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Integral Representation for Solution of Discontinuous Diffusion Operator with Jump Conditions

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Abstract. In this study, the diffusion operator with discontinuity function and the jump conditions is considered. Under certain initial and discontinuity conditions, integral equations have been derived for the solutions. Integral representations, which is too useful for this type equation, have been presented.

Keywords: Diffusion Operator, Jump condition, Sturm-Liouville Equation.

Sıçrama Şartlarına Sahip Süreksiz Difüzyon Operatörünün Çözümleri için Integral Gösterilim

Özet. Bu makalede süreksizlik fonksiyonuna ve sıçrama şartlarına sahip difüzyon operatörü incelenmiştir. Belirli başlangıç ve süreksizlik koşulları altında çözümler için integral denklemler üretilmiştir. Bu tip difüzyon operatörler için oldukça kullanışlı olan integral gösterilimler elde edilmiştir.

Anahtar Kelimeler: Difüzyon Operatör, Sıçrama şartı, Sturm- Liouville denklemi.

1. INTRODUCTION

Let's define the following boundary value problem;

$$l(y) := -y'' + \left[2\lambda p(x) + q(x)\right] y = \lambda^2 \delta(x) y,\ x \in (0, p_1) \cup (p_1, p_2) \cup (p_2, \pi) \tag{1}$$

$$U(y) = y'(0) = 0, \mathrm{V}(y) = y(\pi) = 0 \tag{2}$$

$$y(p_1 + 0) = \alpha_1 y(p_1 - 0) \tag{3}$$

$$y'(p_1 + 0) = \beta_1 y'(p_1 - 0) + i\lambda\gamma_1 y(p_1 - 0) \tag{4}$$

$$y(p_2 + 0) = \alpha_2 y(p_2 - 0) \tag{5}$$

$$y'(p_2 + 0) = \beta_2 y'(p_2 - 0) + i\lambda\gamma_2 y(p_2 - 0) \tag{6}$$

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where λ is a spectral parameter, $q(x) \in L_2[0,\pi]$, $p(x) \in W_2^1[0,\pi]$, $p_1, p_2 \in (0,\pi)$, $p_1 < p_2$,

$$|\alpha_1 - 1|^2 + \gamma_1^2 \neq 0, |\alpha_2 - 1|^2 + \gamma_2^2 \neq 0, \left(\beta_i = \frac{1}{\alpha_i}(i=1,2)\right) \text{and } \delta(x) = \begin{cases} 1, & x \in (0,p_1) \\ \alpha^2, & x \in (p_1,p_2) \\ \beta^2, & x \in (p_2,\pi) \end{cases}$$

where α and β are real numbers.

The spectral theory of differential operators has an important role in applied sciences, mathematics and engineering. In geophysical engineering, determination of underground mines according to the distribution characteristics of the elements underground is example of an inverse problem. In the literature, the Sturm-Liouville equation is defined as follows:

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = \lambda y.$$

There are many direct and inverse problems for this equation studied in various publications. The first study of inverse problem was published by Ambartsumyan in 1929 $[26]$. The next important work was done by Borg in 1945. Borg showed that the coefficients of the operator can be determined by two spectra. This result has been generalized to various $[7-26]$. Direct and inverse spectral problems for the continuous problems have been studied in $[19,22,24]$. For some spectral problems of diffusion equations on a finite interval when $\delta(x)=1$, we refer to $[9,10,16,17]$. In these works the integral representations of the solutions are derived and some properties of the kernel functions are examined. The spectral problem of the Sturm-Liouville equation with discontinuous coefficient is investigated detail in $[21]$. Eigenvalues of an operator with boundary conditions depend on the parameter are defined by Naimark $[5]$. Basic spektral properties of these types boundary value problem for Sturm-Liouville equations was established by Fulton in $[30]$. He defined the operator corresponding to the problem and obtained some properties of the eigenvalues of the operator. Guliyev obtained the Gelfand- Levitan- Marchenko type principal equation for the Sturm-Liouville inverse problem with parameter dependent boundary conditions and gave the complete solution of this problem in 2005 $[20]$.

In the present study, the diffusion operator with discontinuous coefficient and jump conditions in the finite interval is considered. First, integral equations of the solutions were obtained. Second, by using the method of successive approximations, integral representations of solutions were obtained. In addition, the inequalities provided by the kernel functions were obtained. (This work is supported by the Scientific Research Project Fund of Cumhuriyet University under the project number SMYO-020).

Integral equations of solutions.

Let $\phi(x,\lambda)$ be the solutions of (1) satisfying the following conditions;

$i)\phi'(x,\lambda)$ are absolutely continuous in $(0,p_1),(p_1,p_2),(p_2,\pi)$;

$ii) l(\phi)\in L_2(0,\pi)$,

$iii)$ $\phi(x,\lambda)$ satisfies the initial conditions;

$$\phi(0,\lambda)=1,\phi'(0,\lambda)=0 \tag{7}$$

$iv)$ $\phi(x,\lambda)$ satisfies the jump conditions $(3)-(6)$.

Therefore, the following lemma is obtained.

Lemma.

For the function $\phi(x,\lambda)$ the following integral equations holds;

if $0\le x<p_1$;

$$\phi(x,\lambda)=e^{i\lambda x}+\frac{1}{\lambda}\int_0^x \sin\lambda(x-t)J(t)y(\mathrm{t},\lambda)dt \tag{8}$$

if $p_1<x<p_2$;

$$\begin{aligned}\phi(x,\lambda)&=\beta_1^+e^{i\lambda\varsigma^+(x)}+\beta_1^-e^{i\lambda\varsigma^-(x)}+\frac{\gamma_1}{2\alpha}e^{i\lambda\varsigma^+(x)}-\frac{\gamma_1}{2\alpha}e^{i\lambda\varsigma^-(x)}\\&+\beta_1^+\int_0^{p_1}\frac{\sin\lambda(\varsigma^+(x)-t)}{\lambda}J(t)y(\mathrm{t},\lambda)dt+\beta_1^-\int_0^{p_1}\frac{\sin\lambda(\varsigma^-(x)-t)}{\lambda}J(t)y(\mathrm{t},\lambda)dt\\&-i\frac{\gamma_1}{2\alpha}\int_0^{p_1}\frac{\cos\lambda(\varsigma^+(x)-t)}{\lambda}J(t)y(\mathrm{t},\lambda)dt++i\frac{\gamma_1}{2\alpha}\int_0^{p_1}\frac{\cos\lambda(\varsigma^-(x)-t)}{\lambda}J(t)y(\mathrm{t},\lambda)dt\\&+\int_{a_1}^{x}\frac{\sin\lambda(x-t)}{\lambda}J(t)y(\mathrm{t},\lambda)dt\end{aligned} \tag{9}$$

if $p_2<x\le\pi$;

$$
\begin{aligned}
\phi(x,\lambda) &= \xi^{+}e^{i\lambda b^{+}(x)} + \xi^{-}e^{i\lambda b^{-}(x)} + \vartheta^{+}e^{i\lambda s^{+}(x)} + \vartheta^{-}e^{i\lambda s^{-}(x)} \\
&+\left(\beta_1^{+}\beta_2^{+} + \frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}\frac{\sin\lambda\left(b^{+}(x)-t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt + \left(\beta_1^{+}\beta_2^{-} - \frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}\frac{\sin\lambda\left(s^{+}(x)-t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt \\
&+\left(\beta_1^{-}\beta_2^{-} - \frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}\frac{\sin\lambda\left(b^{-}(x)-t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt + \left(\beta_1^{-}\beta_2^{+} + \frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_{p_1}^{p_2}\frac{\sin\lambda\left(s^{-}(x)-t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt \\
&-i\left(\frac{\gamma_1\beta_2^{+}}{2\alpha} + \frac{\gamma_2\beta_1^{+}}{2\beta}\right)\int_0^{p_1}\frac{\cos\lambda\left(b^{+}(x)-t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt - i\left(\frac{\gamma_1\beta_2^{-}}{2\alpha} - \frac{\gamma_2\beta_1^{+}}{2\beta}\right)\int_0^{p_1}\frac{\cos\lambda\left(s^{+}(x)-t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt \\
&+i\left(\frac{\gamma_1\beta_2^{-}}{2\alpha} - \frac{\gamma_2\beta_1^{-}}{2\beta}\right)\int_0^{p_1}\frac{\cos\lambda\left(b^{-}(x)-t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt + i\left(\frac{\gamma_1\beta_2^{+}}{2\alpha} + \frac{\gamma_2\beta_1^{-}}{2\beta}\right)\int_{p}^{ap_2}\frac{\cos\lambda\left(s^{-}(x)-t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt \\
&+\beta_2^{+}\int_{p_1}^{p_2}\frac{\sin\lambda\left(\beta x-\beta p_2+\alpha p_2-\alpha t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt - \beta_2^{-}\int_{p_1}^{p_2}\frac{\sin\lambda\left(\beta x-\beta p_2-\alpha p_2+\alpha t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt \\
&-i\frac{\gamma_2}{2\beta}\int_{p_1}^{p_2}\frac{\cos\lambda\left(\beta x-\beta p_2+\alpha p_2-\alpha t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt + i\frac{\gamma_2}{2\beta}\int_{p_1}^{p_2}\frac{\cos\lambda\left(\beta x-\beta p_2-\alpha p_2+\alpha t\right)}{\lambda}J(t)y(\mathrm{t},\lambda)dt \\
&+\int_{p_2}^{x}\frac{\sin\lambda(x-t)}{\lambda}J(t)y(\mathrm{t},\lambda)dt
\end{aligned}
\tag{10}
$$

where $J(t)=2\lambda p(t)+q(t)$, $\varsigma^{\pm}(x)=\pm\alpha x\mp\alpha p_1+p_1$, $\beta_1^{\pm}=\frac{1}{2}\left(\alpha_1\pm\frac{\beta_1}{\alpha}\right)$, , $b^{\pm}(x)=\beta x-\beta p_2+\varsigma^{\pm}(p_2)$

$s^{\pm}(x)=-\beta x+\beta p_2+\varsigma^{\pm}(p_2)$ $\beta_2^{\mp}=\frac{1}{2}\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}\right)$, $\xi^{\mp}=\frac{1}{2}\left(\beta_1^{\mp}\mp\frac{\gamma_1}{2\alpha}\right)\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}+\frac{\gamma_2}{\beta}\right)$,

$$\vartheta^{\mp}=\frac{1}{2}\left(\beta_1^{\mp}\mp\frac{\gamma_1}{2\alpha}\right)\left(\alpha_2\pm\frac{\alpha\beta_2}{\beta}-\frac{\gamma_2}{\beta}\right).$$

The proof of this lemma is easily follows by using the well-known method of variation of parameters.

Theorem. Let $p(x)\in W_2^1(0,\pi)$, $q(x)\in L_2(0,\pi)$ and let $y_\upsilon(x,\lambda)$ be solutions of (1), satisfying boundary-jump conditions $(2)-(6)$. Then the following relation holds

$$y_\upsilon(x,\lambda)=y_{0\upsilon}(x,\lambda)+\int_{-x}^{x}K_\upsilon(x,\mathrm{t})e^{i\lambda t}dt \quad \left(\upsilon=\overline{1,3}\right)$$

where

$$y_{0\upsilon}(x,\lambda)=\begin{cases} R_0(x)e^{i\lambda x} & ;0\le x<p_1 \\ R_1(x)e^{i\lambda\varsigma^{+}(x)}+R_2(x)e^{i\lambda\varsigma^{-}(x)} & ;p_1<x<p_2 \\ R_3(x)e^{i\lambda b^{+}(x)}+R_4(x)e^{i\lambda b^{-}(x)}+R_5(x)e^{i\lambda s^{+}(x)}+R_6(x)e^{i\lambda s^{-}(x)} & ;p_2<x\le\pi \end{cases}$$

$$R_0(x)=e^{-i\int_0^x p(x)dx},R_1(x)=\left(\beta_1^++\frac{\gamma_1}{2\alpha}\right)R_0(p_1)e^{-\frac{i}{\alpha}\int_{p_1}^x p(t)dt},R_2(x)=\left(\beta_1^--\frac{\gamma_1}{2\alpha}\right)R_0(p_1)e^{\frac{i}{\alpha}\int_{p_1}^x p(t)dt},$$

$$R_3(x)=\left(\beta_2^++\frac{\gamma_2}{2\beta}\right)R_1(p_2)e^{-\frac{i}{\beta}\int_{p_2}^x p(t)dt},R_4(x)=\left(\beta_2^-+\frac{\gamma_2}{2\beta}\right)R_2(p_2)e^{-\frac{i}{\beta}\int_{p_2}^x p(t)dt},$$

$$R_5(x)=\left(\beta_2^--\frac{\gamma_2}{2\beta}\right)R_1(p_2)e^{\frac{i}{\beta}\int_{p_2}^x p(t)dt},\ R_6(x)=\left(\beta_2^+-\frac{\gamma_2}{2\beta}\right)R_2(p_2)e^{\frac{i}{\beta}\int_{p_2}^x p(t)dt}$$

and $\varpi(x)=\int_0^x\left((x-k)|q(k)|+2|p(k)|\right)dk$ and the functions $K_\upsilon(x,t)$ satisfy the inequality

$$\int_{-x}^x|K_\upsilon(x,\lambda)|dt\le e^{c_\upsilon\varpi(x)}-1$$

with $c_1=1$, $c_2=\left(\beta_1^++|\beta_1^-|+\frac{\gamma_1}{\alpha}+\frac{2}{\alpha}\right)$, $c_3=\left(\alpha_2\left(\beta_1^++|\beta_1^-|\right)+\frac{1}{\alpha}\left(\beta_2^++|\beta_2^-|\right)+\frac{\beta^+}{\beta}+\frac{\gamma_2}{\beta}\right)$.

here $\varsigma^\pm(x)=\pm\alpha x\mp\alpha p_1+p_1$, $\beta_1^\pm=\frac{1}{2}\left(\alpha_1\pm\frac{\beta_1}{\alpha}\right)$, $b^\pm(x)=\beta x-\beta p_2+\varsigma^\pm(p_2)$ $s^\pm(x)=-\beta x+\beta p_2+\varsigma^\pm(p_2)$

$\beta_2^\mp=\frac{1}{2}\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}\right)$, $\xi^\mp=\frac{1}{2}\left(\beta_1^\mp\mp\frac{\gamma_1}{2\alpha}\right)\left(\alpha_2\mp\frac{\alpha\beta_2}{\beta}+\frac{\gamma_2}{\beta}\right)$, $\vartheta^\mp=\frac{1}{2}\left(\beta_1^\mp\mp\frac{\gamma_1}{2\alpha}\right)\left(\alpha_2\pm\frac{\alpha\beta_2}{\beta}-\frac{\gamma_2}{\beta}\right)$,

$\beta^\pm=\frac{1}{2}\left(1\pm\frac{1}{\beta}\right)$.

Proof.

For $0\le x<p_1$

$$y(x,\lambda)=e^{i\lambda x}+\int_0^X\frac{\sin\lambda(x-t)}{\lambda}J(t)y(t,\lambda)dt\ ,J(t)=\left(2\lambda p(t)+q(t)\right)$$

$$y_{01}(x,\lambda)+\int_{-x}^x K_1(x,t)e^{i\lambda t}dt$$
$$=e^{i\lambda x}+\int_0^x\frac{e^{i\lambda(x-t)}-e^{-i\lambda(x-t)}}{2i\lambda}\left(2\lambda p(t)+q(t)\right)\left(R_0(t)e^{i\lambda t}+\int_{-t}^t K_1(t,\tau)e^{i\lambda\tau}d\tau\right)dt$$

$$= e^{i\lambda x} - ie^{i\lambda x}\int_0^x R_0(t)p(t)dt + i\int_0^x R_0(t)p(t)e^{i\lambda(2t-x)}dt + \frac{1}{2}\int_0^x R_0(t)q(t)\int_{2t-x}^{x} e^{i\lambda u}dudt$$
$$-i\int_0^x p(t)\int_{-t}^{t} K_1(t,\tau)e^{i\lambda(x-t+\tau)}d\tau dt + i\int_0^x p(t)\int_{-t}^{t} K_1(t,\tau)e^{i\lambda(-x+t+\tau)}d\tau dt$$
$$+\frac{1}{2}\int_0^x q(t)\int_{-t}^{t} K_1(t,\tau)\int_{-x+t+\tau}^{x-t+\tau} e^{i\lambda u}dud\tau dt$$

$$= e^{i\lambda x} - ie^{i\lambda x}\int_0^x R_0(t)p(t)dt + i\int_{-x}^{x} R_0\left(\frac{x+t}{2}\right)p\left(\frac{x+t}{2}\right)e^{i\lambda t}dt$$
$$+\frac{1}{2}\int_{-x}^{x}\int_0^{\frac{x+t}{2}}\left(R_0(u)q(u)du\right)e^{i\lambda t}dt - i\int_{-x}^{x}\left(\int_{\frac{x-t}{2}}^{x} p(u)K_1(\mathrm{u},\mathrm{t}-\mathrm{x}+\mathrm{u})du\right)e^{i\lambda t}dt$$
$$+i\int_{-x}^{x}\left(\int_{\frac{x+t}{2}}^{x} p(u)K_1(\mathrm{u},\mathrm{t}+\mathrm{x}-\mathrm{u})du\right)e^{i\lambda t}dt + \frac{1}{2}\int_{-x}^{x}\left(\int_{-t}^{t} q(u)\int_{t-x+u}^{t+x-u} K_1(\mathrm{u},\tau)d\tau du\right)e^{i\lambda t}dt$$

where $x \in (0, p_1)$

$R_0(x) = 1 - i\int_0^x R_0(t)p(t)dt$ hence

$$R_0(x) = e^{-i\int_0^x p(x)dx} \qquad (11)$$

obtained and

$$K_1(x,k) = iR_0\left(\frac{k+x}{2}\right)p\left(\frac{k+x}{2}\right) + \frac{1}{2}\int_0^{\frac{k+x}{2}}\left(R_0(u)q(u)du\right) - i\int_{\frac{x-t-k}{2}}^{x} p(u)K_1(\mathrm{u},k-\mathrm{x}+\mathrm{u})du$$
$$+i\int_{\frac{x+k}{2}}^{x} p(u)K_1(\mathrm{u},k+\mathrm{x}-\mathrm{u})du + \frac{1}{2}\int_{-k}^{k} q(u)\int_{k-x+u}^{k+x-u} K_1(\mathrm{u},\tau)d\tau du$$

If we apply the integral equation successive approximation method for $K_1(x,k)$;

$$K_1^{(0)}(x,k)=iR_0\left(\frac{k+x}{2}\right)p\left(\frac{k+x}{2}\right)+\frac{1}{2}\int_0^{\frac{k+x}{2}}R_0(u)q(u)du$$

$$K_1^{(1)}(x,k)=-i\int_{\frac{x-k}{2}}^{x}p(u)K_1^{(0)}(\mathrm{u},k-\mathrm{x}+\mathrm{u})du+i\int_{\frac{k+x}{2}}^{x}p(u)K_1^{(0)}(\mathrm{u},k+\mathrm{x}-\mathrm{u})du$$

$$+\frac{1}{2}\int_{-k}^{k}q(u)\int_{k-x+u}^{k+x-u}K_1^{(0)}(\mathrm{u},\tau)d\tau du$$

...

$$K_1^{(n)}(x,k)=-i\int_{\frac{x-k}{2}}^{x}p(u)K_1^{(n-1)}(\mathrm{u},k+\mathrm{u}-\mathrm{x})du+i\int_{\frac{k+x}{2}}^{x}p(u)K_1^{(n-1)}(\mathrm{u},k-\mathrm{u}+\mathrm{x})du$$

$$+\frac{1}{2}\int_{-k}^{k}q(u)\int_{k-x+u}^{k+x-u}K_1^{(n-1)}(\mathrm{u},\tau)d\tau du$$

Hence

$$\int_{-x}^{x}\left|K_1^{(0)}(x,k)\right|dt\le\int_{-x}^{x}\left|R_0\left(\frac{k+x}{2}\right)\right|\left|p\left(\frac{k+x}{2}\right)\right|dt+\frac{1}{2}\int_{-x}^{x}\int_0^{\frac{k+x}{2}}\left|R_0(u)\right|\left|q(u)\right|dudt$$

$$=\int_0^x\left|R_0(u)\right|\left|p(u)\right|du+\frac{1}{2}\int_0^x\left|R_0(u)\right|\left|q(u)\right|\left(\int_{2u-x}^{x}dk\right)du$$

$$\le\int_0^x 2\left|p(k)\right|dk+\int_0^x\left|q(k)\right|(-k+x)dk=\int_0^x\left((x-k)\left|q(k)\right|+2\left|p(k)\right|\right)dk=\varpi(x)$$

$$\int_{-x}^{x}\left|K_1^{(0)}(x,k)\right|dk\le\varpi(x)$$

we get to inequaltiy. For $n=1$ we have

$$\int_{-x}^{x}\left|K_1^{(1)}(x,k)\right|dt\le\int_{-x}^{x}\int_{\frac{x-k}{2}}^{x}\left|p(u)\right|\left|K_1^{(0)}(\mathrm{u},k+\mathrm{u}-\mathrm{x})\right|dudk+\int_{-x}^{x}\int_{\frac{x+k}{2}}^{x}\left|p(u)\right|\left|K_1^{(0)}(\mathrm{u},k-\mathrm{u}+\mathrm{x})\right|dudk$$

$$+\frac{1}{2}\int_{-x}^{x}\int_{-k}^{k}\left|q(u)\right|\int_{k-x+u}^{k+x-u}\left|K_1^{(0)}(\mathrm{u},\tau)\right|d\tau dudk$$

$$\le\int_0^x\left|p(u)\right|\int_{x-2u}^{x}\left|K_1^{(0)}(\mathrm{u},k+\mathrm{u}-\mathrm{x})\right|dkdu+\int_0^x\left|p(u)\right|\int_{-x}^{2u-x}\left|K_1^{(0)}(\mathrm{u},k-\mathrm{u}+\mathrm{x})\right|dkdu$$

$$+\frac{1}{2}\int_0^x |q(u)| \int_{-x}^{x} \int_{k-x+u}^{k+x-u} \left|K_1^{(0)}(\mathrm{u},\tau)\right| d\tau dk du$$

$$\leq \int_0^x |p(k)| \int_{-t}^{t} \left|K_1^{(0)}(k,\tau)\right| d\tau dk + \int_0^x |p(k)| \int_{-t}^{t} \left|K_1^{(0)}(k,\tau)\right| d\tau dk$$

$$+\int_0^x (x-u)|q(u)| \int_{u-2x}^{2x-u} \left|K_1^{(0)}(k,\tau)\right| d\tau du$$

$$\leq \int_0^x \varpi(k)|p(k)|dk + \int_0^x \varpi(k)|p(k)|dk + \int_0^x (x-k)|q(k)|\varpi(k)dk$$

$$= \int_0^x \left(2|p(k)| + (x-k)|q(k)|\right)\varpi(k)dk$$

$$= \int_0^x \varpi(k) d\left(\int_0^x \left((x-k)|q(k)| + 2|p(k)|\right)dk\right) = \frac{\varpi^2(x)}{2!}$$

$$\int_{-x}^{x} \left|K_1^{(1)}(x,k)\right| dk \leq \frac{\varpi^2(x)}{2!}$$

Suppose thhat for $n = k-1$, the inequality

$\int_{-x}^{x} \left|K_1^{(k-1)}(x,k)\right| dk \leq \frac{\varpi^k(x)}{\mathrm{k}!}$ be true. Then for $n = k$, we have

$$\int_{-x}^{x} \left|K_1^{(k)}(x,k)\right| dk \leq \frac{\varpi^{k+1}(x)}{(k+1)!}$$

let's examine the truth.

$$\int_{-x}^{x} \left|K_1^{(k)}(x,k)\right| dk \leq \int_{-x}^{x} \int_{\frac{x-k}{2}}^{x} |p(u)| \left|K_1^{(k-1)}(\mathrm{u},k+\mathrm{u}-\mathrm{x})\right| du dk + \int_{-x}^{x} \int_{\frac{k+x}{2}}^{x} |p(u)| \left|K_1^{(k-1)}(\mathrm{u},k-\mathrm{u}+\mathrm{x})\right| du dk$$

$$+\frac{1}{2}\int_{-x}^{x} \int_{-k}^{k} |q(u)| \int_{k-x+u}^{k+x-u} \left|K_1^{(k-1)}(\mathrm{u},\tau)\right| d\tau du dk$$

$$\leq \int_0^x |p(u)| \int_{x-2u}^{x} \left|K_1^{(k-1)}(\mathrm{u},k+\mathrm{u}-\mathrm{x})\right| dk du + \int_0^x |p(u)| \int_{-x}^{2u-x} \left|K_1^{(k-1)}(\mathrm{u},k-\mathrm{u}+\mathrm{x})\right| dk du$$

$$+\frac{1}{2}\int_0^x |q(u)| \int_{-x}^{x} \int_{k-x+u}^{k+x-u} \left|K_1^{(k-1)}(\mathrm{u},\tau)\right| d\tau dk du$$

$$\leq \int_0^x |p(k)| \int_{-k}^{k} \left|K_1^{(k-1)}(k,\tau)\right| d\tau dk + \int_0^x |p(k)| \int_{-k}^{k} \left|K_1^{(k-1)}(k,\tau)\right| d\tau dk$$

$$+\int_0^x (x-u)|q(u)|\int_{u-2x}^{2x-u}\left|K_1^{(k-1)}(k,\tau)\right|d\tau du$$

$$\le \int_0^x \frac{\varpi^k(k)}{k!}|p(k)|dk+\int_0^x \frac{\varpi^k(k)}{k!}|p(k)|dk+\int_0^x |q(k)|(x-k)\frac{\varpi^k(k)}{k!}dk$$

$$=\int_0^x \left((x-k)|q(k)|+2|p(k)|\right)\frac{\varpi^k(t)}{k!}dt$$

$$=\int_0^x \frac{\varpi^k(k)}{k!}d\left(\int_0^x \left((x-k)|q(k)|+2|p(k)|\right)dk\right)=\frac{\varpi^{k+1}(x)}{(k+1)!}$$

Then we have

$$\int_{-x}^{x}|K_1(x,k)|dk \le \sum_{n=0}^{\infty}\int_{-x}^{x}\left|K_1^{(k)}(x,k)\right|dk \le 1-1+\varpi(x)+\frac{\varpi^2(x)}{2!}+\ldots+\frac{\varpi^{n+1}(x)}{(n+1)!}+\ldots=\mathrm{e}^{\varpi(x)}-1$$

$$\int_{-x}^{x}|K_1(x,k)|dk \le \mathrm{e}^{\varpi(x)}-1=e^{\int_0^x \left((x-k)|q(k)|+2|p(k)|\right)dk}-1 \qquad (12)$$

Similarly, successive approximation method for $p_1<x<p_2$ and $p_2<x\le\pi$,

Gives us

$$R_1(x)=\beta_1^+ +\frac{\gamma_1}{2\alpha}+i\beta_1^+\int_0^{p_1}R_0(t)p(t)dt-i\frac{\gamma_1}{2\alpha}\int_0^{p_1}R_0(t)p(t)dt-\frac{i}{\alpha}\int_{p_1}^{x}R_1(t)p(t)dt$$

$$R_1(x)=c\cdot e^{-\frac{i}{\alpha}\int_{p_1}^{x}p(t)dt}\quad ;c=\left(\beta_1^+ +\frac{\gamma_1}{2\alpha}\right)R_0(p_1), \qquad (13)$$

$$R_2(x)=\beta_1^- -\frac{\gamma_1}{2\alpha}-i\beta_1^-\int_0^{p_1}R_0(t)p(t)dt+i\frac{\gamma_1}{2\alpha}\int_0^{p_1}R_0(t)p(t)dt+\frac{i}{\alpha}\int_{p_1}^{x}R_2(t)p(t)dt$$

$$R_2(x)=c\cdot e^{\frac{i}{\alpha}\int_{p_1}^{x}p(t)dt}\quad ;c=\left(\beta_1^- -\frac{\gamma_1}{2\alpha}\right)R_0(p_1) \qquad (14)$$

and

$$K_2(x,t)=i\cdot\frac{\beta_1^+}{2}p\left(\frac{\varsigma^+(x)+t}{2}\right)R_0\left(\frac{\varsigma^+(x)+t}{2}\right)+\frac{\beta_1^+}{2}\int_0^{\frac{\varsigma^+(x)+t}{2}}q(u)R_0(u)du+\frac{\beta_1^+}{2}\int_0^{p_1}q(u)R_0(u)du$$

$$-i\cdot\beta_1^+\int_{\frac{\varsigma^+(x)-t}{2}}^{p_1}p(u)K_2\left(u,t-\varsigma^+(x)+u\right)du+i\cdot\beta_1^+\int_{\frac{\varsigma^+(x)+t}{2}}^{p_1}p(u)K_2\left(u,t+\varsigma^+(x)-u\right)du$$

$$+\frac{\beta_1^+}{2}\int_0^{p_1}q(u)\int_{t-\varsigma^+(x)+u}^{t+\varsigma^+(x)-u}K_2(u,\tau)d\tau du+i\cdot\frac{\beta_1^-}{2}p\left(\frac{\varsigma^-(x)+t}{2}\right)R_0\left(\frac{\varsigma^-(x)+t}{2}\right)$$

$$+\frac{\beta_1^-}{2}\int_0^{\frac{\varsigma^-(x)+t}{2}}q(u)R_0(u)du+\frac{\beta_1^-}{2}\int_0^{p_1}q(u)R_0(u)du-i\beta_1^-\int_{\frac{\varsigma^-(x)-t}{2}}^{p_1}p(u)K_2\left(u,t-\varsigma^-(x)+u\right)du$$

$$+i\cdot\beta_1^-\int_{\frac{\varsigma^-(x)+t}{2}}^{p_1}p(u)K_2\left(u,t+\varsigma^-(x)-u\right)du+\frac{\beta_1^-}{2}\int_0^{p_1}q(u)\int_{t-\varsigma^-(x)+u}^{t+\varsigma^-(x)-u}K_2(u,\tau)d\tau du$$

$$-i\frac{\gamma_1}{2\alpha}p\left(\frac{\varsigma^+(x)+t}{2}\right)R_0\left(\frac{\varsigma^+(x)+t}{2}\right)+\frac{\gamma_1}{4\alpha}\int_0^s q(s)R_0(s)ds-\frac{\gamma_1}{4\alpha}\int_0^s q\left(\frac{\varsigma^+(x)+s}{2}\right)R_0\left(\frac{\varsigma^+(x)+s}{2}\right)ds$$

$$-i\frac{\gamma_1}{2\alpha}\int_{\frac{\varsigma^+(x)-t}{2}}^{p_1}p(u)K_2\left(u,t-\varsigma^+(x)+u\right)du-i\frac{\gamma_1}{2\alpha}\int_{\frac{\varsigma^+(x)+t}{2}}^{p_1}p(u)K_2\left(u,t+\varsigma^+(x)-u\right)du$$

$$+\frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^+(x)-s}{2}}^{p_1}q(u)\int_0^s K_2\left(u,s-\varsigma^+(x)+u\right)duds+\frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^+(x)+s}{2}}^{p_1}q(u)\int_0^s K_2\left(u,s+\varsigma^+(x)-u\right)duds$$

$$+i\frac{\gamma_1}{2\alpha}p\left(\frac{\varsigma^-(x)+t}{2}\right)R_0\left(\frac{\varsigma^-(x)+t}{2}\right)-\frac{\gamma_1}{4\alpha}\int_0^s q(s)R_0(s)ds-\frac{\gamma_1}{4\alpha}\int_0^s q\left(\frac{\varsigma^-(x)+s}{2}\right)R_0\left(\frac{\varsigma^-(x)+s}{2}\right)ds$$

$$+i\frac{\gamma_1}{2\alpha}\int_{\frac{\varsigma^-(x)-t}{2}}^{p_1}p(u)K_2\left(u,t-\varsigma^-(x)+u\right)du+i\frac{\gamma_1}{2\alpha}\int_{\frac{\varsigma^-(x)+t}{2}}^{p_1}p(u)K_2\left(u,t+\varsigma^-(x)-u\right)du$$

$$-\frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^-(x)-s}{2}}^{p_1}q(u)\int_0^s K_2\left(u,s-\varsigma^-(x)+u\right)duds-\frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^-(x)+s}{2}}^{p_1}q(u)\int_0^s K_2\left(u,s+\varsigma^-(x)-u\right)duds$$

$$+\frac{i}{\alpha^2}\cdot p\left(\frac{t+\alpha p_1+p_1}{\alpha}\right)R_1\left(\frac{t+\alpha p_1+p_1}{\alpha}\right)+\frac{1}{2\alpha}\int_{p_1}^{\frac{u-\alpha p_1+p_1}{\alpha}}q(u)R_1(u)du$$

$$+\frac{1}{2\alpha}\int_{p_1}^x q(u)R_2(u)du+\frac{1}{2\alpha}\int_{p_1}^{\frac{\alpha x-\alpha p_1+p_1-u}{\alpha+1}}q(u)R_2(u)du-\frac{i}{\alpha}\cdot\int_{\frac{\alpha p_1-p_1-t}{\alpha}}^x p(u)K_2\left(u,\mathrm{t}-\alpha(x-t)\right)du$$

$$-\frac{i}{\alpha}\cdot\int_{p_1}^x p(u)K_2\left(u,\alpha(t-x)+t\right)du-\frac{i}{\alpha}\cdot\int_{p_1}^{\frac{\alpha x+\varsigma^+(x)-t}{\alpha}}p(u)K_2\left(u,\alpha(t-x)+t\right)du$$

$$+\frac{i}{\alpha}\cdot\int_{p_1}^{\frac{t+\varsigma^+(x)+\alpha x}{\alpha}} p(u)K_2(u,\alpha(-t+x)+t)du+\frac{i}{\alpha}\cdot\int_{p_1}^{x} p(u)K_2(u,\alpha(-t+x)+t)du$$

$$+\frac{i}{\alpha}\cdot\int_{\frac{t+\alpha p_1-p_1}{\alpha}}^{x} p(u)K_2(u,\alpha(-t+x)+t)du+\frac{1}{2\alpha}\int_{p_1}^{\frac{\varsigma^+(x)+\alpha x+t}{\alpha}} q(u)\int_{t-\alpha x+\alpha u}^{t+\alpha x-\alpha u} K_2(u,\tau)d\tau du$$

$$+\frac{1}{2\alpha}\int_{p_1}^{\frac{-t+\varsigma^+(x)-\alpha x}{\alpha}} q(u)\int_{t-\alpha x-\alpha u}^{t-\alpha x+\alpha u} K_2(u,\tau)d\tau du-\frac{i}{\alpha(\alpha-1)}\cdot p\left(\frac{\varsigma^+(x)-t}{1-\alpha}\right)R_2\left(\frac{\varsigma^+(x)-t}{1-\alpha}\right)$$

Let us apply the integral equation successive approximation method for $K_2(x,t)$;

$$K_2^{(0)}(x,t)=i\cdot\frac{\beta_1^+}{2}p\left(\frac{\varsigma^+(x)+t}{2}\right)R_0\left(\frac{\varsigma^+(x)+t}{2}\right)+\frac{\beta_1^+}{2}\int_0^{\frac{\varsigma^+(x)+t}{2}} q(u)R_0(u)du+\frac{\beta_1^+}{2}\int_0^{p_1} q(u)R_0(u)du$$

$$+i\cdot\frac{\beta_1^-}{2}p\left(\frac{\varsigma^-(x)+t}{2}\right)R_0\left(\frac{\varsigma^-(x)+t}{2}\right)+\frac{\beta_1^-}{2}\int_0^{\frac{\varsigma^-(x)+t}{2}} q(u)R_0(u)du+\frac{\beta_1^-}{2}\int_0^{p_1} q(u)R_0(u)du$$

$$-i\frac{\gamma_1}{2\alpha}p\left(\frac{\varsigma^+(x)+t}{2}\right)R_0\left(\frac{\varsigma^+(x)+t}{2}\right)+\frac{\gamma_1}{4\alpha}\int_0^s q(s)R_0(s)ds-\frac{\gamma_1}{4\alpha}\int_0^s q\left(\frac{\varsigma^+(x)+s}{2}\right)R_0\left(\frac{\varsigma^+(x)+s}{2}\right)ds$$

$$+i\frac{\gamma_1}{2\alpha}p\left(\frac{\varsigma^-(x)+t}{2}\right)R_0\left(\frac{\varsigma^-(x)+t}{2}\right)-\frac{\gamma_1}{4\alpha}\int_0^s q(s)R_0(s)ds-\frac{\gamma_1}{4\alpha}\int_0^s q\left(\frac{\varsigma^-(x)+s}{2}\right)R_0\left(\frac{\varsigma^-(x)+s}{2}\right)ds$$

$$+\frac{i}{\alpha^2}\cdot p\left(\frac{t+\alpha p_1+p_1}{\alpha}\right)R_1\left(\frac{t+\alpha p_1+p_1}{\alpha}\right)+\frac{1}{2\alpha}\int_{p_1}^{\frac{u-\alpha p_1+p_1}{\alpha}} q(u)R_1(u)du+\frac{1}{2\alpha}\int_{p_1}^{x} q(u)R_2(u)du$$

$$+\frac{1}{2\alpha}\int_{p_1}^{\frac{\alpha x-\alpha p_1+p_1-u}{\alpha+1}} q(u)R_2(u)du-\frac{i}{\alpha(\alpha-1)}\cdot p\left(\frac{t-\varsigma^+(x)}{\alpha-1}\right)R_2\left(\frac{t-\varsigma^+(x)}{\alpha-1}\right)$$

$$K_2^{(1)}(x,t) = -i\cdot\beta_1^+ \int_{\frac{\varsigma^+(x)-t}{2}}^{p_1} p(u)K_2^{(0)}\left(u,u+t-\varsigma^+(x)\right)du + i\cdot\beta_1^+ \int_{\frac{\varsigma^+(x)+t}{2}}^{a_1} p(u)K_2^{(0)}\left(u,-u+t+\varsigma^+(x)\right)du$$

$$+\frac{\beta_1^+}{2}\int_0^{p_1} q(u)\int_{t-\varsigma^+(x)+u}^{t+\varsigma^+(x)-u} K_2^{(0)}(u,\tau)d\tau du - i\beta_1^- \int_{\frac{\varsigma^-(x)-t}{2}}^{p_1} p(u)K_2^{(0)}\left(u,t-\varsigma^-(x)+u\right)du$$

$$+i\cdot\beta_1^- \int_{\frac{\varsigma^-(x)+t}{2}}^{p_1} p(u)K_2^{(0)}\left(u,t+\varsigma^-(x)-u\right)du + \frac{\beta_1^-}{2}\int_0^{p_1} q(u)\int_{t-\varsigma^-(x)+u}^{t+\varsigma^-(x)-u} K_2^{(0)}(u,\tau)d\tau du$$

$$-i\frac{\gamma_1}{2\alpha}\int_{\frac{\mu^+(x)-t}{2}}^{a_1} p(u)K_2^{(0)}\left(u,t-\varsigma^+(x)+u\right)du - i\frac{\gamma_1}{2\alpha}\int_{\frac{\varsigma^+(x)+t}{2}}^{p_1} p(u)K_2^{(0)}\left(u,t+\varsigma^+(x)-u\right)du$$

$$+\frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^+(x)-s}{2}}^{p_1} q(u)\int_0^s K_2^{(0)}\left(u,s-\varsigma^+(x)+u\right)duds + \frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^+(x)+s}{2}}^{p_1} q(u)\int_0^s K_2^{(0)}\left(u,s+\varsigma^+(x)-u\right)duds$$

$$+i\frac{\gamma_1}{2\alpha}\int_{\frac{\varsigma^-(x)-t}{2}}^{p_1} p(u)K_2^{(0)}\left(u,t-\varsigma^-(x)+u\right)du + i\frac{\gamma_1}{2\alpha}\int_{\frac{\varsigma^-(x)+t}{2}}^{p_1} p(u)K_2^{(0)}\left(u,t+\varsigma^-(x)-u\right)du$$

$$-\frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^-(x)-s}{2}}^{p_1} q(u)\int_0^s K_2^{(0)}\left(u,s-\varsigma^-(x)+u\right)duds - \frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^-(x)+s}{2}}^{p_1} q(u)\int_0^s K_2^{(0)}\left(u,s+\varsigma^-(x)-u\right)duds$$

$$-\frac{i}{\alpha}\cdot\int_{\frac{\alpha p_1-p_1-t}{\alpha}}^{x} p(u)K_2^{(0)}\left(u,t-\alpha(x-t)\right)du - \frac{i}{\alpha}\cdot\int_{p_1}^{x} p(u)K_2^{(0)}\left(u,t-\alpha(x-t)\right)du$$

$$-\frac{i}{\alpha}\cdot\int_{p_1}^{\frac{\alpha x+\varsigma^+(x)-t}{\alpha}} p(u)K_2^{(0)}\left(u,t-\alpha(x-t)\right)du + \frac{i}{\alpha}\cdot\int_{p_1}^{\frac{t+\varsigma^+(x)+\alpha x}{\alpha}} p(u)K_2^{(0)}\left(u,\alpha(-t+x)+t\right)du$$

$$+\frac{i}{\alpha}\cdot\int_{p_1}^{x} p(u)K_2^{(0)}\left(u,\alpha(x-t)+t\right)du + \frac{i}{\alpha}\cdot\int_{\frac{t+\alpha p_1-p_1}{\alpha}}^{x} p(u)K_2^{(0)}\left(u,t+\alpha(x-t)\right)du$$

$$+\frac{1}{2\alpha}\int_{p_1}^{\frac{\varsigma^+(x)+\alpha x+t}{\alpha}} q(u)\int_{t-\alpha x+\alpha u}^{t-\alpha u+\alpha x} K_2^{(0)}(u,\tau)d\tau du + \frac{1}{2\alpha}\int_{p_1}^{\frac{\varsigma^+(x)-\alpha x-t}{\alpha}} q(u)\int_{t-\alpha x-\alpha u}^{t+\alpha u-\alpha x} K_2^{(0)}(u,\tau)d\tau du$$

...

$$K_2^{(n)}(x,t) = -i\cdot\beta_1^+ \int_{\frac{\varsigma^+(x)-t}{2}}^{p_1} p(u)K_2^{(n-1)}(u,t-\varsigma^+(x)+u)du + i\cdot\beta_1^+ \int_{\frac{\varsigma^+(x)+t}{2}}^{a_1} p(u)K_2^{(n-1)}(u,t+\varsigma^+(x)-u)du$$

$$+\frac{\beta_1^+}{2}\int_0^{p_1} q(u)\int_{t-\varsigma^+(x)+u}^{t+\varsigma^+(x)-u} K_2^{(n-1)}(u,\tau)d\tau du - i\beta_1^- \int_{\frac{\varsigma^-(x)-t}{2}}^{p_1} p(u)K_2^{(n-1)}(u,t-\varsigma^-(x)+u)du$$

$$+i\cdot\beta_1^- \int_{\frac{\varsigma^-(x)+t}{2}}^{p_1} p(u)K_2^{(n-1)}(u,t+\varsigma^-(x)-u)du + \frac{\beta_1^-}{2}\int_0^{p_1} q(u)\int_{t-\varsigma^-(x)+u}^{t+\varsigma^-(x)-u} K_2^{(n-1)}(u,\tau)d\tau du$$

$$-i\frac{\gamma_1}{2\alpha}\int_{\frac{\mu^+(x)-t}{2}}^{a_1} p(u)K_2^{(n-1)}(u,t+u-\varsigma^+(x)+u)du - i\frac{\gamma_1}{2\alpha}\int_{\frac{\varsigma^+(x)+t}{2}}^{p_1} p(u)K_2^{(n-1)}(u,t-u+\varsigma^+(x))du$$

$$+\frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^+(x)-s}{2}}^{p_1} q(u)\int_0^s K_2^{(n-1)}(u,s-\varsigma^+(x)+u)duds + \frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^+(x)+s}{2}}^{p_1} q(u)\int_0^s K_2^{(n-1)}(u,s+\varsigma^+(x)-u)duds$$

$$+i\frac{\gamma_1}{2\alpha}\int_{\frac{\varsigma^-(x)-t}{2}}^{p_1} p(u)K_2^{(n-1)}(u,u+t-\varsigma^-(x))du + i\frac{\gamma_1}{2\alpha}\int_{\frac{\varsigma^-(x)+t}{2}}^{p_1} p(u)K_2^{(n-1)}(u,t+\varsigma^-(x)-u)du$$

$$-\frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^-(x)-s}{2}}^{p_1} q(u)\int_0^s K_2^{(n-1)}(u,s-\varsigma^-(x)+u)duds - \frac{\gamma_1}{4\alpha}\int_{\frac{\varsigma^-(x)+s}{2}}^{p_1} q(u)\int_0^s K_2^{(n-1)}(u,s+\varsigma^-(x)-u)duds$$

$$-\frac{i}{\alpha}\cdot\int_{\frac{\alpha p_1-p_1-t}{\alpha}}^{x} p(u)K_2^{(n-1)}(u,t-\alpha(x-t))du - \frac{i}{\alpha}\cdot\int_{p_1}^{x} p(u)K_2^{(n-1)}(u,\alpha(t-x)+t)du$$

$$-\frac{i}{\alpha}\cdot\int_{p_1}^{\frac{\alpha x+\varsigma^+(x)-t}{\alpha}} p(u)K_2^{(n-1)}(u,t-\alpha(x-t))du + \frac{i}{\alpha}\cdot\int_{p_1}^{\frac{t+\alpha x+\varsigma^+(x)}{\alpha}} p(u)K_2^{(n-1)}(u,\alpha(x-t)+t)du$$

hence

$$+\frac{i}{\alpha}\cdot\int_{p_1}^{x} p(u)K_2^{(n-1)}(u,t+\alpha(x-t))du + \frac{i}{\alpha}\cdot\int_{\frac{t+\alpha p_1-p_1}{\alpha}}^{x} p(u)K_2^{(n-1)}(u,t+\alpha(x-t))du$$

$$+\frac{1}{2\alpha}\int_{p_1}^{\frac{t+\varsigma^+(x)+\alpha x}{\alpha}} q(u)\int_{t-\alpha x+\alpha u}^{t+\alpha x-\alpha u} K_2^{(n-1)}(u,\tau)d\tau du + \frac{1}{2\alpha}\int_{p_1}^{\frac{-t+\varsigma^+(x)-\alpha x}{\alpha}} q(u)\int_{t-\alpha x-\alpha u}^{t-\alpha x+\alpha u} K_2^{(n-1)}(u,\tau)d\tau du$$

$$\int_{-x}^{x}\left|K_2^{(0)}(x,t)\right|dt \le \left(\beta_1^+ + \left|\beta_1^-\right| + \frac{\gamma_1}{\alpha} + \frac{2}{\alpha}\right)\varpi(x) = \varpi_1(x)$$

$$\int_{-x}^{x}\left|K_2^{(0)}(x,t)\right|dt \le \varpi_1(x)\ .$$

As a result;

$$\int_{-x}^{x}\left|K_2(x,t)\right|dt \le \sum_{n=0}^{\infty}\int_{-x}^{x}\left|K_2^{(n)}(x,t)\right|dt \le 1-1+\varpi_1(x)+\frac{\varpi_1^{\,2}(x)}{2!}+\ldots+\frac{\varpi_1^{\,n+1}(x)}{(n+1)!}+\ldots=\mathrm{e}^{\varpi_1(x)}-1$$

$$\int_{-x}^{x}\left|K_2(x,t)\right|dt \le \mathrm{e}^{\varpi_1(x)}-1=e^{\left(\beta_1^{+}+\left|\beta_1^{-}\right|+\frac{\gamma_1}{\alpha}+\frac{2}{\alpha}\right)\int_0^x\left(2\left|p(t)\right|+(x-t)\left|q(t)\right|\right)dt}-1 \quad . \tag{15}$$

For $x\in(p_2,\pi)$

$$R_3(x)=\xi^{+}-i\left(\beta_1^{+}\beta_2^{+}+\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}p(t)R_0(t)\,dt-i\left(\frac{\gamma_1\beta_2^{+}}{2\alpha}+\frac{\gamma_2\beta_1^{+}}{2\beta}\right)\int_0^{p_1}p(t)R_0(t)\,dt$$
$$-i\left(\beta_2^{+}+\frac{\gamma_2}{2\beta}\right)\int_{p_1}^{p_2}p(t)R_1(t)\,dt-\frac{i}{\beta}\int_{p_2}^{x}p(t)R_3(t)\,dt$$

$$R_3(x)=c\cdot e^{-\frac{i}{\beta}\int_{p_2}^{x}p(t)dt}\;;c=\left(\beta_2^{+}+\frac{\gamma_2}{2\beta}\right)R_1(p_2) \tag{16}$$

$$R_4(x)=\xi^{-}-i\left(\beta_1^{-}\beta_2^{-}-\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}p(t)R_0(t)\,dt+i\left(\frac{\gamma_1\beta_2^{-}}{2\alpha}-\frac{\gamma_2\beta_1^{-}}{2\beta}\right)\int_0^{p_1}p(t)R_0(t)\,dt$$
$$+i\left(\beta_2^{+}+\frac{\gamma_2}{2\beta}\right)\int_{p_1}^{p_2}p(t)R_1(t)\,dt-\frac{i}{\beta}\int_{p_2}^{x}p(t)R_4(t)\,dt$$

$$R_4(x)=c\cdot e^{-\frac{i}{\beta}\int_{p_2}^{x}p(t)dt}\;;\;c=\left(\beta_2^{-}+\frac{\gamma_2}{2\beta}\right)R_2(p_2) \tag{17}$$

$$R_5(x)=\vartheta^{+}-i\left(\beta_1^{+}\beta_2^{-}-\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}p(t)R_0(t)\,dt-i\left(\frac{\gamma_1\beta_2^{-}}{2\alpha}-\frac{\gamma_2\beta_1^{+}}{2\beta}\right)\int_0^{p_1}p(t)R_0(t)\,dt$$
$$-i\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)\int_{p_1}^{p_2}p(t)R_1(t)\,dt+\frac{i}{\beta}\int_{p_2}^{x}p(t)R_5(t)\,dt$$

$$R_5(x)=c\cdot e^{\frac{i}{\beta}\int_{p_2}^{x}p(t)dt}\;;c=\left(\beta_2^{-}-\frac{\gamma_2}{2\beta}\right)R_1(p_2) \tag{18}$$

$$R_6(x)=\vartheta^{-}-i\left(\beta_1^{-}\beta_2^{+}+\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\int_0^{p_1}p(t)R_0(t)\,dt+i\left(\frac{\gamma_1\beta_2^{+}}{2\alpha}+\frac{\gamma_2\beta_1^{-}}{2\beta}\right)\int_0^{p_1}p(t)R_0(t)\,dt$$
$$+i\left(\beta_2^{+}-\frac{\gamma_2}{2\beta}\right)\int_{p_1}^{p_2}p(t)R_1(t)\,dt+\frac{i}{\beta}\int_{p_2}^{x}p(t)R_6(t)\,dt$$

$$R_6(x)=c\cdot e^{\frac{i}{\beta}\int_{p_2}^{x} p(t)dt}\ ;\ c=\left(\beta_2^{+}-\frac{\gamma_2}{2\beta}\right)R_2(p_2) \tag{19}$$

obtained. Let us apply the integral equation successive approximation method for $K_3(x,t)$;

$$K_3^{(0)}(x,t)=\left(\beta_1^{+}\beta_1^{+}+\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\left[iR_0\left(\frac{t+b^{+}(x)}{2}\right)p\left(\frac{t+b^{+}(x)}{2}\right)+\frac{1}{2}\int_0^{\frac{t+b^{+}(x)}{2}} R_0(u)q(u)du\right.$$
$$\left.+\frac{1}{2}\int_{\frac{t+b^{+}(x)}{2}}^{p_1} R_0(u)q(u)\right]+\left(\beta_1^{-}\beta_1^{-}-\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\left[iR_0\left(\frac{t+b^{-}(x)}{2}\right)p\left(\frac{t+b^{-}(x)}{2}\right)+\frac{1}{2}\int_0^{\frac{t+b^{-}(x)}{2}} R_0(u)q(u)du\right.$$
$$\left.+\frac{1}{2}\int_{\frac{t+b^{-}(x)}{2}}^{p_1} R_0(u)q(u)\right]+\left(\beta_1^{+}\beta_1^{-}-\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\left[iR_0\left(\frac{t+s^{+}(x)}{2}\right)p\left(\frac{t+s^{+}(x)}{2}\right)+\frac{1}{2}\int_0^{\frac{t+s^{+}(x)}{2}} R_0(u)q(u)du\right.$$
$$\left.+\frac{1}{2}\int_{\frac{t+s^{+}(x)}{2}}^{p_1} R_0(u)q(u)\right]+\left(\beta_1^{-}\beta_1^{+}+\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\left[iR_0\left(\frac{t+s^{-}(x)}{2}\right)p\left(\frac{t+s^{-}(x)}{2}\right)+\frac{1}{2}\int_0^{\frac{t+s^{-}(x)}{2}} R_0(u)q(u)du\right.$$
$$\left.+\frac{1}{2}\int_{\frac{t+s^{-}(x)}{2}}^{p_1} R_0(u)q(u)\right]-i\left(\frac{\gamma_1\beta_2^{+}}{2\alpha}+\frac{\gamma_2\beta_1^{+}}{2\beta}\right)\left[R_0\left(\frac{t+b^{+}(x)}{2}\right)p\left(\frac{t+b^{+}(x)}{2}\right)+\frac{1}{2}\int_0^{p_1} R_0(u)q(u)du\right.$$

$$+\frac{1}{2}\int_{-b^+(x)}^{2p_1-b^+(x)} R_0\left(\frac{s+b^+(x)}{2}\right)p\left(\frac{s+b^+(x)}{2}\right)ds\Bigg]+i\left(\frac{\gamma_1\beta_2^-}{2\alpha}-\frac{\gamma_2\beta_1^-}{2\beta}\right)\Bigg[R_0\left(\frac{t+b^-(x)}{2}\right)p\left(\frac{t+b^-(x)}{2}\right)$$

$$+\frac{1}{2}\int_0^{p_1} R_0(u)q(u)du+\frac{1}{2}\int_{-b^-(x)}^{2p_1-b^-(x)} R_0\left(\frac{s+b^-(x)}{2}\right)p\left(\frac{s+b^-(x)}{2}\right)ds\Bigg]$$

$$-i\left(\frac{\gamma_1\beta_2^-}{2\alpha}-\frac{\gamma_2\beta_1^+}{2\beta}\right)\Bigg[R_0\left(\frac{t+s^+(x)}{2}\right)p\left(\frac{t+s^+(x)}{2}\right)+\frac{1}{2}\int_0^{p_1} R_0(u)q(u)du$$

$$+\frac{1}{2}\int_{-s^+(x)}^{2p_1-s^+(x)} R_0\left(\frac{s+s^+(x)}{2}\right)p\left(\frac{s+s^+(x)}{2}\right)ds\Bigg]+i\left(\frac{\gamma_1\beta_2^+}{2\alpha}+\frac{\gamma_2\beta_1^-}{2\beta}\right)\Bigg[R_0\left(\frac{t+s^-(x)}{2}\right)p\left(\frac{t+s^-(x)}{2}\right)$$

$$+\frac{1}{2}\int_0^{p_1} R_0(u)q(u)du+\frac{1}{2}\int_{-s^-(x)}^{2p_1-s^-(x)} R_0\left(\frac{s+s^-(x)}{2}\right)p\left(\frac{s+s^-(x)}{2}\right)ds\Bigg]$$

$$+\beta_2^+\Bigg[R_1\left(\frac{t+(1+\alpha)p_1+(\alpha+\beta)p_2+\beta x}{2\alpha}\right)p\left(\frac{t+(1+\alpha)p_1+(\alpha+\beta)p_2+\beta x}{2\alpha}\right)$$

$$-iR_2\left(\frac{(1+\alpha)p_1+(\alpha-\beta)p_2+\beta x-t}{2\alpha}\right)p\left(\frac{(1+\alpha)p_1+(\alpha-\beta)p_2+\beta x-t}{2\alpha}\right)$$

$$+\frac{1}{2}\int_{p_1}^{p_2} R_1(u)q(u)du+\frac{1}{2}\int_{p_1}^{\frac{p_1+\alpha(p_1+p_2)+\beta(x-p_2)-t}{2\alpha}} R_2(u)q(u)du\Bigg]$$

$$+\beta_2^-\Bigg[-iR_1\left(\frac{(\alpha-1)p_1+(\alpha-\beta)p_2+\beta x+t}{2\alpha}\right)p\left(\frac{(\alpha-1)p_1+(\alpha-\beta)p_2+\beta x+t}{2\alpha}\right)$$

$$+iR_2\left(\frac{(1+\alpha)p_1+(\alpha+\beta)p_2-\beta x-t}{2\alpha}\right)p\left(\frac{(1+\alpha)p_1+(\alpha+\beta)p_2-\beta x-t}{2\alpha}\right)$$

$$+\frac{1}{2}\int_{p_1}^{p_2} R_1(u)q(u)du+\frac{1}{2}\int_{p_1}^{\frac{(1+\alpha)p_1+(\alpha+\beta)p_2-\beta x-t}{2\alpha}} R_2(u)q(u)du+\frac{1}{2}\int_{\frac{(1+\alpha)p_1+(\alpha+\beta)p_2-\beta x-t}{2\alpha}}^{p_1} R_2(u)q(u)du$$

$$+\frac{1}{2}\int_{p_1}^{p_2} R_1(u)q(u)du+\frac{1}{2}\int_{p_1}^{\frac{(1+\alpha)p_1+(\alpha+\beta)p_2-\beta x-t}{2\alpha}} R_2(u)q(u)du\Bigg]$$

$$+\frac{i}{\beta^2}R_3\left(\frac{t+\alpha(p_1-p_2)+\beta(x-p_2)-p_1}{2\beta}\right)p\left(\frac{t+\alpha(p_1-p_2)+\beta(x-p_2)-p_1}{2\beta}\right)$$

$$+\frac{i}{\beta^2}R_4\left(\frac{t+\alpha(p_2-p_1)+\beta(x+p_2)-p_1}{2\beta}\right)p\left(\frac{t+\alpha(p_2-p_1)+\beta(x+p_2)-p_1}{2\beta}\right)$$

$$+\frac{i}{\beta^2}R_5\left(\frac{p_1+\alpha(p_1-p_2)+\beta(x-p_2)-t}{2\beta}\right)p\left(\frac{p_1+\alpha(p_1-p_2)+\beta(x-p_2)-t}{2\beta}\right)$$

$$+\frac{i}{\beta^2}R_6\left(\frac{p_1+\alpha(p_1-p_2)+\beta(x+p_2)-t}{2\beta}\right)p\left(\frac{p_1+\alpha(p_1-p_2)+\beta(x+p_2)-t}{2\beta}\right)$$

$$+\frac{1}{2\beta}\int_{p_1}^{\frac{t+\alpha(p_1-p_2)+\beta(x+p_2)-p_1}{2\beta}}R_3(u)q(u)du+\frac{1}{2\beta}\int_{p_1}^{\frac{t+\alpha(p_2-p_1)+\beta(x+p_2)-p_1}{2\beta}}R_4(u)q(u)du$$

$$+\frac{1}{2\beta}\int_{p_1}^{\frac{(1-\alpha)p_1+(\alpha+\beta)p_2+\beta x-t}{2\beta}}R_5(u)q(u)du+\frac{1}{2\beta}\int_{p_1}^{\frac{(1+\alpha)p_1-(\alpha-\beta)p_2+\beta x-t}{2\beta}}R_6(u)q(u)du$$

$$-i\frac{\gamma_2}{2\beta}\Bigg[R_1\left(\frac{t+\alpha(p_1+p_2)+\beta(x-p_2)-p_1}{2\alpha}\right)p\left(\frac{t+\alpha(p_1+p_2)+\beta(x-p_2)-p_1}{2\alpha}\right)$$

$$+\frac{1}{2}\int_{p_1}^{p_2}R_1(u)q(u)du+\int_{s^-(x)}^{s^+(x)}R_1\left(\frac{s+\alpha(p_1+p_2)+\beta(x-p_2)-p_1}{2\alpha}\right)p\left(\frac{s+\alpha(p_1+p_2)+\beta(x-p_2)-p_1}{2\alpha}\right)ds$$

$$+R_2\left(\frac{p_1+\alpha(p_1+p_2)-\beta(x-p_2)-t}{2\alpha}\right)p\left(\frac{p_1+\alpha(p_1+p_2)-\beta(x-p_2)-t}{2\alpha}\right)+\frac{1}{2}\int_{p_1}^{p_2}R_2(u)q(u)du$$

$$\int_{b^-(x)}^{b^+(x)}R_1\left(\frac{s+\alpha(p_1+p_2)+\beta(x-p_2)-a_1}{2\alpha}\right)p\left(\frac{s+\alpha(p_1+p_2)+\beta(x-p_2)-p_1}{2\alpha}\right)ds\Bigg]$$

$$+i\frac{\gamma_2}{2\beta}\Bigg[R_1\left(\frac{t+\alpha(p_1+p_2)+\beta(p_2-x)+p_1}{2\alpha}\right)p\left(\frac{t+\alpha(p_1+p_2)+\beta(p_2-x)+p_1}{2\alpha}\right)$$

$$+\frac{1}{2}\int_{p_1}^{p_2}R_1(u)q(u)du+\int_{s^-(x)}^{s^+(x)}R_1\left(\frac{-s+\alpha(p_1+p_2)+\beta(x-p_2)+p_1}{2\alpha}\right)p\left(\frac{-s+\alpha(p_1+p_2)+\beta(x-p_2)+p_1}{2\alpha}\right)ds$$

$$+R_2\left(\frac{-t+\alpha(p_1+p_2)+\beta(p_2-x)+p_1}{2\alpha}\right)p\left(\frac{-t+\alpha(p_1+p_2)+\beta(p_2-x)+p_1}{2\alpha}\right)+\frac{1}{2}\int_{p_1}^{p_2}R_2(u)q(u)du$$

$$\int_{b^-(x)}^{b^+(x)}R_1\left(\frac{-s+\alpha(p_1+p_2)+\beta(p_2-x)+p_1}{2\alpha}\right)p\left(\frac{-s+\alpha(p_1+p_2)+\beta(p_2-x)+p_1}{2\alpha}\right)ds\Bigg]$$

$$K_3^{(1)}(x,t)=$$

$$\left(\beta_1^{+}\beta_1^{+}+\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\left[-i\int_{\frac{b^{+}(x)-t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,t-b^{+}(x)+u\right)du+i\int_{\frac{b^{+}(x)+t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,t+b^{+}(x)-u\right)du\right.$$

$$\left.+\frac{1}{2}\int_0^{p_1}q(u)\int_{t+b^{+}(x)-u}^{t-b^{+}(x)+u}K_3^{(0)}(u,\tau)d\tau du\right]+\left(\beta_1^{-}\beta_1^{-}-\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\left[-i\int_{\frac{b^{-}(x)-t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,t-b^{-}(x)+u\right)du\right.$$

$$\left.+i\int_{\frac{b^{-}(x)+t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,t+b^{-}(x)-u\right)du+\frac{1}{2}\int_0^{p_1}q(u)\int_{t+b^{-}(x)-u}^{t-b^{-}(x)+u}K_3^{(0)}(u,\tau)d\tau du\right]$$

$$+\left(\beta_1^{+}\beta_1^{-}-\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\left[-i\int_{\frac{s^{+}(x)-t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,u+t-s^{+}(x)\right)du+i\int_{\frac{s^{+}(x)+t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,-u+t+s^{+}(x)\right)du\right.$$

$$\left.+\frac{1}{2}\int_0^{p_1}q(u)\int_{t+s^{+}(x)-u}^{t-s^{+}(x)+u}K_3^{(0)}(u,\tau)d\tau du\right]+\left(\beta_1^{-}\beta_1^{+}+\frac{\gamma_1\gamma_2}{4\alpha\beta}\right)\left[-i\int_{\frac{s^{-}(x)-t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,t-s^{-}(x)+u\right)du\right.$$

$$\left.+i\int_{\frac{s^{-}(x)+t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,t+s^{-}(x)-u\right)du+\frac{1}{2}\int_0^{p_1}q(u)\int_{t+s^{-}(x)-u}^{t-s^{-}(x)+u}K_3^{(0)}(u,\tau)d\tau du\right]$$

$$-i\left(\frac{\gamma_1\beta_2^{+}}{2\alpha}+\frac{\gamma_2\beta_1^{+}}{2\beta}\right)\left[\int_{\frac{b^{+}(x)-t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,t-b^{+}(x)+u\right)du+\int_{\frac{b^{+}(x)+t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,t+b^{+}(x)-u\right)du\right.$$

$$\left.+\frac{1}{2}\int_{b^{+}(x)-2a_1}^{b^{+}(x)}q(u)\int_{\frac{b^{+}(x)+s}{2}}^{a_1}K_3^{(0)}\left(u,s-b^{+}(x)+u\right)duds+\frac{1}{2}\int_{-b^{+}(x)}^{2p_1-b^{+}(x)}q(u)\int_{\frac{b^{+}(x)+s}{2}}^{p_1}K_3^{(0)}\left(u,s-b^{+}(x)+u\right)duds\right]$$

$$+i\left(\frac{\gamma_1\beta_2^{-}}{2\alpha}-\frac{\gamma_2\beta_1^{-}}{2\beta}\right)\left[\int_{\frac{b^{-}(x)-t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,u+t-b^{-}(x)\right)du+\int_{\frac{b^{-}(x)+t}{2}}^{p_1}p(u)K_3^{(0)}\left(u,t+b^{-}(x)-u\right)du\right.$$

$$\left.+\frac{1}{2}\int_{b^{-}(x)-2a_1}^{b^{-}(x)}q(u)\int_{\frac{b^{-}(x)+s}{2}}^{p_1}K_3^{(0)}\left(u,s-b^{-}(x)+u\right)duds+\frac{1}{2}\int_{-b^{-}(x)}^{2p_1-b^{-}(x)}q(u)\int_{\frac{b^{-}(x)+s}{2}}^{p_1}K_3^{(0)}\left(u,s-p^{-}(x)+u\right)duds\right]$$

$$-i\left(\frac{\gamma_1\beta_2^-}{2\alpha}-\frac{\gamma_2\beta_1^+}{2\beta}\right)\left[\int_{\frac{s^+(x)-t}{2}}^{p_1} p(u)K_3^{(0)}\left(u,u+t-s^+(x)\right)du+\int_{\frac{s^+(x)+t}{2}}^{p_1} p(u)K_3^{(0)}\left(u,-u+t+s^+(x)\right)du\right.$$

$$\left.+\frac{1}{2}\int_{s^+(x)-2p_1}^{s^+(x)} q(u)\int_{\frac{s^+(x)+s}{2}}^{p_1} K_3^{(0)}\left(u,s-s^+(x)+u\right)duds+\frac{1}{2}\int_{-s^+(x)}^{2p_1-s^+(x)} q(u)\int_{\frac{s^+(x)+s}{2}}^{p_1} K_3^{(0)}\left(u,s-s^+(x)+u\right)duds\right]$$

$$+i\left(\frac{\gamma_1\beta_2^+}{2\alpha}+\frac{\gamma_2\beta_1^-}{2\beta}\right)\left[\int_{\frac{s^-(x)-t}{2}}^{p_1} p(u)K_3^{(0)}\left(u,u+t-s^-(x)\right)du+\int_{\frac{s^-(x)+t}{2}}^{p_1} p(u)K_3^{(0)}\left(u,-u+t+s^-(x)\right)du\right.$$

$$\left.+\frac{1}{2}\int_{s^-(x)-2p_1}^{s^-(x)} q(u)\int_{\frac{s^-(x)+s}{2}}^{p_1} K_3^{(0)}\left(u,s-s^-(x)+u\right)duds+\frac{1}{2}\int_{-s^-(x)}^{2p_1-s^-(x)} q(u)\int_{\frac{s^-(x)+s}{2}}^{p_1} K_3^{(0)}\left(u,s-s^-(x)+u\right)duds\right]$$

$$+\beta_2^+\left[-i\int_{p_1}^{p_2} p(u)K_3^{(0)}(u,u-E)du-i\int_{p_1}^{\frac{\beta(x-p_2)+\alpha(p_1+p_2)+p_1-t}{2\alpha}} p(u)K_3^{(0)}(u,u-E)du\right.$$

$$\left.+i\int_{p_1}^{\frac{\beta(x+p_2)+\alpha(p_1+p_2)-p_1+t}{2\alpha}} p(u)K_3^{(0)}(u,u+E)du+i\int_{p_1}^{p_2} p(u)K_3^{(0)}(u,u+E)du+\frac{1}{2}\int_{p_1}^{p_2} q(u)\int_{u+E}^{u-E} K_3^{(0)}(t,\tau)d\tau du\right]$$

$$+\beta_2^-\left[-i\int_{\frac{-\beta(x-p_2)+\alpha(p_1+p_2)+p_1-t}{2\alpha}}^{p_2} p(u)K_3^{(0)}(u,u-F)du+i\int_{\frac{\beta(p_2-x)+\alpha(p_1+p_2)+p_1-t}{2\alpha}}^{p_2} p(u)K_3^{(0)}(u,u-F)du\right.$$

$$\left.+\frac{1}{2}\int_{p_1}^{p_2} q(u)\int_{u-F}^{u+F} K_3^{(0)}(t,\tau)d\tau du\right]-\frac{i}{\beta}\int_{p_2}^{\frac{\beta x-\alpha p_1-p_1-t}{\alpha-\beta}} p(u)K_3^{(0)}\left(u,u+\beta(t-x)\right)du$$

$$-\frac{i}{\beta}\int_{p_2}^{x} p(u)K_3^{(0)}\left(u,u+\beta(t-x)\right)du-\frac{i}{\beta}\int_{p_2}^{\frac{\beta x-\alpha p_1+p_1-t}{\alpha-\beta}} p(u)K_3^{(0)}\left(u,u+\beta(t-x)\right)du$$

$$+\frac{i}{\beta}\int_{p_2}^{\frac{-\beta x-\alpha p_1+p_1-t}{\alpha-\beta}} p(u)K_3^{(0)}\left(u,u+\beta(x-t)\right)du+\frac{i}{\beta}\int_{p_2}^{x} p(u)K_3^{(0)}\left(u,u+\beta(x-t)\right)du$$

$$+\frac{i}{\beta}\int_{p_2}^{\frac{-\beta x-\alpha p_1+p_1-t}{\beta-\alpha}} p(u)K_3^{(0)}\left(u,u+\beta(x-t)\right)du$$

$$+\frac{1}{2\beta}\int_{p_2}^{\frac{-\beta x-\alpha p_1+p_1-t}{\alpha-\beta}} q(u)\int_{u+\beta(t-x)}^{u+\beta(x-t)} K_3^{(0)}(u,\tau)d\tau du+\frac{1}{2\alpha}\int_{p_2}^{x} q(u)\int_{u+\beta(t-x)}^{u+\beta(x-t)} K_3^{(0)}(u,\tau)d\tau du$$

$$+\frac{1}{2\alpha}\int_{a_2}^{\frac{\beta x-\alpha p_1+p_1-t}{\beta-\alpha}} q(u)\int_{u+\beta(t-x)}^{u+\beta(x-t)} K_3^{(0)}(u,\tau)d\tau du - i\frac{\gamma_2}{2\beta}\left[\int_{p_1}^{p_2} p(u)K_3^{(0)}(u,u-E)du+\int_{p_1}^{p_2} p(u)K_3^{(0)}(u,u+E)du\right.$$

$$+\int_{p_1}^{\frac{(\alpha-1)p_1+(\alpha+\beta)p_2+\beta x+t}{2\alpha}} p(u)K_3^{(0)}(u,u+E)du+\int_{p_1}^{\frac{\beta(x-p_2)+\alpha(p_1+p_2)+p_1-t}{2\alpha}} p(u)K_3^{(0)}(u,u-E)du$$

$$+\frac{1}{2}\int_{E+\varsigma^+(x)}^{-s^+(t)}\left(\int_{p_1}^{p_2} p(u)K_3^{(0)}(u,s+E)du\right)ds+\frac{1}{2}\int_{-s^+(x)}^{-s^-(x)}\left(\int_{p_1}^{\frac{\beta(x-p_2)+\alpha(p_1+p_2)+p_1-s}{2\alpha}} p(u)K_3^{(0)}(u,s-E)du\right)ds$$

$$\left.+\frac{1}{2}\int_{s^-(x)}^{s^+(x)}\left(\int_{p_1}^{\frac{\beta(x+p_2)+\alpha(p_1+p_2)-s}{2\alpha}} p(u)K_3^{(0)}(u,s+E)du\right)ds+\frac{1}{2}\int_{p_1}^{p_2} q(u)\int_{u+E}^{u-E} K_3^{(0)}(t,\tau)d\tau du\right]$$

$$+i\frac{\gamma_2}{2\beta}\left[\int_{\frac{-\beta(x-p_2)+\alpha(p_1+p_2)+a-p_1-t}{2\alpha}}^{p_2} p(u)K_3^{(0)}(u,u-F)du+\int_{\frac{\beta(p_2-x)+\alpha(p_1+p_2)+p_1-t}{2\alpha}}^{a_2} p(u)K_3^{(0)}(u,u+F)du\right.$$

$$+\frac{1}{2}\int_{b^-(x)}^{b^+(x)}\left(\int_{\frac{-\beta(x-p_2)+\alpha(p_1+p_2)+p_1-s}{2\alpha}}^{p_2} p(u)K_3^{(0)}(u,s-F)du\right)ds$$

$$\left.+\frac{1}{2}\int_{b^-(x)}^{b^+(x)}\left(\int_{\frac{\beta(p_2-x)+\alpha(p_1-p_2)+p_1-s}{2\alpha}}^{p_2} p(u)K_3^{(0)}(u,s+F)du\right)ds\right]$$

where $E=\beta x-\beta p_2+\alpha p_2-\alpha t$, $F=\beta x-\beta p_2-\alpha p_2+\alpha t$. In that case

$$\int_{-x}^{x}\left|K_3^{(0)}(x,t)\right|dt\le\left(\alpha_2\left(\beta_1^++\left|\beta_1^-\right|\right)+\frac{1}{\alpha}\left(\beta_2^++\left|\beta_2^-\right|\right)+\frac{\beta^+}{\beta}+\frac{\gamma_2}{\beta}\right)\varpi(x)=\varpi_2(x).$$

As a result,

$$\int_{-x}^{x}\left|K_3(x,t)\right|dt\le\sum_{n=0}^{\infty}\int_{-x}^{x}\left|K_{3_n}(x,t)\right|dt\le 1-1+\varpi_2(x)+\frac{\varpi_2{}^2(x)}{2!}+\ldots+\frac{\varpi_2{}^{n+1}(x)}{(n+1)!}+\ldots=e^{\varpi_2(x)}-1$$

$$\int_{-x}^{x}\left|K_3(x,t)\right|dt\le e^{\varpi_2(x)}-1=e^{\left(\alpha_2\left(\beta_1^++\left|\beta_1^-\right|\right)+\frac{1}{\alpha}\left(\beta_2^++\left|\beta_2^-\right|\right)+\frac{\beta^+}{\beta}+\frac{\gamma_2}{\beta}\right)\int_0^x\left((x-k)\left|q(k)\right|+2\left|p(k)\right|\right)dk}-1 \quad (20)$$

we get the needed inequalities .

REFERENCES

[1]. Jdanovich,B. F. Formulae for the zeros of Drichlet polynomials and quasi-polynomials, Doklady Akad. Nauk SSSR. 135 (1960), 1046–1049.

[2]. Levitan,B. M. and Gasymov,M. G. Determination of a differantial equation by two spctra, Uspekhi Mathematicheskikh Nauk. 19 (1964), 3-63.

[3]. Levitan,B. M. and Sargsyan ,I. S. Introduction to Spectral Theory. Amer. Math. Soc., (1975).

[4]. Levitan,B. M. Inverse Sturm-Liouville Problems. Nauka, Moscow, Russia, (1987).

[5]. Naimark, M. A. Linear Differantial Operators. London, Toronto, Harrap, (1968).

[6]. Yang,C. F. Reconstruction of the diffusion operator from nodal data. Zeitschrift für Naturforschung, 65 (2010), 100–106.

[7]. Borg,G. Eine Umkehrung der Sturm-Lİouvilleschen Eigenwertaufgabe. Acta Mathematica, 78 (1946), 1-96.

[8]. Freiling,G. and Yurko,V. Inverse spectral problems for singular non-selfadjoint differential operators with discontinuities in an interior point. Inverse Problems, 18 (2002), 757–773.

[9]. Guseinov,G. S. On the spectral analysis of a quadratic pencil of Sturm-Liouville operators. Dodlady Akad. Nauk SSSR, 285 (1985), 1292-1296.

[10]. Guseinov,G. S. Inverse spectral problems for a quadratic pencil of Sturm-Lİouville operators on a finite interval. Spectral Theory of Operators and Its Applications. Elm, Baku, Azerbaijan (1986) 51-101.

[11]. Koyunbakan,H. and Panakhov,E. S. Half-inverse problem for diffusion operators on the finite interval. J. of Math. Anal. and App., 326 (2007), 1024–1030.

[12]. McLaughin ,J. R. Analyical methods for recovering coefficients in differantial equations from spectral data. SIAM 28 (1986) , 53-72.

[13]. Aydemir,K. and Muhtaroglu,O. Asymptotic distribution of eigenvalues and eigenfunctions for a multi point discontinuous Sturm Liouville Problem. Electronic Journal Of Differential Equations, 131 (2016), 1-14.

[14]. Anderson,L. Inverse eigenvalue problems with discontiuous coefficients. Inverse Problems 4 (1998), 353-397.

[15]. Gasymov,M. G. and Guseinov,G. S. Determination of a diffusion operatör from spectral data. Akad. Nauk Azerbaidzhanskoi SSR, 37 (1981), 19-23.

[16]. Nabiev,M. I. Inverse periodic Problem for a diffusion operatör. Transactions of Acad. of science of Azerbaijan, 23 (2003), 125-130.

[17]. Nabiev,M. I. The inverse spectral problem for he diffusion operatör on an interval. Mathematicheskaya Fizika, Analiz, Geometriya, 11 (2004), 302-313.

[18]. Krein,M. and Levin,B. Ya. On entire almost periodic functions of exponential type. Doklady Akad. Nauk SSSR, 64 (1949), 285–287.

[19]. Guliyev,N. J. Inverse eigenvalue problems for Sturm–Liouville equations with spectral parameter linearly contained in one of the boundary conditions. *Inverse Problems*, 21:4 (2005), 1315–1330

[20]. Hald,O. H. Discontinuous inverse eigenvalue problems. Comm. on Pure and Appl. Math., 37 (1984), 539–577.

[21]. Carlson,R. An inverse spectral problem for Sturm-Liouville operators with discontinuous coefficients. Proceedings of the Amer. Math. Soc., 120 (1994), 475-484.

[22]. Amirov,R. Kh. and Yurko,V. A. On differential operators with a singularity and discontinuity conditions inside an interval. Ukrainian Math. J., 53 (2001), 1443–1457.

[23]. Amirov,R. Kh. and Nabiev,A. A. Inverse problems for quadratic pencil of the Sturm-Liouville Equations with Impulse. J. of Abst. and Anal. (2013), doi:10.1155/2013/361989.

[24]. Amirov,R. Kh. and Ergun,A. İki noktada süreksizlik koşullarına sahip difüzyon denkleminin çözümleri için integral gösterilim. Cumhuriyet Sci. J. 36-5 (2015), 71-85. DOI: 10.17776/csj.46267

[25]. Buterin,S. A. and Yurko,V. A. Inverse spectral problem for pencils of differential operators on a finite interval. Vesn. Bashkir. Univ. 4 (2006), 8–12.

[26]. Ambartsumyan,V. A. Über eine frage der eigenwerttheorie. Zeitschrift für Physik,53 (1929), 690-695.

[27]. Marchenko,V. A. Sturm-Liouville Operators and Applications. AMS Chelsea Publishing, (1986).

[28]. Yurko,V. A. Inverse spectral problems for linear differential operators and their applications. Gordon and Breach, New York, (2000).

[29]. Yurko,V. A. Introduction to the Theory of Inverse Spectral Problems. Fizmatlit, Moscow, Russia, (2007).

[30]. Fulton,C. T.Singular eigenvalue problems with eigenvalue parameter contained in the boundary conditions, Proc. R. Soc. Edinburgh, A87, (1980),1-34.