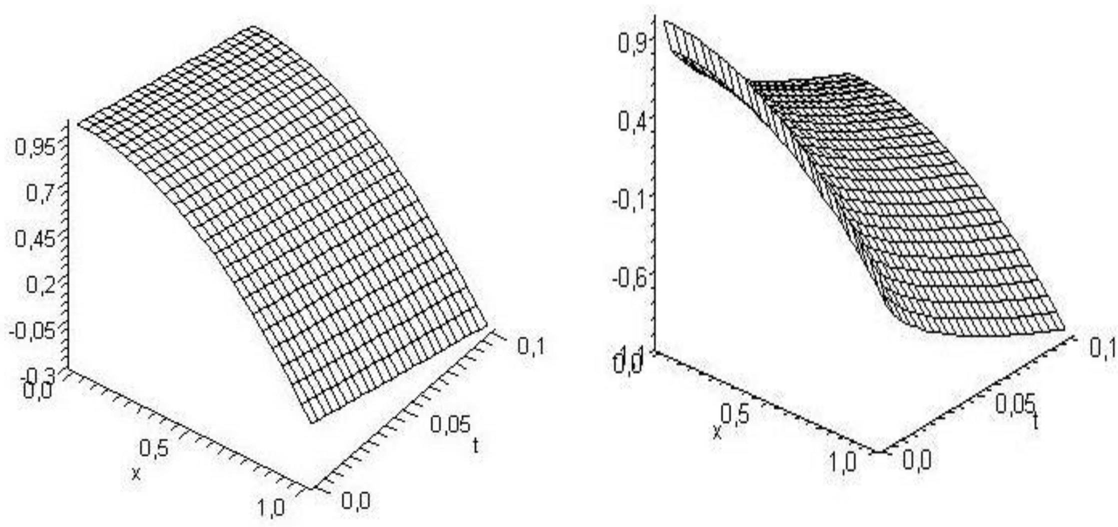


Figure 1. The surfaces show the solution $\varphi(r,t)$ for equation (23) when $p=1$: (a) $\beta=1$; (b) $\beta=0.5$.

Example 3. 2.

Suppose that the function φ is a solution of the equation

$$\frac{\partial^{\beta}\varphi}{\partial t^{\beta}} = \frac{\partial^2\varphi}{\partial r^2}+\frac{2}{r}\frac{\partial\varphi}{\partial r}, \quad 0<r<1\,, \tag{30}$$

and satisfies the initial condition

$$\varphi = 1-\mathrm{r}^2 \ \text{ when } \ \mathrm{t}=0,\ 0\le \mathrm{r}\le 1, \tag{31}$$

and the boundary conditions

$$\frac{\partial\varphi}{\partial r}=0 \ \ at \ \ r=0,\ t>0;\ \varphi=0 \ \ at \ \ r=1,\ t>0\,. \tag{32}$$

For non-zero values of r there is no difficulty in expressing each derivative in terms of standard finite difference approximation, but at $r=0$ we have singularity in the right side of equation (30). This can be dealt with by replacing the polar-coordinate form of $\nabla^2\varphi$ by its Cartesian equivalent [34]. As known clearly, two dimensional problem in Cylindrical coordinates possesses circular symmetry, then $\nabla^2\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r}$.

Let $\frac{\partial\varphi}{\partial r} = 0$ at $r = 0$ then, $\frac{1}{r}\frac{\partial\varphi}{\partial r}$ assumes the indeterminate form $0/0$ at this point if the problem is symmetrical with respect to the origin. Therefore as a result by Maclaurin's expansion of $\varphi'(r)$, so $\lim_{r\to 0}\frac{1}{r}\frac{\partial\varphi}{\partial r} = \lim_{r\to 0}\frac{\partial^2\varphi}{\partial r^2}$ at $r = 0$. Hence the equation $\frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r}$ at $r = 0$ can be replaced by

$$\frac{\partial\varphi}{\partial t} = 3\frac{\partial^2\varphi}{\partial r^2}. \tag{33}$$

After discretization we have the main equation, initial and boundary conditions as follows:

$$\frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)}\Phi_{k+1}^{\beta}(0) = 6\left(\frac{\Phi_k^{\beta}(1) - \Phi_k^{\beta}(0)}{(\Delta r)^2}\right) \quad \text{for } r = 0\,, \tag{34}$$

$$\begin{aligned}\frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)}\Phi_{k+1}^{\beta}(i) = &\frac{\Phi_k^{\beta}(i+1) - 2\Phi_k^{\beta}(i) + \Phi_k^{\beta}(i-1)}{(\Delta r)^2} \\ &+ \frac{2}{i(\Delta r)}\left(\frac{\Phi_k^{\beta}(i+1) - \Phi_k^{\beta}(i-1)}{2(\Delta r)}\right)\end{aligned} \quad \text{for } r \neq 0\,, \tag{35}$$

$$\Phi_0^{\beta}(i) = 1 - i^2(\Delta r)^2\,, \tag{36}$$

$$\Phi_k^{\beta}(M) = 0\,. \tag{37}$$

After evaluating these equations, simulation results are easily obtained by using the inverse transformation rule in GDTM. Numerical solutions for this problem with various β values are shown in Figure 2.

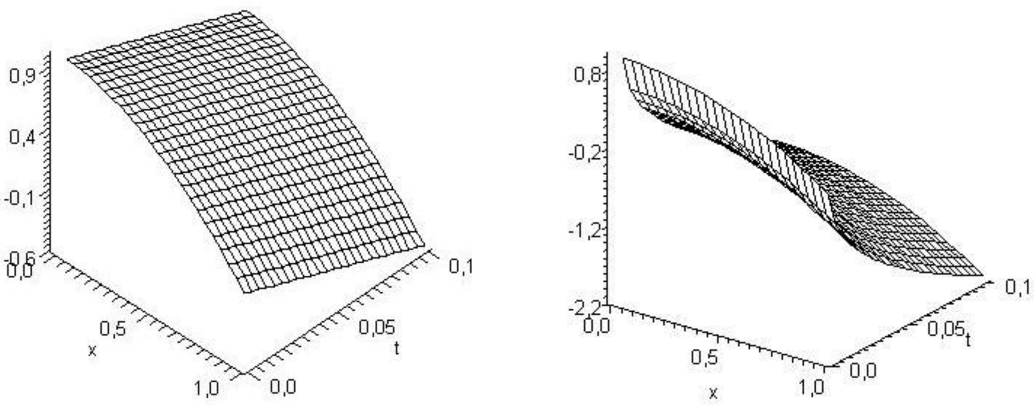


Figure 2. The surfaces show the solution $\varphi(r,t)$ for equation (31): (a) $\beta = 1$; (b) $\beta = 0.5$.

Comparison of present solutions and finite difference solutions [34] is shown in Table 1 for this problem for $\beta=1$. As can be seen from the Table 1, it is clear that the present work gives good approximation as well as FDM.

Table1. Comparison of numerical results with finite difference solution at $t=0.01$ for $\Delta t=0.001$.

x	[34]	Present	Error
0	0.94	0.94	0
0.1	0.93	0.93	10^{-10}
0.2	0.9	0.9	2.5×10^{-9}
0.3	0.85	0.85	6.44×10^{-8}
0.4	0.780001	0.78	0.000001
0.5	0.690013	0.69	0.000013

Example 3. 3.

The function φ is a solution of the equation

$$\frac{\partial^\beta \varphi}{\partial t^\beta}=\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2},\quad 0<r<1\ , \tag{38}$$

at every point $P(r,\theta,t)$ of the open bounded domain $0<r<1,\ t>0$ and satisfies the initial condition $\varphi=r\sin\left(\frac{1}{2}\theta\right),\ 0\le r\le 1,\ t=0$ and the boundary condition $\frac{\partial\varphi}{\partial r}=-\varphi$ at $r=1,\ t>0$, where (r,θ,t) are the cylindrical polar coordinates of P .

By using GDTM in time direction and central FDM in spatial directions, the transformed form of equation (38) and initial condition are as follows:

$$\begin{aligned}\frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)}\Phi^\beta_{k+1}(i,j)=&\frac{\Phi^\beta_k(i+1,j)-2\Phi^\beta_k(i,j)+\Phi^\beta_k(i-1,j)}{(\Delta r)^2}\\&+\frac{2}{i(\Delta r)}\left(\frac{\Phi^\beta_k(i+1,j)-\Phi^\beta_k(i-1,j)}{2(\Delta r)}\right)\\&+\frac{1}{i^2(\Delta r)^2}\left(\frac{\Phi^\beta_k(i,j+1)-2\Phi^\beta_k(i,j)+\Phi^\beta_k(i,j-1)}{(\Delta\theta)^2}\right)\end{aligned} \tag{39}$$

$$\Phi^\beta_0(i,j)=i(\Delta r)\sin\left(\frac{1}{2}j(\Delta\theta)\right). \tag{40}$$

At $r=0$ the right hand side of equation (38) appears to contain singularities. To overcome this complication, a circle of radius Δr is constructed at the origin**.** Let φ_0 be the value of φ at the origin and φ_* be the mean value of φ round the circle.

Rotation of the axes through a small angle clearly leads to

$$\nabla^2\varphi=\frac{4\left(\varphi_*-\varphi_0\right)}{\left(\Delta r\right)^2}. \tag{41}$$

Here,

$$\varphi_*=\frac{\Delta r}{2\pi}\int_0^{2\pi}\sin\left(\frac{\theta}{2}\right)d\theta=\frac{2\Delta r}{\pi}. \tag{42}$$

Then, we get

$$\frac{\Gamma\left(\beta\left(k+1\right)+1\right)}{\Gamma\left(\beta k+1\right)}\Phi_{k+1}^{\beta}\left(0,j\right)=\frac{4}{\left(\Delta r\right)^2}\left(\frac{2\Delta r}{\pi}-0\right)=\frac{8}{\pi\left(\Delta r\right)}. \tag{43}$$

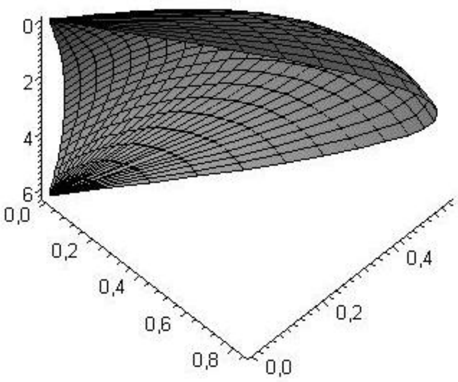
The transformed form of boundary conditions are defined by

$$\begin{aligned}\Phi_{k+1}^{\beta}\left(0,j\right)&=\frac{\Gamma\left(\beta k+1\right)}{\Gamma\left(\beta\left(k+1\right)+1\right)}\frac{8}{\pi\left(\Delta r\right)},\\ \Phi_{k}^{\beta}\left(M,j\right)&=\frac{\Gamma\left(\beta k+1\right)}{\Gamma\left(\beta\left(k+1\right)+1\right)}\left\{\frac{\left(-2\Delta r-2\right)\Phi_{k}^{\beta}\left(M,j\right)+2\Phi_{k}^{\beta}\left(M-1,j\right)}{\left(\Delta r\right)^2}\right.\\ &\left.-\frac{\Phi_{k}^{\beta}\left(M,j\right)}{M\left(\Delta r\right)}+\frac{\Phi_{k}^{\beta}\left(M,j+1\right)-2\Phi_{k}^{\beta}\left(M,j\right)+\Phi_{k}^{\beta}\left(M,j-1\right)}{M^2\left(\Delta r\right)^2\left(\Delta\theta\right)^2}\right\}.\end{aligned} \tag{44}$$

Using equations (39), (40) and (44) differential transform coefficients are evaluated up to certain number of terms and then using the inverse transformation rule $\varphi\left(r_i,\theta_j,t\right)$ is evaluated as follows:

$$\varphi\left(r_i,\theta_j,t\right)=\sum_{k=0}^{10}\Phi_{k}^{\beta}\left(i,j\right)t^{k\beta}. \tag{45}$$

Following figure shows the graphs of the solution of equation (38) for various β values.



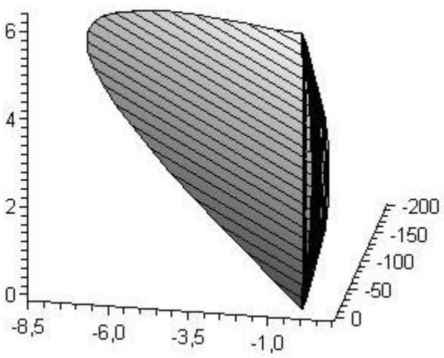


Figure 3. The surfaces show the solution $\varphi\left(r,\theta,t\right)$ for equation (38): (a) $\beta=1$; (b) $\beta=0.5$.

4. CONCLUSION

In this study a new GDTM/FDM are studied for the solution of a time-fractional Navier-Stokes equation in a tube for three different cases. In the first case; we found the same solution with ADM in series form. In the second case; we obtained singularity in the equation and we overcame this difficulty by using hybrid method. Then, we compared our results with FDM. In the third case; we solved two-dimensional fractional differential equations. We had convergent solution easily because the recurrence relation of hybrid method did not contain any complexity rather than GDTM in the higher dimension. Therefore, results showed that the hybrid approach is more convenient than GDTM for computational purpose because of reducing the execution time and memory requirements for large scale computations. Numerical calculations illustrate the effectiveness and the performance of this method. Also, the surfaces of the solution of equations (22), (30) and (38) are showed by graphically for different values of Beta and they are compatible with literature for $\beta = 1$.

REFERENCES

[1]. Ahmad, W. M., El-Khazali, R., Fractional-Order Dynamical Models of Love, Chaos, Solitons & Fractals 33 (4) (2007) 1367–1375.

[2]. Padovan J., Computational algorithms for FE formulations involving fractional operators, Comput. Mech. 5 (1987) 271–287.

[3]. Momani S., Non-perturbative analytical solutions of the space- and time-fractional Burgers equations, Chaos Solitons & Fractals 28 (4) (2006) 930–937.

[4]. Momani S. and Odibat Z., Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method, Appl. Math. Comput. 177 (2006) 488–494.

[5]. Odibat Z. and Momani S., Approximate solutions for boundary value problems of time-fractional wave equation, Appl. Math. Comput. 181 (2006) 1351–1358.

[6]. Momani S., An explicit and numerical solution of the fractional KdV equation, Math. Comput. Simulation 70 (2) (2005) 110–118.

[7]. Momani S. And Odibat Z., Analytical approach to linear fractional partial differential equations arising in fluid mechanics, Phys. Lett. A 355 (2006) 271–279.

[8]. Momani S. And Odibat Z., Numerical comparison of methods for solving linear differential equations of fractional order, Chaos, Solitons & Fractals, 31 (5) (2007) 1248–1255.

[9]. He J.H., Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Engrg. 167 (1998) 57–68.

[10]. Odibat Z. and Momani S., Application of variational iteration method to nonlinear differential equations of fractional order, Int. J. Nonlinear Sci. Numer. Simul., 7 (1) (2006) 15–27.

[11]. Odibat Z. and Momani S., Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order, Chaos, Solitons & Fractals 36 (2008) 167-174.

[12]. Momani S. and Odibat Z., Comparison between homotopy perturbation method and the variational iteration method for linear fractional partial differential equations, Comput. Math. Appl. 54 (2007) 910-919.

[13]. Podlubny I., Fractional Differential Equations, Academic Press, New York, 1999.

[14]. Baskonus, H. M.and Bulut H., On the Numerical Solutions of Some Fractional Ordinary Differential Equations by Fractional Adams-Bashforth-Moulton Method, Open Mathematics, 13(1)(2015), 547–556.

[15]. Odibat Z. and Momani S., A generalized differential transform method for linear partial differential equations of fractional order, Applied Mathematics Letters 21 (2) (2008) 194–199.

[16]. El-Shahed M. and Salem A., On the generalized Navier–Stokes equations, Appl. Math. Comput. 156 (1) (2004) 287–293.
[17]. Odibat Z.M. and Shawagfeh N.T., Generalized Taylor's formula, Appl. Math. Comp. 186 (2007) 286–293.
[18]. Zhou J.K., Differential Transformation and its Applications for Electrical Circuits, Huazhong University Press, Wuhan, P. R., China 1986.
[19]. Arikoglu A. and Ozkol I., Solution of Difference Equations by Using Differential Transform Method, Appl. Math. Comput. 174(2) (2006) 1216-1228.
[20]. Ayaz F., Solutions of the system of differential equations by differential transform method, Appl. Math. Comput. 147 (2004) 547–567.
[21]. Arikoglu A. and Ozkol I., Solution of boundary value problems for integro-differential equations by using differential transform method, Appl. Math. Comput. 168(2) (2005) 1145–1158.
[22]. Bildik N., Konuralp A.,Orak Bek F., Kucukarslan S., Solution of different type of the partial differential equation by differential transform method and Adomian's decomposition method, Appl. Math. Comput. 172 (2006) 551–567.
[23]. Abdel-Halim Hassan I.H., Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems, Chaos, Solitons & Fractals 36 (2008) 53-65.
[24]. Jang M.J., Chen C.L., Liy Y.C., On Solving the Initial-Value Problems Using the Differential Transformation Method, Appl. Math. Comput., 115 (2000) 145–160.
[25]. Rida S.Z., El-Sayed A.M.A., Arafa A.A.M., On the solutions of time-fractional reaction–diffusion equations, Communications in Nonlinear Science and Numerical Simulation 15 (2010) 3847-3854.
[26]. Odibat Z., Bertelle C., Aziz-Alaoui M.A., Duchamp G.H.E., A multi-step differential transform method and application to non-chaotic and chaotic systems, Comput. Math. Appl. 59 (2010) 1462–1472.
[27]. Zou L., Wang Z., Zong Z., Zou D., Zhang S., Solving shock wave with discontinuity by enhanced differential transform method (EDTM), Appl. Math. Mech. -Engl. Ed., 33(12) (2012) 1569–1582.
[28]. Liu H. and Song Y., Differential transform method applied to high index differential–algebraic equations, Appl. Math. Comp. 184 (2007) 748–753.
[29]. Chen X., Dai Y., Differential transform method for solving Richards' equation, Appl. Math. Mech. -Engl. Ed., 37(2) (2016) 169–180.
[30]. Yu L.T. and Chen C.K., Application of the Hybrid Method to the Transient Thermal Stresses Response in Isotropic Annular Fins, J. Appl. Mech. 66 (1999) 340-347.
[31]. Kuo B.L. and Chen C.K., Application of a Hybrid Method to the Solution of the Nonlinear Burgers' Equation, J. Appl. Mech. 70 (2003) 926-930.
[32]. Chen C.K., Lai H.Y., Liu C.C., Application of hybrid differential transformation/finite difference method to nonlinear analysis of micro fixed-fixed beam, Microsyst Technol. 15 (2009) 813–820.
[33]. Chu H.P. and Chen C.L., Hybrid differential transform and finite difference method to solve the nonlinear heat conduction problem, Communications in Nonlinear Science and Numerical Simulation 13 (8) (2008) 1605–1614.
[34]. Smith G.D., Numerical Solution of Partial Differential Equations Finite Difference Methods, Oxford University Press, 1978.