Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.39-3(2018) 581-596

Approximation by Double Deferred Nörlund Means of Double Fourier Series for Lipschitz Functions

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Received: 02.07.2018; Accepted: 24.09.2018 *http://dx.doi.org/10.17776/csj.439832*

Abstract. In this paper, the concept of Double Deferred Nörlund means is defined and some important results are obtained. Particularly, we investigate the rate of uniform approximation by Double Deferred Nörlund means of the rectangular partial sums of the double Fourier series of a function $f(x,y)$ belong to Lip_α $0<\alpha\le 1$ on the two dimensional region $-\pi \le x,y \le \pi$. We also obtain the rate of uniform approximation by Double Deferred Cesáro means.

2010 Mathematics Subject Classification: 40C05, 42A25, 42A10, 44A99

***Keywords*:** Functions of bounded variation, Ostrowski type inequalities, Riemann-Stieltjes integrals.

Lipschitz Fonksiyonları için Çift İndisli Fourier Serilerinin Double Deferred Nörlund Ortalamasıyla Yaklaşım

Özet. Bu çalışmada, Double Deferred Nörlund ortalaması kavramı tanımlandı ve bazı önemli sonuçlar elde edildi. Özellikle, iki boyutlu $-\pi \le x,y \le \pi$ torus bölgesinde Lip_α $0<\alpha\le 1$ sınıfına ait $f(x,y)$ fonksiyonunun çift indisli Fourier serisinin dikdörtgensel kısmi toplamlarının çift indisli Deferred Nörlund ortalamasıyla düzgün yaklaşım oranını araştırıyoruz. Ayrıca; Double Deferred Cesáro ortalaması yardımı ile düzgün yaklaşım oranı elde ediyoruz.

***Anahtar Kelimeler*:** Sınırlı varyasyonlu fonksiyon, Ostrowski tipli eşitsizlikler, Riemann-Stieltjes integralleri.

1. INTRODUCTION

In [3], the first study on double sequences was examined by Bromwich. And then it was investigated by many authors such as Hardy [7], Moricz [14], Tripathy [19], Başarır and Sonalcan [2]. The concept of regular convergence for double sequences was defined by Hardy [7]. Many recent improvements containing the summability by four dimensional matrices might be found in [25]. Various approaches of the Fourier series have been studied by various authors [8-11, 17, 21-23]. In [4], Deferred Cesáro mean $D_{\beta,\gamma}$ (Double Deferred Cesáro mean) for a double sequence $x=(x_{nm})$ is defined and several theorems on this subject are given. Let $\{p_{jk} : j,k=0,1,\dots\}$ be a double sequence of nonnegative numbers $p_{00}>0$. Its partial sum is defined as

$$P_{nm}=\sum_{j=0}^{n}\sum_{k=0}^{m} p_{jk} \qquad (m,n=0,1,\dots).$$

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Let $\{s_{jk} : j,k = 0,1,\ldots\}$ be a double sequence of complex numbers. The Nörlund means N_{nm} are defined by

$$N_{nm} = \sum_{j=0}^{n}\sum_{k=0}^{m} p_{n-j,m-k} s_{jk}.$$

Let $f(x,y)$ be a complex valued function of period 2π with respect to each of the variables, and integrable defined on the two dimensional real torus $Q := \{(x,y)\epsilon\mathbb{R}^2 : -\pi < x \leq \pi, -\pi < y \leq \pi\}$, i.e., $f \in L^1(Q)$. We consider the double Fourier series of f defined by

$$f(x,y) \approx \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)} \tag{1.1}$$

where

$$c_{jk} = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v) e^{-i(ju+kv)} du dv \qquad (j,k = \ldots,-1,0,1,\ldots).$$

We write the double sequence of (symmetric) rectangular partial sums for the series (1.1) as follows

$$s_{nm}(x,y) = \sum_{j=-n}^{n}\sum_{k=-m}^{m} c_{jk} e^{i(jx+ky)} \qquad (m,n = 0,1,\ldots).$$

We say that the function f belongs to Lipschitz class of order α for some $\alpha > 0$, if

$$\begin{aligned} w(\delta; f) &= \sup_{(x,y)\in Q}\ \sup_{\{u^2+v^2\}^{1/2} \leq \delta} \left| f(x+u, y+v) - f(x,y) \right| \\ &\leq C\delta^{\alpha} \qquad (\delta > 0) \end{aligned} \tag{1.2}$$

where the constant C does not depend on δ. The quantity $w(\delta; f)$ is called the (total) modulus of continuity of the function f.

Clearly, if $f \in Lip\alpha$ for some $\alpha > 0$, then f is necessarily continuous everywhere. Only the case $0 < \alpha \leq 1$ is interesting. If $\alpha > 1$, then $\partial f/\partial x$ and $\partial f/\partial y$ exist and are zero everywhere, so f must be a constant.

Condition (1.2) can be rewritten as

$$\left| f(x+u, y+v) - f(x,y) \right| \leq C\{u^2+v^2\}^{\alpha/2}$$

for every real x, y, u, v; or equivalently,

$$\left| f(x+u, y+v) - f(x,y) \right| \leq C\left(|u|^{\alpha} + |v|^{\alpha}\right).$$

From above inequalities yield

$$\left|\phi(u,v)\right| \leq C\left(|u|^{\alpha} + |v|^{\alpha}\right). \tag{1.3}$$

We will use some well-known estimates.

$D_j(u)$ is the Dirichlet kernels in terms of u

$$D_j(u)=\frac{1}{2}+\sum_{\sigma=1}^{j}\cos\sigma u=\frac{\sin\left(j+\frac{1}{2}\right)u}{2\sin\frac{1}{2}u} \qquad (j=0,1,\ldots).$$

For $j=0,1,\ldots$,

$$\left|D_j(u)\right|< j+1 \qquad \text{for every } u. \tag{1.4}$$

For $a,b=0,1,\ldots;\ a\le b$,

$$\sum_{j=a}^{b}\sin\left(j+\frac{1}{2}\right)u=\frac{\cos au-\cos(b+1)u}{2\sin\frac{1}{2}u}.$$

By the following inequalities

$$\frac{\sin u}{u}\ge\frac{2}{\pi} \qquad 0<u\le\frac{\pi}{2}, \tag{1.5}$$

we have

$$\left|\sum_{j=a}^{b}\sin\left(j+\frac{1}{2}\right)u\right|\le\frac{\pi}{u} \qquad \text{for } 0<u\le\pi \tag{1.6}$$

and we obtain

$$\left|\sum_{j=a}^{b}D_j(u)\right|=\frac{\cos au-\cos(b+1)u}{\left(2\sin\frac{1}{2}u\right)^2}\le\frac{\pi^2}{2u^2} \qquad \text{for } 0<u\le\pi. \tag{1.7}$$

2. MAIN RESULTS

Motivating by Moricz [13] we make the following definition.

Let $(a_n),(b_n),(t_m)$ and (r_m) are sequences of nonnegative integers satisfying the conditions $a_n<b_n,\ t_m<r_m$ and $\lim_{n\to\infty}b_n=\infty,\ \lim_{m\to\infty}r_m=\infty$. Let $\{p_{jk}:\ j,k=0,1,\ldots\}$ be a double sequence of nonnegative numbers such that $p_{00}>0$. Then $P_{nm}^{\beta\gamma}$ is defined as follows,

$$P_{nm}^{\beta\gamma}=\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m}p_{jk} \qquad (n,m=0,1,\ldots)$$

where $\beta(n)=(a_n,b_n)$ and $\gamma(m)=(t_m,r_m)$.

Now, we have the following definition

Definition 2.1. *Let $\{s_{jk}:\ j,k=0,1,\ldots\}$ be a double sequence of complex numbers; Double Deferred Nörlund means $D_\beta^\gamma N_{nm}$ is defined as follows*

$$D_\beta^\gamma N_{nm}=\frac{1}{P_{nm}^{\beta\gamma}}\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m}p_{b_n-j,r_m-k}s_{jk}.$$

It is clear that $D_{\beta}^{\gamma}N_{nm}$ method is regular.

The $D_{\beta}^{\gamma}N_{nm}$ method for the double sequence $\left\{s_{nm}\left(x,y\right)\right\}$ are defined as

$$D_{\beta}^{\gamma}N_{nm}\left(x,y\right)=\frac{1}{P_{nm}^{\beta\gamma}}\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m}p_{b_n-j,r_m-k}s_{jk}\left(x,y\right) \qquad \left(n,m=0,1,\ldots\right).$$

The following representation is important, for $n,m=0,1,\ldots$

$$D_{\beta}^{\gamma}N_{nm}\left(x,y\right)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f\left(x+u,y+v\right)D_{\beta}^{\gamma}K_{nm}\left(u,v\right)dudv \qquad (2.1)$$

where $D_{\beta}^{\gamma}N_{nm}$ -kernel of $D_{\beta}^{\gamma}K_{nm}\left(u,v\right)$ is defined by

$$D_{\beta}^{\gamma}K_{nm}\left(u,v\right)=\frac{1}{P_{nm}^{\beta\gamma}}\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m}p_{b_n-j,r_m-k}D_j\left(u\right)D_k\left(v\right) \qquad \left(n,m=0,1,\ldots\right).$$

From (2.1), we have

$$D_{\beta}^{\gamma}N_{nm}\left(x,y\right)-f\left(x,y\right)=\frac{4}{\pi^2}\int_{0}^{\pi}\int_{0}^{\pi}\phi\left(u,v\right)D_{\beta}^{\gamma}K_{nm}\left(u,v\right)dudv \qquad (2.2)$$

where

$$\phi\left(u,v\right)=\frac{1}{4}\left\{f\left(x+u,y+v\right)+f\left(x-u,y+v\right)+f\left(x+u,y-v\right)+f\left(x-u,y-v\right)-4f\left(x,y\right)\right\}.$$

We will use the notations for $j,k=0,1,\ldots,$

$$\Delta_{10}p_{jk}=p_{jk}-p_{j+1,k},$$
$$\Delta_{01}p_{jk}=p_{jk}-p_{j,k+1},$$

and

$$\Delta_{11}p_{jk}=p_{jk}-p_{j+1,k}-p_{j,k+1}+p_{j+1,k+1}.$$

The double sequence $\left\{p_{jk}\right\}$ is nondecreasing if $\Delta_{10}p_{jk}\le 0$ and $\Delta_{01}p_{jk}\le 0$, and is nonincreasing if $\Delta_{10}p_{jk}\ge 0$ and $\Delta_{01}p_{jk}\ge 0$ for every $j,k=0,1,\ldots$. We also set, for $n,m=0,1,\ldots$

$$D_{\beta}^{\gamma}q_{nm}=\frac{1}{P_{nm}^{\beta\gamma}}\sum_{k=t_m+1}^{r_m}p_{b_n k}, \qquad D_{\beta}^{\gamma}r_{nm}=\frac{1}{P_{nm}^{\beta\gamma}}\sum_{j=a_n+1}^{b_n}p_{jr_m}.$$

Let us consider the case where $\left\{p_{jk}\right\}$ is nondecreasing. Then

$$\begin{aligned}(b_n-a_n)D_\beta^\gamma q_{nm} &= \frac{(b_n-a_n)}{P_{nm}^{\beta\gamma}}\sum_{k=t_m+1}^{r_m} p_{b_nk} \\ &\geq \frac{1}{P_{nm}^{\beta\gamma}}\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m} p_{jk} = 1\end{aligned} \tag{2.3}$$

and similarly,

$$(r_m-t_m)D_\beta^\gamma r_{nm} \geq 1. \tag{2.4}$$

We also have

$$P_{nm}^{\beta\gamma} \leq (b_n-a_n)(r_m-t_m)\,p_{b_nr_m} \qquad (n,m=0,1,\ldots).$$

In the sequel, we need the opposite inequality:

$$\frac{(b_n-a_n)(r_m-t_m)\,p_{b_nr_m}}{P_{nm}^{\beta\gamma}} = O(1) \quad . \tag{2.5}$$

The opposite inequality is proved for the following sequence defined below,

$$p_{jk} = (b_n-j+1)^\sigma (r_m-k+1)^\rho \qquad (\sigma,\rho \geq 0).$$

Condition (2.5) implies that

$$\begin{aligned}(b_n-a_n)D_\beta^\gamma q_{nm} &= \frac{(b_n-a_n)}{P_{nm}^{\beta\gamma}}\sum_{k=t_m+1}^{r_m} p_{b_nk} \\ &\leq \frac{(b_n-a_n)(r_m-t_m)}{P_{nm}^{\beta\gamma}}\,p_{b_nr_m} = O(1)\end{aligned} \tag{2.6}$$

and

$$(r_m-t_m)D_\beta^\gamma r_{nm} = O(1). \tag{2.7}$$

In particular, the conditions of regularity are satisfied:

$$\lim_{n,m\to\infty} D_\beta^\gamma q_{nm} = \lim_{n,m\to\infty} D_\beta^\gamma r_{nm} = 0.$$

Thus, we may assume that

$$D_\beta^\gamma q_{nm} < \pi \quad \text{and} \quad D_\beta^\gamma r_{nm} < \pi \qquad (n,m=0,1,\ldots).$$

If $\{p_{jk}\}$ is nonincreasing, then

$$(b_n-a_n)D_\beta^\gamma q_{nm} \leq 1 \quad \text{and} \quad (r_m-t_m)D_\beta^\gamma r_{nm} \leq 1. \tag{2.8}$$

In the special case where

$$\lim_{n,m\to\infty} p_{nm} > 0, \tag{2.9}$$

we have

$$\frac{1}{(b_n-a_n)D_\beta^\gamma q_{nm}} \le \frac{p_{a_n+1,t_m+1}}{p_{b_n r_m}} = O(1) \quad \text{and} \quad \frac{1}{(r_m-t_m)D_\beta^\gamma r_{nm}} \le \frac{p_{a_n+1,t_m+1}}{p_{b_n r_m}} = O(1). \qquad (2.10)$$

The following result is analogue of a result of Moore $[12, \textit{page } 39]$ which follows almost the same lines, so the details are omitted.

Theorem 2.2. *If* $\{p_{jk} : j,k=0,1,\ldots,\ p_{00}>0\}$, *then the necessary and sufficient conditions for the regularity of the* $D_\beta^\gamma N_{nm}$ *-method of summability are*

$$\lim_{n,m\to\infty} \frac{1}{P_{nm}^{\beta\gamma}} \sum_{k=t_m+1}^{r_m} p_{b_n-j,k} = 0 \qquad (j=0,1,\ldots;b_n \ge j)$$

and

$$\lim_{n,m\to\infty} \frac{1}{P_{nm}^{\beta\gamma}} \sum_{j=a_n+1}^{b_n} p_{j,r_m-k} = 0 \qquad (k=0,1,\ldots;r_m \ge k).$$

Lemma 2.3. *Let* $\{p_{jk}>0 : j,k=0,1,\ldots\}$ *be a nondecreasing double sequence such that* $\Delta_{11}p_{jk}$ *is of fixed sign. Then*

$$\left|D_\beta^\gamma K_{nm}(u,v)\right| \le (b_n+1)+(r_m+1), \qquad \forall u,v \qquad (2.11)$$

$$\le \frac{\pi^2}{2u^2 P_{nm}^{\beta\gamma}} \sum_{k=t_m+1}^{r_m} (k+1)\left[p_{b_n-a_n-1,r_m-k-1} - p_{0,r_m-k}\right], \qquad \forall v \text{ and } 0<u\le\pi \qquad (2.12)$$

$$\le \frac{\pi^2}{2v^2 P_{nm}^{\beta\gamma}} \sum_{j=a_n+1}^{b_n} (j+1)\left[p_{b_n-j-1,r_m-t_m-1} - p_{b_n-j,0}\right], \qquad \forall u \text{ and } 0<v\le\pi \qquad (2.13)$$

$$\le \frac{5\pi^4 p_{b_n r_m}}{4u^2v^2 P_{nm}^{\beta\gamma}}, \qquad 0<u,v\le\pi. \qquad (2.14)$$

Proof. From (1.4),

$$\begin{aligned}
\left|D_\beta^\gamma K_{nm}(u,v)\right| &\le \frac{1}{P_{nm}^{\beta\gamma}} \sum_{j=a_n+1}^{b_n} \sum_{k=t_m+1}^{r_m} p_{b_n-j,r_m-k} \left|D_j(u)\right|\left|D_k(v)\right| \\
&\le \frac{1}{P_{nm}^{\beta\gamma}} \sum_{j=a_n+1}^{b_n} \sum_{k=t_m+1}^{r_m} (j+1)(k+1)\, p_{b_n-j,r_m-k} \\
&\le \frac{(b_n+1)(r_m+1)}{P_{nm}^{\beta\gamma}} \sum_{j=a_n+1}^{b_n} \sum_{k=t_m+1}^{r_m} p_{jk} \\
&= (b_n+1)(r_m+1)
\end{aligned}$$

which is (2.11).

Again from (1.4), we have

$$P_{nm}^{\beta\gamma}\left|D_{\beta}^{\gamma}K_{nm}(u,v)\right| \leq \sum_{k=t_m+1}^{r_m}\left|\sum_{j=a_n+1}^{b_n} p_{b_n-j,r_m-k}D_j(u)\right|\left|D_k(v)\right| \leq \sum_{k=t_m+1}^{r_m}(k+1)\left|\sum_{j=a_n+1}^{b_n} p_{b_n-j,r_m-k}D_j(u)\right|. \tag{2.15}$$

If we apply the Abel's transformation to the inner sum, for each k

$$\sum_{j=a_n+1}^{b_n} p_{b_n-j,r_m-k}D_j(u) = \left(\sum_{j=0}^{b_n}-\sum_{k=0}^{r_m}\right)p_{b_n-j,r_m-k}D_j(u) = -\sum_{j=1}^{b_n}\Delta_{10}p_{b_n-j,r_m-k}\sum_{l=0}^{j-1}D_l(u)$$
$$+p_{0,r_m-k}\sum_{l=0}^{b_n}D_l(u)+\sum_{j=1}^{a_n}\Delta_{10}p_{b_n-j,r_m-k}\sum_{l=0}^{j-1}D_l(u)+p_{0,r_m-k}\sum_{l=0}^{a_n}D_l(u).$$

Since we consider $\{p_{jk}\}$ as nondecreasing in j and (1.7), we obtain

$$\left|\sum_{j=a_n+1}^{b_n} p_{b_n-j,r_m-k}D_j(u)\right| = \frac{\pi^2}{2u^2}\left(\sum_{j=1}^{a_n}\Delta_{10}p_{b_n-j,r_m-k}-p_{0,r_m-k}+p_{0,r_m-k}-\sum_{j=1}^{b_n}\Delta_{10}p_{b_n-j,r_m-k}\right)$$
$$\leq \frac{\pi^2}{2u^2}\sum_{j=a_n+1}^{b_n}\Delta_{10}p_{b_n-j,r_m-k} = \frac{\pi^2}{2u^2}\left[p_{b_n-a_n-1,r_m-k}-p_{0,r_m-k}\right]. \tag{2.16}$$

If we write (2.15) again, we have

$$\left|D_{\beta}^{\gamma}K_{nm}(u,v)\right| \leq \frac{\pi^2}{2P_{nm}^{\beta\gamma}u^2}\sum_{k=t_m+1}^{r_m}(k+1)\left[p_{b_n-a_n-1,r_m-k}-p_{0,r_m-k}\right]. \tag{2.17}$$

(2.13) can be shown similarly. We perform the double Abel's transformation to prove (2.14)

$$P_{nm}^{\beta\gamma}D_{\beta}^{\gamma}K_{nm}(u,v) = \sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m} p_{b_n-j,r_m-k}D_j(u)D_k(v)$$
$$= \sum_{j=a_n+1}^{b_n-1}\sum_{k=t_m+1}^{r_m-1}\Delta_{11}p_{b_n-j,r_m-k}\sum_{i=0}^{j-1}D_i(u)\sum_{l=0}^{k-1}D_l(v)+\sum_{j=a_n+1}^{b_n-1}\Delta_{10}p_{b_n-j,0}\sum_{i=0}^{j-1}D_i(u)\sum_{l=0}^{r_m}D_l(v)$$
$$-\sum_{j=a_n+1}^{b_n-1}\Delta_{10}p_{b_n-j,r_m-t_m-1}\sum_{i=0}^{j-1}D_i(u)\sum_{l=0}^{t_m}D_l(v)+\sum_{k=t_m+1}^{r_m-1}\Delta_{01}p_{0,r_m-k}\sum_{i=0}^{b_n}D_i(u)\sum_{l=0}^{k-1}D_l(v)$$
$$-\sum_{k=t_m+1}^{r_m-1}\Delta_{01}p_{b_n-a_n-1,r_m-k}\sum_{i=0}^{a_n}D_i(u)\sum_{l=0}^{k-1}D_l(v)+p_{b_n r_m}\sum_{i=0}^{b_n}D_i(u)\sum_{l=0}^{r_m}D_l(v)$$
$$-p_{b_n-a_n-1,r_m}\sum_{i=0}^{a_n}D_i(u)\sum_{l=0}^{r_m}D_l(v)-p_{b_n,r_m-t_m-1}\sum_{i=0}^{b_n}D_i(u)\sum_{l=0}^{t_m}D_l(v)$$
$$+p_{b_n-a_n-1,r_m-t_m-1}\sum_{i=0}^{a_n}D_i(u)\sum_{l=0}^{t_m}D_l(v),$$

from (1.7)

$$P_{nm}^{\beta\gamma}\left|D_\beta^\gamma K_{nm}(u,v)\right| \le \frac{\pi^4}{4u^2v^2}\Bigg(\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m}\left|\Delta_{11}p_{b_n-j,r_m-k}\right| + \sum_{j=a_n+1}^{b_n}\Delta_{10}p_{b_n-j,0}$$
$$-\sum_{j=a_n+1}^{b_n}\Delta_{10}p_{b_n-j,r_m-t_m-1} + \sum_{k=t_m+1}^{r_m}\Delta_{01}p_{0,r_m-k} - \sum_{k=t_m+1}^{r_m}\Delta_{01}p_{b_n-a_n-1,r_m-k}$$
$$+p_{b_n r_m} - p_{b_n,r_m-t_m-1} - p_{b_n-a_n-1,r_m-t_m-1} - p_{b_n-a_n-1,r_m}\Bigg). \tag{2.18}$$

Since $\Delta_{11}p_{jk}$ is fixed, we get

$$\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m}\left|\Delta_{11}p_{b_n-j,r_m-k}\right| = \left|p_{b_n-a_n,r_m-t_m} - p_{b_n-a_n,0} - p_{0,r_m-t_m} + p_{00}\right|.$$

According to (2.18), if $\Delta_{11}p_{jk} \ge 0$, we obtained

$$P_{nm}^{\beta\gamma}\left|D_\beta^\gamma K_{nm}(u,v)\right| \le \frac{\pi^4}{4u^2v^2}\left(p_{b_n-a_n,r_m-t_m} + p_{b_n,r_m-t_m-1} + p_{b_n-a_n-1,r_m} + p_{b_n-a_n-1,r_m-t_m-1} + p_{b_n,r_m}\right)$$
$$\le \frac{5\pi^4}{4u^2v^2}p_{b_n,r_m}$$

and if $\Delta_{11}p_{jk} \le 0$, we obtained

$$P_{nm}^{\beta\gamma}\left|D_\beta^\gamma K_{nm}(u,v)\right| \le \frac{3\pi^4}{4u^2v^2}p_{b_n r_m}.$$

Lemma 2.4. *Let* $\{p_{jk} > 0 : j,k = 0,1...\}$ *be a nonincreasing double sequence such that* $\Delta_{11}p_{jk}$ *is fixed and let* $p=[1/u], \eta=[1/v]$ *where* $[.]$ *means the integral part. Then*

$$\left|D_\beta^\gamma K_{nm}(u,v)\right| \le (b_n+1)(r_m+1) \quad , \quad \forall u,v \tag{2.19}$$
$$\le \frac{\pi(\pi+1)}{2}\frac{1}{P_{nm}^{\beta\gamma}u}\sum_{k=t_m+1}^{r_m}(k+1)\sum_{j=0}^{p}p_{j,r_m-k}, \quad \forall v \text{ and } 0<u\le\pi \tag{2.20}$$
$$\le \frac{\pi(\pi+1)}{2}\frac{1}{P_{nm}^{\beta\gamma}v}\sum_{j=a_n+1}^{b_n}(j+1)\sum_{k=0}^{\eta}p_{b_n-j,k}, \quad \forall u \text{ and } 0<v\le\pi \tag{2.21}$$
$$\le \frac{\pi^2(1+2\pi+3\pi^2)}{4}\frac{p_{p\eta}}{P_{nm}^{\beta\gamma}uv} \quad , \quad 0<u,v\le\pi. \tag{2.22}$$

Proof. Equation (2.19) coincides with (2.11) as we remarked in the proof of Lemma 2.3.

From (2.15) and (1.5),

$$p_{nm}^{\beta\gamma}\left|D_\beta^\gamma K_{nm}(u,v)\right| \le \sum_{k=t_m+1}^{r_m}(k+1)\left|\sum_{j=a_n+1}^{b_n}p_{b_n-j,r_m-k}D_j(u)\right|$$
$$\le \frac{\pi}{2u}\sum_{k=t_m+1}^{r_m}(k+1)\left|\sum_{j=0}^{b_n-a_n-1}p_{b_n-j,r_m-k}\sin(b_n-j+\frac{1}{2})u\right|. \tag{2.23}$$

By a simple estimate the inner sum can be written as follows for each k,

$$\left|\sum_{j=0}^{b_n-a_n-1} p_{b_n-j,r_m-k}\sin(b_n-j+\frac{1}{2})u\right|$$
$$\leq \sum_{j=0}^{p} p_{b_n-j,r_m-k}+\left|\sum_{j=p+1}^{b_n-a_n-1} p_{j,r_m-k}\sin(b_n-j+\frac{1}{2})u\right|. \tag{2.24}$$

From the Abel's transformation, we have

$$\sum_{j=p+1}^{b_n-a_n-1} p_{j,r_m-k}\sin(b_n-j+\frac{1}{2})u=\sum_{j=p+1}^{b_n-a_n-2}\Delta_{10}p_{j-r_m-k}\sum_{l=p+1}^{j}\sin(b_n-j+\frac{1}{2})u$$
$$+p_{b_n-a_n-1,r_m-k}\sum_{l=p+1}^{b_n}\sin(b_n-j+\frac{1}{2})u. \tag{2.25}$$

From (1.6), $\{p_{jk}\}$ is nonincreasing in j and $\frac{1}{u}<\rho+1$ we have

$$\left|\sum_{j=p+1}^{b_n-a_n-1} p_{b_n-j,r_m-k}\sin(b_n-j+\frac{1}{2})u\right|\leq\frac{\pi}{u}p_{\rho+1,r_m-k}\leq\pi(\rho+1)p_{\rho+1,r_m-k}\leq\pi\sum_{j=0}^{\rho}p_{j,r_m-k} \tag{2.26}$$

Combining (2.18), (2.24) and (2.26) yield (2.20).

(2.21) can be proved similarly. Now, let's prove (2.22) beginning the following inequality,

$$\begin{aligned}P_{nm}^{\beta\gamma}\left|D_\beta^\gamma K_{nm}(u,v)\right| &=\left|\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m}p_{b_n-j,r_m-k}D_j(u)D_k(v)\right|\\ &=\left|\sum_{j=0}^{b_n-a_n-1}\sum_{k=0}^{r_m-t_m-1}p_{jk}D_{b_n-j}(u)D_{r_m-k}(v)\right|\\ &\leq\frac{\pi^2}{4uv}\left|\sum_{j=0}^{b_n-a_n-1}\sum_{k=0}^{b_n-a_n-1}p_{jk}\sin\left(b_n-j+\frac{1}{2}\right)u\sin\left(r_m-k+\frac{1}{2}\right)v\right|.\end{aligned} \tag{2.27}$$

We divide the double sum into four parts:

$$\begin{aligned}&\left|\sum_{j=0}^{b_n-a_n-1}\sum_{k=0}^{b_n-a_n-1}p_{jk}\sin\left(b_n-j+\frac{1}{2}\right)u\sin\left(r_m-k+\frac{1}{2}\right)v\right|\leq\sum_{j=0}^{\rho}\sum_{k=0}^{\eta}p_{jk}+\sum_{k=0}^{\eta}\left|\sum_{j=\rho+1}^{b_n-a_n-1}p_{jk}\sin\left(b_n-j+\frac{1}{2}\right)u\right|\\ &+\sum_{j=0}^{\rho}\left|\sum_{k=\eta+1}^{r_m-t_m-1}p_{jk}\sin\left(r_m-k+\frac{1}{2}\right)v\right|+\left|\sum_{j=\rho+1}^{b_n-a_n-1}\sum_{k=\eta+1}^{r_m-t_m-1}p_{jk}\sin\left(b_n-j+\frac{1}{2}\right)u\sin\left(r_m-k+\frac{1}{2}\right)v\right|\\ &=P_{\rho\eta}+A_1+A_2+A_3.\end{aligned} \tag{2.28}$$

For A_1 we can perform the Abel's transformation similar to (2.25) and conclude that

$$\sum_{j=p+1}^{b_n-a_n-1} p_{b_n-j,r_m-k}\sin(b_n-j+\frac{1}{2})u \le \sum_{j=\rho+1}^{b_n-a_n-2}\Delta_{10}p_{jk}\left|\sum_{l=p+1}^{j}\sin(b_n-l+\frac{1}{2})u\right| + p_{b_n-a_n-1,k}\left|\sum_{l=p+1}^{b_n-a_n-1}\sin(b_n-l+\frac{1}{2})u\right|$$

$$\le \frac{\pi}{u}p_{\rho+1,k} \le \pi(\rho+1)p_{\rho+1,k} \le \pi\sum_{j=0}^{\rho}p_{jk}$$

Thus we have the following inequation

$$A_1 \le \pi P_{\rho\eta}. \tag{2.29}$$

So similarly,

$$A_2 \le \pi P_{\rho\eta}. \tag{2.30}$$

For A3, we perform the double Abel's transformation:

$$\sum_{j=\rho+1}^{b_n-a_n-1}\sum_{k=\eta+1}^{r_m-t_m-1} p_{jk}\sin\left(b_n-j+\frac{1}{2}\right)u\sin\left(r_m-k+\frac{1}{2}\right)v$$
$$=\sum_{j=\rho+1}^{b_n-a_n-2}\sum_{k=\eta+1}^{r_m-t_m-2}\Delta_{11}p_{jk}\sum_{i=\rho+1}^{j}\sin\left(b_n-i+\frac{1}{2}\right)u\sum_{l=\eta+1}^{k}\sin\left(r_m-l+\frac{1}{2}\right)v$$
$$+\sum_{j=\rho+1}^{b_n-a_n-2}\Delta_{10}p_{jr_m}\sum_{i=\rho+1}^{j}\sin\left(b_n-i+\frac{1}{2}\right)u\sum_{l=\eta+1}^{r_m-t_m-2}\sin\left(r_m-l+\frac{1}{2}\right)v$$
$$+\sum_{k=\eta+1}^{r_m-t_m-2}\Delta_{01}p_{jk}\sum_{i=\rho+1}^{b_n-a_n-2}\sin\left(b_n-i+\frac{1}{2}\right)u\sum_{l=\eta+1}^{k}\sin\left(r_m-l+\frac{1}{2}\right)v$$
$$+p_{b_nr_m}\sum_{i=\rho+1}^{b_n-a_n-1}\sin\left(b_n-i+\frac{1}{2}\right)u\sum_{l=\eta+1}^{r_m-t_m-1}\sin\left(r_m-l+\frac{1}{2}\right)v.$$

By (1.6),

$$\left|\sum_{j=\rho+1}^{b_n-a_n-1}\sum_{k=\eta+1}^{r_m-t_m-1} p_{jk}\sin\left(b_n-j+\frac{1}{2}\right)u\sin\left(r_m-k+\frac{1}{2}\right)v\right|$$
$$\le\frac{\pi^2}{uv}\left\{\left|\sum_{j=\rho+1}^{b_n-a_n-1}\sum_{k=\eta+1}^{r_m-t_m-1}\Delta_{11}p_{jk}\right|+\sum_{j=\rho+1}^{b_n-a_n-2}\Delta_{10}p_{jr_m}+\sum_{k=\eta+1}^{r_m-t_m-2}\Delta_{01}p_{b_nk}+p_{b_nr_m}\right\}$$
$$=\frac{\pi^2}{uv}p_{\rho+1,\eta+1}\qquad\left(\Delta_{11}p_{jk}\ge 0\right)$$
$$=\frac{\pi^2}{uv}\left(-2p_{b_nr_m}+2p_{\rho+1,r_m}+2p_{b_n,\eta+1}-p_{\rho+1,\eta+1}\right)\le\frac{3\pi^2}{uv}p_{\rho+1,\eta+1}\qquad\left(\Delta_{11}p_{jk}\le 0\right).$$

Hence, the following inequality is obtained

$$A_3\le\frac{3\pi^2}{uv}p_{\rho+1,\eta+1}\le 3\pi^2(\rho+1)(\eta+1)p_{\rho+1,\eta+1}\le 3\pi^2\sum_{j=0}^{\rho}\sum_{k=0}^{\eta}p_{jk}=3\pi^2P_{\rho\eta}. \tag{2.31}$$

If we consider (2.28)-(2.31) together, we obtain

$$\left|\sum_{j=0}^{b_n-a_n-1}\sum_{k=0}^{r_m-t_m-1} p_{jk}\sin\left(b_n-j+\frac{1}{2}\right)u\sin\left(r_m-k+\frac{1}{2}\right)v\right|\leq\left(1+2\pi+3\pi^2\right)P_{\rho\eta}.$$

Thus (2.27) implies (2.22).

Theorem 2.5. *Let* $\{p_{jk}>0: j,k=0,1...\}$ *be a nondecreasing double sequence such that* $\Delta_{11}p_{jk}$ *is fixed and condition (2.5) is satisfied. If* $f\in Lip\alpha$ *for some* $0<\alpha<1$, *then*

$$\sup_{(x,y)\in\mathbb{R}}\left|D_\beta^\gamma N_{nm}(x,y)-f(x,y)\right|$$
$$=\begin{cases} O\left(\left(D_\beta^\gamma q_{nm}\right)^\alpha+\left(D_\beta^\gamma r_{nm}\right)^\alpha\right) & , \quad 0<\alpha<1 \\ O\left(D_\beta^\gamma q_{nm}\ln\frac{\pi}{D_\beta^\gamma q_{nm}}+D_\beta^\gamma r_{nm}\ln\frac{\pi}{D_\beta^\gamma r_{nm}}\right) & , \quad \alpha=1. \end{cases} \tag{2.32}$$

Proof. We start with representation (2.2). Let

$$\frac{\pi^2}{4}\left|D_\beta^\gamma N_{nm}(x,y)-f(x,y)\right|$$
$$\leq\left\{\int_0^{D_\beta^\gamma q_{nm}}\int_0^{D_\beta^\gamma r_{nm}}+\int_{D_\beta^\gamma q_{nm}}^{\pi}\int_0^{D_\beta^\gamma r_{nm}}+\int_0^{D_\beta^\gamma q_{nm}}\int_{D_\beta^\gamma r_{nm}}^{\pi}+\int_{D_\beta^\gamma q_{nm}}^{\pi}\int_{D_\beta^\gamma r_{nm}}^{\pi}\right\}\left|\phi(u,v)\right|\left|K_\beta^\gamma(u,v)\right|dudv$$
$$=I_1+I_2+I_3+I_4. \tag{2.33}$$

$\phi(u,v)$ will be taken as estimated in (1.3) and the appropriate estimate of Lemma 2.3 is substituted for the kernel $D_\beta^\gamma K_{nm}(u,v)$. From (2.11), for $\alpha>0$

$$I_1\leq(b_n+1)(r_m+1)\int_0^{D_\beta^\gamma q_{nm}}\int_0^{D_\beta^\gamma r_{nm}}\left(u^\alpha+v^\alpha\right)dudv$$
$$=\frac{1}{\alpha+1}(b_n+1)(r_m+1)D_\beta^\gamma q_{nm}D_\beta^\gamma r_{nm}\left(\left(D_\beta^\gamma q_{nm}\right)^\alpha+\left(D_\beta^\gamma r_{nm}\right)^\alpha\right).$$

By (2.6) and (2.7),

$$I_1=O\left(\left(D_\beta^\gamma q_{nm}\right)^\alpha+\left(D_\beta^\gamma r_{nm}\right)^\alpha\right). \tag{2.34}$$

By (2.12),

$$I_2\leq\frac{\pi^2}{2P_{nm}^{\beta\gamma}}\sum_{k=t_m+1}^{r_m}(k+1)\left[p_{b_n-a_n-1,r_m-k}-p_{0,r_m-k}\right]\int_{D_\beta^\gamma q_{nm}}^{\pi}\int_0^{D_\beta^\gamma r_{nm}}\frac{u^\alpha+v^\alpha}{u^2}dudv$$

for $0<\alpha<1$,

$$I_2 \le \frac{\pi^2}{2P_{nm}^{\beta\gamma}} \frac{D_\beta^\gamma r_{nm}}{D_\beta^\gamma q_{nm}} \sum_{k=t_m+1}^{r_m} (k+1)\left[p_{b_n-a_n-1,r_m-k} - p_{0,r_m-k} \right] \left(\frac{\left(D_\beta^\gamma q_{nm}\right)^\alpha}{1-\alpha} + \frac{\left(D_\beta^\gamma q_{nm}\right)^\alpha}{1+\alpha} \right),$$

for $\alpha = 1$,

$$I_2 \le \frac{\pi^2}{2P_{nm}^{\beta\gamma}} \frac{D_\beta^\gamma r_{nm}}{D_\beta^\gamma q_{nm}} \sum_{k=t_m+1}^{r_m} (k+1)\left[p_{b_n-a_n-1,r_m-k} - p_{0,r_m-k} \right] \left(D_\beta^\gamma q_{nm} \ln \frac{\pi}{D_\beta^\gamma q_{nm}} + \frac{1}{2} D_\beta^\gamma r_{nm} \right).$$

With the help of (2.7)

$$\begin{aligned} &\frac{\pi^2}{2P_{nm}^{\beta\gamma}} \frac{D_\beta^\gamma r_{nm}}{D_\beta^\gamma q_{nm}} \sum_{k=t_m+1}^{r_m} (k+1)\left[p_{b_n-a_n-1,r_m-k} - p_{0,r_m-k} \right] \\ &\le \frac{\pi^2}{2P_{nm}^{\beta\gamma}} \frac{D_\beta^\gamma r_{nm}}{D_\beta^\gamma q_{nm}} \sum_{k=t_m+1}^{r_m} (k+1)\, p_{b_n,r_m-k} = (r_m+1) \frac{\pi^2}{2P_{nm}^{\beta\gamma}} \frac{D_\beta^\gamma r_{nm}}{D_\beta^\gamma q_{nm}} \sum_{k=t_m+1}^{r_m} p_{b_n,r_m-k} \\ &= (r_m+1) D_\beta^\gamma r_{nm} = O(1). \end{aligned}$$

So,

$$I_2 = \begin{cases} O\left(\left(D_\beta^\gamma q_{nm}\right)^\alpha + \left(D_\beta^\gamma r_{nm}\right)^\alpha \right) & , \quad 0 < \alpha < 1 \\ O\left(D_\beta^\gamma q_{nm} \ln \frac{\pi}{D_\beta^\gamma q_{nm}} + D_\beta^\gamma r_{nm} \right) & , \quad \alpha = 1. \end{cases} \tag{2.35}$$

Similarly, using (2.13),

$$I_3 = \begin{cases} O\left(\left(D_\beta^\gamma q_{nm}\right)^\alpha + \left(D_\beta^\gamma r_{nm}\right)^\alpha \right) & , \quad 0 < \alpha < 1 \\ O\left(D_\beta^\gamma q_{nm} + D_\beta^\gamma r_{nm} \ln \frac{\pi}{D_\beta^\gamma r_{nm}} \right) & , \quad \alpha = 1. \end{cases} \tag{2.36}$$

By (2.14),

$$I_4 \le \frac{5\pi^4}{4} \frac{p_{b_n r_m}}{P_{nm}^{\beta\gamma}} \int_{D_\beta^\gamma q_{nm}}^{\pi} \int_{D_\beta^\gamma r_{nm}}^{\pi} \frac{u^\alpha + v^\alpha}{u^2 v^2} \, du dv$$

for $\alpha = 1$,

$$\frac{5\pi^4}{4(1-\alpha)} \frac{p_{b_n r_m}}{D_\beta^\gamma q_{nm} D_\beta^\gamma r_{nm} P_{nm}^{\beta\gamma}} \left(D_\beta^\gamma q_{nm} \ln \frac{\pi}{D_\beta^\gamma q_{nm}} + D_\beta^\gamma r_{nm} \ln \frac{\pi}{D_\beta^\gamma r_{nm}} \right).$$

From (2.3), (2.4) and (2.5),

$$\frac{p_{b_n r_m}}{D_\beta^\gamma q_{nm} D_\beta^\gamma r_{nm} P_{nm}^{\beta\gamma}} = \frac{(b_n - a_n)(r_m - t_m)\, p_{b_n r_m}}{(b_n - a_n) D_\beta^\gamma q_{nm} (r_m - t_m) D_\beta^\gamma r_{nm}} = O(1).$$

As a result,

$$I_4 = \begin{cases} O\left(\left(D_\beta^\gamma q_{nm}\right)^\alpha + \left(D_\beta^\gamma r_{nm}\right)^\alpha\right) & , \quad 0<\alpha<1 \\ O\left(D_\beta^\gamma q_{nm} \ln \frac{\pi}{D_\beta^\gamma q_{nm}} + D_\beta^\gamma r_{nm} \ln \frac{\pi}{D_\beta^\gamma r_{nm}}\right) & , \quad \alpha=1. \end{cases} \tag{2.37}$$

Collecting (2.33)-(2.37) together yields (2.32).

Theorem 2.6. *Let* $\{p_{jk}>0: j,k=0,1...\}$ *be a nondecreasing double sequence such that* $\Delta_{11}p_{jk}$ *is fixed and condition (2.5) is satisfied. If* $f \in Lip\alpha$ *for some* $0<\alpha<1$, *then*

$$\sup_{(x,y)\in\square} \left|D_\beta^\gamma N_{nm}(x,y) - f(x,y)\right|$$
$$= \left\{\frac{1}{P_{nm}^{\beta\gamma}} \sum_{j=0}^{b_n}\sum_{k=0}^{r_m}\left(\frac{P_{jk}}{(j+1)^{\alpha+1}(k+1)} + \frac{P_{jk}}{(j+1)(k+1)^{\alpha+1}}\right)\right\}. \tag{2.38}$$

Proof . We use (2.33) with $D_\beta^\gamma q_{nm}$ and $D_\beta^\gamma r_{nm}$ that are replaced by $\pi/(b_n+1)$ and $\pi/(r_m+1)$, respectively. The right-hand side of (2.38) is denoted by Ψ_{nm}.

From (2.19), for $\alpha>0$

$$I_1 \le (b_n+1)(r_m+1)\int_0^{\pi/(b_n+1)}\int_0^{\pi/(r_m+1)}\left(u^\alpha+v^\alpha\right)dudv$$
$$= \frac{\pi^{\alpha+2}}{\alpha+1}\left(\frac{1}{(b_n+1)^\alpha} + \frac{1}{(r_m+1)^\alpha}\right). \tag{2.39}$$

Since p_{jk} is nonincreasing, we clearly have

$$P_{jk} \ge (b_j+1)(r_k+1)\,p_{jk} \qquad (j,k=0,1,\ldots).$$

Thus

$$\frac{1}{(b_n+1)^\alpha} = \frac{1}{(b_n+1)^\alpha}\frac{1}{P_{nm}^{\beta\gamma}}\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m} p_{jk}$$
$$\le \frac{1}{(b_n+1)^\alpha}\frac{1}{P_{nm}^{\beta\gamma}}\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m}\frac{P_{jk}}{(b_j+1)(r_k+1)}$$
$$\le \frac{1}{P_{nm}^{\beta\gamma}}\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m}\frac{P_{jk}}{(b_j+1)^{\alpha+1}(r_k+1)}$$

and similarly,

$$\frac{1}{(r_m+1)^\alpha} \le \frac{1}{P_{nm}^{\beta\gamma}}\sum_{j=a_n+1}^{b_n}\sum_{k=t_m+1}^{r_m}\frac{P_{jk}}{(b_j+1)(r_k+1)^{\alpha+1}}.$$

Combining (2.39) with the last two inequalities, we have

$$I_1 = O(\Psi_{nm}). \tag{2.40}$$

By (2.20),

$$\begin{aligned} I_2 &\le \frac{\pi(\pi+1)}{2P_{nm}^{\beta\gamma}} \sum_{k=t_m+1}^{r_m} \int_{\pi/(b_n+1)}^{\pi} \int_0^{\pi/(r_m+1)} \frac{u^\alpha + v^\alpha}{u} \sum_{j=0}^{\rho} p_{j,r_m-k}\, dvdu \\ &= \frac{\pi(\pi+1)}{2P_{nm}^{\beta\gamma}} \sum_{k=t_m+1}^{r_m} (k+1) \left\{ \frac{\pi}{r_m+1} \int_{\pi/(b_n+1)}^{\pi} u^{\alpha-1} \sum_{j=0}^{\rho} p_{j,r_m-k}\, du \right. \\ &\quad \left. + \frac{\pi^{\alpha+1}}{(\alpha+1)(r_m+1)^{\alpha+1}} \int_{\pi/(b_n+1)}^{\pi} \frac{1}{u} \sum_{j=0}^{\rho} p_{j,r_m-k}\, du \right\} \end{aligned} \tag{2.41}$$

In each integration, replace u by $1/w$ (remembering that $\rho = [1/u]$) to get

$$\begin{aligned} I_2 &= \frac{O(1)}{P_{nm}^{\beta\gamma}} \sum_{k=t_m+1}^{r_m} (k+1) \left\{ \frac{1}{r_m+1} \sum_{l=0}^{b_n} \frac{1}{(l+1)^{\alpha+1}} \sum_{j=0}^{l} p_{j,r_m-k} \right. \\ &\quad \left. + \frac{1}{(r_m+1)^{\alpha+1}} \sum_{l=0}^{b_n} \frac{1}{l+1} \sum_{j=0}^{l} p_{j,r_m-k} \right\}. \end{aligned} \tag{2.42}$$

The first sum on the right is equal to

$$\begin{aligned} A &= \frac{1}{(r_m+1)P_{nm}^{\beta\gamma}} \sum_{k=t_m+1}^{r_m} (k+1) \sum_{l=0}^{b_n} \frac{1}{(l+1)^{\alpha+1}} \sum_{j=0}^{l} p_{j,r_m-k} \\ &= \frac{1}{(r_m+1)P_{nm}^{\beta\gamma}} \sum_{l=0}^{b_n} \frac{1}{(l+1)^{\alpha+1}} \sum_{j=0}^{l} \sum_{k=t_m+1}^{r_m} (k+1) p_{j,r_m-k}. \end{aligned}$$

Using the following inequality

$$\sum_{k=t_m+1}^{r_m} (k+1) p_{j,r_m-k} \le \sum_{k=0}^{r_m} \sum_{s=0}^{r_m-k} p_{js},$$

we can write

$$\begin{aligned} A &= \frac{1}{(r_m+1)P_{nm}^{\beta\gamma}} \sum_{l=0}^{b_n} \frac{1}{(l+1)^{\alpha+1}} \sum_{k=0}^{r_m} \sum_{j=0}^{l} \sum_{s=0}^{r_m-k} p_{js} = \frac{1}{(r_m+1)P_{nm}^{\beta\gamma}} \sum_{l=0}^{b_n} \sum_{k=0}^{r_m} \frac{P_{l,r_m-k}}{(l+1)^{\alpha+1}} \\ &= \frac{1}{(r_m+1)P_{nm}^{\beta\gamma}} \sum_{l=0}^{b_n} \sum_{k=0}^{r_m} \frac{P_{lk}}{(l+1)^{\alpha+1}} \le \frac{1}{P_{nm}^{\beta\gamma}} \sum_{l=0}^{b_n} \sum_{k=0}^{r_m} \frac{P_{lk}}{(l+1)^{\alpha+1}(r_k+1)}. \end{aligned} \tag{2.43}$$

The second sum at the right hand side of (2.42) can be dominated in a similar manner:

$$\frac{1}{(r_m+1)^{\alpha+1} P_{nm}^{\beta\gamma}} \sum_{k=0}^{r_m} (k+1) \sum_{l=0}^{b_n} \frac{1}{l+1} \sum_{j=0}^{l} p_{j,r_m-k} \le \frac{1}{P_{nm}^{\beta\gamma}} \sum_{l=0}^{b_n} \sum_{k=0}^{r_m} \frac{P_{lk}}{(l+1)(r_k+1)^{\alpha+1}}. \tag{2.44}$$

From (2.42)-(2.44) it follows that

$$I_2 = O(\Psi_{nm}). \tag{2.45}$$

Similarly, by (2.21)

$$I_3 = O(\Psi_{nm}). \qquad (2.46)$$

Using that (2.22),

$$I_4 = \frac{O(1)}{P_{nm}^{\beta\gamma}} \int_{\pi/(b_n+1)}^{\pi} \int_{\pi/(r_m+1)}^{\pi} \left(\frac{u^{\alpha} + v^{\alpha}}{uv} \right) P_{\rho\eta} dudv.$$

We replace y by $1/y$, v by $1/w$, keeping in mind that $\rho = [1/u]$ and $\eta = [1/v]$. As a result, we obtain

$$I_4 = \frac{O(1)}{P_{nm}^{\beta\gamma}} \int_{1/\pi}^{(b_n+1)/\pi} \int_{1/\pi}^{(r_m+1)/\pi} \left(\frac{1}{y^{\alpha+1} w} + \frac{1}{y w^{\alpha+1}} \right) P_{[y][w]} dydw.$$

$$I_4 = \frac{O(1)}{P_{nm}^{\beta\gamma}} \sum_{j=0}^{b_n} \sum_{k=0}^{r_m} \left(\frac{1}{(j+1)^{\alpha+1}(k+1)} + \frac{1}{(j+1)(k+1)^{\alpha+1}} \right) P_{jk} = O(\Psi_{nm})$$

Combining (2.33), (2.40), (2.45)-(2.47) results in (2.38).

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