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Solution of Vector Equation of Transfer in a Finite Plane Parallel Media

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Abstract. H_N method is used for the solution of vector equation of transfer, which includes two Stokes parameters, with a combination of Rayleigh and isotropic scattering in a finite plane-parallel atmosphere. The behavior of albedo and transmission factor are obtained according to the slab thickness and reflection coefficient.

Keywords: Vector equation of transfer, Rayleigh scattering, Stokes parameters, albedo, transmission factor.

Sonlu Bir Düzlem Paralel Ortamda Vektör Transfer Denkleminin Çözümü

Özet. Bir sonlu düzlem parallel atmosferde Rayleigh ve izotropik saçılma durumunun bir kombinasyonunda, iki Stokes parametresini içeren vektör transfer denkleminin çözümü için H_N metodu kullanılmıştır. Albedonun ve iletim faktörünün davranışı slab kalınlığına ve yansıma katsayına bağlı olarak elde edilmiştir.

Anahtar Kelimeler: Vektör transfer denklemi, Rayleigh saçılması, Stokes parametreleri, albedo, iletim faktörü

1. INTRODUCTION

Radiative transfer equation has many applications on the atmosphere and ocean physics. The radiative transfer equation with polarization is a vector equation and its solution is more complicated than using one-dimensional radiative transfer equation. This equation is important to define the polarization of light in the studies including molecular atmosphere, aerosols and clouds. There are many studies on the calculations of albedo, transmission factor and two-Stokes parameters in finite plane-parallel atmosphere with different methods [1-5].

The equation of transfer with polarization is given by [1-3].

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau,\mu) + \mathbf{I}(\tau,\mu) = \frac{\omega}{2} \mathbf{K}(\mu) \int_{-1}^{+1} \mathbf{K}^{T}(\mu') \mathbf{I}(\tau,\mu') d\mu' \tag{1}$$

Here, $\mathbf{I}(\tau,\mu)$ is the vector intensity whose components are $I_\ell(\tau,\mu)$, $I_r(\tau,\mu)$ and these components are related by the two Stokes parameters as

$$\mathrm{I}(\tau,\mu) = \mathrm{I}_\ell(\tau,\mu) + \mathrm{I}_r(\tau,\mu)$$
$$\mathrm{Q}(\tau,\mu) = \mathrm{I}_\ell(\tau,\mu) - \mathrm{I}_r(\tau,\mu) \ .$$

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μ is the direction cosine of the propagating radiation, τ is the optical variable. The scattering process is characterized by single scattering albedo $\omega\epsilon[0,1]$ and the square matrix $\mathbf{K}(\mu)$. $\mathbf{K}^{T}(\mu)$ is the transpose of $\mathbf{K}(\mu)$ and

$$\mathbf{K}(\mu)=\frac{3}{2}\,(c+2)^{-1/2}\begin{pmatrix} c\mu^2+\frac{2}{3}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0 \end{pmatrix}. \quad (2)$$

$c=1$ and $\omega=1$ would yield Chandrasekhar's conservative Rayleigh scattering case, $c=1$ and $\omega\epsilon[0,1]$ allows the general Rayleigh-scattering, $c\epsilon[0,1]$ and $\omega\epsilon[0,1]$ yields the general mixture of Rayleigh- and isotropic-scattering [1].

The general solution of Eq.(1) is given by [6]

$$\mathbf{I}(\tau,\mu)=A(\eta_0)\mathbf{\Phi}(\eta_0,\mu)e^{-\tau/\eta_0}+A(-\eta_0)\mathbf{\Phi}(-\eta_0,\mu)e^{\tau/\eta_0}+\int_{-1}^{+1}\mathbf{\Psi}(\eta,\mu)\,\boldsymbol{A}(\eta)e^{-\tau/\eta}\,d\eta\ . \quad (3)$$

where $A(\pm\eta_0)$ and $\boldsymbol{A}(\eta)$ are the arbitrary expansion coefficients to be determined from the boundary conditions of a given problem [2]. The vectors $\boldsymbol{A}(\eta)$ and $\mathbf{\Psi}(\eta,\mu)$ are 2×1 and 2×2 matrices, respectively and they are

$$\boldsymbol{A}(\eta)=\begin{pmatrix} A_1(\eta) \\ A_2(\eta) \end{pmatrix}, \qquad \mathbf{\Psi}(\eta,\mu)=(\mathbf{\Phi_1}(\eta,\mu)\quad \mathbf{\Phi_2}(\eta,\mu))\ . \quad (4)$$

$\mathbf{\Phi}(\pm\eta_0,\mu)$ and $\mathbf{\Phi}_{\boldsymbol{\alpha}}(\eta,\mu)$, $\alpha=1$ or 2, are the discrete and continuum eigenvectors, respectively. These eigenvectors can be written as

$$\mathbf{\Phi}(\pm\eta_0,\mu)=\frac{3}{2}\omega\eta_0\,\frac{1}{\eta_0\mp\mu}\begin{pmatrix} c(1-\mu^2)\Lambda_2(\eta_0)+\varpi_2(\eta_0) \\ \varpi_2(\eta_0) \end{pmatrix},$$

$$\mathbf{\Phi_1}(\eta,\mu)=\begin{pmatrix} \frac{3}{2}\omega c\eta(1-\eta^2)(1-\mu^2)\frac{\mathcal{P}}{\eta-\mu}+\varpi_1(\eta)\delta(\eta-\mu) \\ -\varpi_2(\eta)\delta(\eta-\mu) \end{pmatrix}, \quad (5)$$

$$\mathbf{\Phi_2}(\eta,\mu)=\begin{pmatrix} \frac{3}{2}\omega\eta(1-\eta^2)\frac{\mathcal{P}}{\eta-\mu}+\lambda_1(\eta)\delta(\eta-\mu) \\ \frac{3}{2}\omega\eta(1-\eta^2)\frac{\mathcal{P}}{\eta-\mu}+\lambda_2(\eta)\delta(\eta-\mu) \end{pmatrix}.$$

Here, the symbol $\mathcal{P}$ denotes that all the integrals are to be evaluated with Cauchy principal value and

$$\begin{gathered} \lambda_i(\eta)=(-1)^i+3(1-\eta^2)\lambda_0(\eta)-(-1)^i 3\eta^2(1-\omega)\,,\quad i=1 \text{ or } 2 \\ \lambda_0(\eta)=1-\omega\eta\tanh^{-1}(\eta)\,, \\ \varpi_1(\eta)=c(1-\eta^2)\lambda_1(\eta)+\varpi_2(\eta)\,, \\ \varpi_2(\eta)=\frac{4}{3}(1-c)+2c(1-\omega)\eta^2\,, \end{gathered} \quad (6)$$

$$\Lambda_i(\eta_0) = (-1)^i + 3(1-{\eta_0}^2)\Lambda_0(\eta_0) - (-1)^i 3{\eta_0}^2(1-\omega)\,, \quad i = 1 \text{ or } 2$$
$$\Lambda_0(\eta_0) = 1 - \omega\eta_0 \tanh^{-1}(1/\eta_0)\,,$$
$$\varpi_2(\eta_0) = \frac{4}{3}(1-c) + 2c(1-\omega){\eta_0}^2\,.$$

The orthogonality relations of the discrete and continuum eigenvectors are [7-9]

$$N(\pm\eta_0) = \int_{-1}^{+1} \mu\; \mathbf{\Phi}^{\boldsymbol{T}}(\pm\eta_0,\mu)\; \mathbf{\Phi}(\pm\eta_0,\mu)\; d\mu\;, \tag{7}$$

$$\langle\alpha|\beta\rangle = \int_{-1}^{+1} \mu\, \mathbf{\Phi}_{\boldsymbol{\alpha}}{}^{\boldsymbol{T}\dagger}(\eta,\mu)\; \mathbf{\Phi}_{\boldsymbol{\beta}}(\eta,\mu)\, d\mu, \qquad \alpha,\beta = 1 \text{ and } 2 \tag{8}$$
$$\langle\alpha|\beta\rangle = 0 \quad \alpha \neq \beta,$$
$$\langle 1|1\rangle = \langle 2|2\rangle = N(\eta)\delta(\eta-\eta')\,.$$

Here,

$$N(\pm\eta_0) = \pm 12\omega\eta_0^2[c(1-\eta_0^2)\Lambda_2(\eta_0) + \varpi_2(\eta_0)]\frac{d}{dz}\Lambda(z)\Big|_{z=\eta_0}\quad , \tag{9}$$
$$N(\eta) = N_{11}(\eta)N_{22}(\eta) - N_{12}(\eta)N_{21}(\eta)\;, \tag{10}$$

and

$$\mathbf{\Phi_1}{}^{\dagger}(\eta,\mu) = N_{22}(\eta)\;\mathbf{\Phi_1}(\eta,\mu) - N_{12}(\eta)\mathbf{\Phi_2}(\eta,\mu)\quad ,$$

$$\mathbf{\Phi_2}{}^{\dagger}(\eta,\mu) = N_{11}(\eta)\;\mathbf{\Phi_2}(\eta,\mu) - N_{21}(\eta)\mathbf{\Phi_1}(\eta,\mu)\quad ,$$

$$N_{11}(\eta) = c^2(1-\eta^2)^2\left[\lambda_1^2(\eta) + \frac{9}{4}\omega^2\eta^2(1-\eta^2)^2\pi^2\right] + 2\varpi_2(\eta)[c(1-\eta^2)\lambda_1(\eta) + \varpi_2(\eta)]\,, \tag{11}$$

$$N_{12}(\eta) = N_{21}(\eta) = c(1-\eta^2)\left[\lambda_1^2(\eta) + \frac{9}{4}\omega^2\eta^2(1-\eta^2)^2\pi^2\right] - 2\varpi_2(\eta)[1-3(1-\omega)\eta^2]\,,$$

$$N_{22}(\eta) = 2[1-3(1-\omega)\eta^2]^2 + 18(1-\eta^2)^2\left[\lambda_0^2(\eta) + \frac{1}{4}\omega^2\eta^2\pi^2\right].$$

2. ANALYSIS

To apply the H_N method to the problem on the finite plane-parallel atmosphere, the unknown coefficients in Eq.(3), $A(\pm\eta_0)$ and $\boldsymbol{A}(\eta)$, $\eta\epsilon(0,1)$, are calculated [10, 11]. To get these coefficients, firstly, we consider the surface intensities at $\tau = 0$ and $\tau = \tau_0$ which can be written from Eq.(3) as

$$\mathbf{I}(0,-\mu)=\begin{pmatrix}\mathrm{I}_\ell(0,-\mu)\\ \mathrm{I}_r(0,-\mu)\end{pmatrix},$$

$$\mathbf{I}(0,-\mu)=A(\eta_0)\mathbf{\Phi}(\eta_0,-\mu)+A(-\eta_0)\mathbf{\Phi}(-\eta_0,-\mu)+\int_{-1}^{+1}A_1(\eta)\mathbf{\Phi_1}(\eta,-\mu)\,d\eta+\int_{-1}^{+1}A_2(\eta)\mathbf{\Phi_2}(\eta,-\mu)\,d\eta\ , \tag{12a}$$

$$\mathbf{I}(\tau_0,\mu)=\begin{pmatrix}\mathrm{I}_\ell(\tau_0,\mu)\\ \mathrm{I}_r(\tau_0,\mu)\end{pmatrix},$$

$$\mathbf{I}(\tau_0,\mu)=A(\eta_0)\mathbf{\Phi}(\eta_0,\mu)e^{-\tau_0/\eta_0}+A(-\eta_0)\,\mathbf{\Phi}(-\eta_0,\mu)e^{\tau_0/\eta_0}+\int_{-1}^{+1}A_1(\eta)\mathbf{\Phi_1}(\eta,\mu)\,e^{-\tau_0/\eta}d\eta+\int_{-1}^{+1}A_2(\eta)\mathbf{\Phi_2}(\eta,\mu)\,e^{-\tau_0/\eta}\,d\eta\ . \tag{12b}$$

The boundary conditions and the surface intensities are given by [2-4]

$$\mathbf{I}(0,\mu)=\begin{pmatrix}\mathrm{I}_\ell(0,\mu)\\ \mathrm{I}_r(0,\mu)\end{pmatrix}=\frac{1}{2}\,\delta(\mu-\mu_0)\boldsymbol{F}\ ,\quad \mu,\mu_0>0\,,$$

$$\mathbf{I}(\tau_0,-\mu)=\begin{pmatrix}\mathrm{I}_\ell(\tau_0,-\mu)\\ \mathrm{I}_r(\tau_0,-\mu)\end{pmatrix}=\lambda_0\boldsymbol{D}\int_0^1\mathbf{I}(\tau_0,\mu')\,\mu'\,d\mu'\ ,\quad \mu>0\,,$$

$$\mathbf{I}(0,-\mu)=\begin{pmatrix}\mathrm{I}_\ell(0,-\mu)\\ \mathrm{I}_r(0,-\mu)\end{pmatrix}=\mathbf{K}(\mu)\sum_{\boldsymbol{\alpha}=\mathbf{0}}^{\boldsymbol{N}}\boldsymbol{a}_{\boldsymbol{\alpha}}\mu^{\alpha},\quad \mu>0\,, \tag{13}$$

$$\mathbf{I}(\tau_0,\mu)=\begin{pmatrix}\mathrm{I}_\ell(\tau_0,\mu)\\ \mathrm{I}_r(\tau_0,\mu)\end{pmatrix}=\frac{1}{2}\,\delta(\mu-\mu_0)e^{-\tau_0/\mu_0}\boldsymbol{F}+\mathbf{K}(\mu)\sum_{\boldsymbol{\alpha}=\mathbf{0}}^{\boldsymbol{N}}\boldsymbol{b}_{\boldsymbol{\alpha}}\mu^{\alpha}\ ,\quad \mu>0\,,$$

where λ_0 is the reflection coefficient, $\boldsymbol{D}$ is the reflective coupling matrix between the two components and $\boldsymbol{F}$ is a constant matrix. These matrices are given by

$$\boldsymbol{D}=\begin{pmatrix}1&1\\1&1\end{pmatrix}\quad ,\qquad \boldsymbol{F}=\begin{pmatrix}1/2\\1/2\end{pmatrix}. \tag{14}$$

Here, the unknown constants $\boldsymbol{a}_{\boldsymbol{\alpha}}$ and $\boldsymbol{b}_{\boldsymbol{\alpha}}$ are to be determined by using the H_N method.

To get the unknown coefficients $A(\pm\eta_0)$, we multiply both sides of Eqs. (12a-b) by $\mu\mathbf{\Phi}^{\boldsymbol{T}}(\eta_0,-\mu)$ and integrate over μ, $\mu\in[-1,1]$. Using the orthogonality relations for the discrete and continuum modes, the boundary conditions and surface intensities in Eqs.(13), we obtain

$$A(\eta_0) = \frac{\mu_0}{2\,N(\eta_0)}\boldsymbol{\Phi}^{\boldsymbol{T}}(\eta_0,\mu_0)\boldsymbol{F} - \frac{1}{N(\eta_0)}\sum_{\alpha=1}^{N}\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}(\eta_0)\boldsymbol{a}_{\boldsymbol{\alpha}} \tag{15a}$$

$$\begin{aligned} A(-\eta_0) = {} & \lambda_0\mu_0\frac{e^{-\frac{\tau_0}{\eta_0}}e^{-\frac{\tau_0}{\mu_0}}}{2N(\eta_0)}\boldsymbol{B}_{\boldsymbol{0}}^{\boldsymbol{T}}(\eta_0)\boldsymbol{D}\boldsymbol{F} + \lambda_0\frac{e^{-\frac{\tau_0}{\eta_0}}}{N(\eta_0)}\boldsymbol{B}_{\boldsymbol{0}}^{\boldsymbol{T}}(\eta_0)\boldsymbol{D}\sum_{\alpha=1}^{N}\int_0^1 \mu'^{\,\alpha+1}\mathbf{K}(\mu')\,d\mu'\,\boldsymbol{b}_{\boldsymbol{\alpha}} \\ & - \mu_0\frac{e^{-\frac{\tau_0}{\eta_0}}e^{-\frac{\tau_0}{\mu_0}}}{2N(\eta_0)}\boldsymbol{\Phi}^{\boldsymbol{T}}(\eta_0,-\mu_0)\boldsymbol{F} - \frac{e^{-\frac{\tau_0}{\eta_0}}}{N(\eta_0)}\sum_{\alpha=1}^{N}\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}(\eta_0)\boldsymbol{b}_{\boldsymbol{\alpha}}\,. \end{aligned} \tag{15b}$$

Where,

$$\boldsymbol{B}_{\alpha}(\eta_0) = \int_0^1 \mu^{\alpha+1}\boldsymbol{\Phi}(\eta_0,\mu)d\mu\,, \tag{16a}$$

$$\boldsymbol{B}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}(\eta) = \int_0^1 \mu^{\alpha+1}\boldsymbol{\Phi}_{\boldsymbol{\beta}}(\eta,\mu)d\mu\,, \quad \beta = 1{,}2 \tag{16b}$$

$$\boldsymbol{A}_{\alpha}(\eta_0) = \int_0^1 \mu^{\alpha+1}\boldsymbol{\Phi}(\eta_0,-\mu)d\mu\,, \tag{16c}$$

$$\boldsymbol{A}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}(\eta) = \int_0^1 \mu^{\alpha+1}\boldsymbol{\Phi}_{\boldsymbol{\beta}}(\eta,-\mu)d\mu\,, \quad \beta = 1{,}2 \tag{16d}$$

$$\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}(\eta_0) = \int_0^1 \mu^{\alpha+1}\,\boldsymbol{\Phi}^{\boldsymbol{T}}(\eta_0,-\mu)\,\mathbf{K}(\mu)\,d\mu\,, \tag{16e}$$

$$\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}(\eta) = \int_0^1 \mu^{\alpha+1}\,\boldsymbol{\Phi}_{\boldsymbol{\beta}}^{\boldsymbol{T}}(\eta,-\mu)\,\mathbf{K}(\mu)\,d\mu\,, \quad \beta = 1{,}2\,. \tag{16f}$$

Similarly, we find the unknown coefficients $A_1(\pm\eta)$ and $A_2(\pm\eta)$ multiplying Eqs.(12a-b) by $\mu\boldsymbol{\Phi}_{\boldsymbol{\beta}}^{\boldsymbol{T}\dagger}(\eta,-\mu)$, $\beta = 1{,}2$, and integrating over μ, $\mu \in [-1,1]$ as,

$$\begin{aligned} A_1(\eta) = {} & \frac{\mu_0}{2\,N(\eta)}\left[N_{22}(\eta)\boldsymbol{\Phi}_{\boldsymbol{1}}^{\boldsymbol{T}}(\eta,\mu_0) - N_{12}(\eta)\boldsymbol{\Phi}_{\boldsymbol{2}}^{\boldsymbol{T}}(\eta,\mu_0)\right]\boldsymbol{F} \\ & - \frac{1}{N(\eta_0)}\sum_{\alpha=1}^{N}\left[N_{22}(\eta)\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}^{\boldsymbol{1}}(\eta) - N_{12}(\eta)\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}^{\boldsymbol{2}}(\eta)\right]\boldsymbol{a}_{\boldsymbol{\alpha}} \end{aligned} \tag{17a}$$

$$\begin{aligned} A_2(\eta) = {} & \frac{\mu_0}{2\,N(\eta)}\left[-N_{21}(\eta)\boldsymbol{\Phi}_{\boldsymbol{1}}^{\boldsymbol{T}}(\eta,\mu_0) + N_{11}(\eta)\boldsymbol{\Phi}_{\boldsymbol{2}}^{\boldsymbol{T}}(\eta,\mu_0)\right]\boldsymbol{F} \\ & - \frac{1}{N(\eta_0)}\sum_{\alpha=1}^{N}\left[-N_{21}(\eta)\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}^{\boldsymbol{1}}(\eta) + N_{11}(\eta)\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}^{\boldsymbol{2}}(\eta)\right]\boldsymbol{a}_{\boldsymbol{\alpha}} \end{aligned} \tag{17b}$$

$$
\begin{aligned}
A_1(-\eta) = {} & \lambda_0\mu_0 \frac{e^{-\frac{\tau_0}{\eta}} e^{-\frac{\tau_0}{\mu_0}}}{2N(\eta)} \left[N_{22}(\eta)\mathbf{B}_{\mathbf{0}}^{\mathbf{1}\,\boldsymbol{T}}(\eta) - N_{12}(\eta)\mathbf{B}_{\mathbf{0}}^{\mathbf{2}\,\boldsymbol{T}}(\eta)\right] \boldsymbol{DF} \\
& + \lambda_0 \frac{e^{-\frac{\tau_0}{\eta}}}{N(\eta)} \left[N_{22}(\eta)\mathbf{B}_{\mathbf{0}}^{\mathbf{1}\,\boldsymbol{T}}(\eta) - N_{12}(\eta)\mathbf{B}_{\mathbf{0}}^{\mathbf{2}\,\boldsymbol{T}}(\eta)\right]\boldsymbol{D} \sum_{\alpha=1}^{N} \int_0^1 \mu'^{\,\alpha+1}\mathbf{K}(\mu')\, d\mu'\, \boldsymbol{b}_{\boldsymbol{\alpha}} \\
& - \mu_0 \frac{e^{-\frac{\tau_0}{\eta}} e^{-\frac{\tau_0}{\mu_0}}}{2N(\eta)} \left[N_{22}(\eta)\boldsymbol{\Phi}_{\mathbf{1}}^{\boldsymbol{T}}(\eta,-\mu_0) - N_{12}(\eta)\boldsymbol{\Phi}_{\mathbf{2}}^{\boldsymbol{T}}(\eta,-\mu_0)\right]\boldsymbol{F} \\
& - \frac{e^{-\frac{\tau_0}{\eta}}}{N(\eta)} \sum_{\alpha=1}^{N} [N_{22}(\eta)\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}^{\mathbf{1}}(\eta) - N_{12}(\eta)\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}^{\mathbf{2}}(\eta)]\boldsymbol{b}_{\boldsymbol{\alpha}}.
\end{aligned} \tag{17c}
$$

$$
\begin{aligned}
A_2(-\eta) = {} & \lambda_0\mu_0 \frac{e^{-\frac{\tau_0}{\eta}} e^{-\frac{\tau_0}{\mu_0}}}{2N(\eta)} \left[-N_{12}(\eta)\mathbf{B}_{\mathbf{0}}^{\mathbf{1}\,\boldsymbol{T}}(\eta) + N_{11}(\eta)\mathbf{B}_{\mathbf{0}}^{\mathbf{2}\,\boldsymbol{T}}(\eta)\right] \boldsymbol{DF} \\
& + \lambda_0 \frac{e^{-\frac{\tau_0}{\eta}}}{N(\eta)} \left[-N_{12}(\eta)\mathbf{B}_{\mathbf{0}}^{\mathbf{1}\,\boldsymbol{T}}(\eta)\right. \\
& \left. + N_{11}(\eta)\mathbf{B}_{\mathbf{0}}^{\mathbf{2}\,\boldsymbol{T}}(\eta)\right]\boldsymbol{D} \sum_{\alpha=1}^{N} \int_0^1 \mu'^{\,\alpha+1}\mathbf{K}(\mu')\, d\mu'\, \boldsymbol{b}_{\boldsymbol{\alpha}} \\
& - \mu_0 \frac{e^{-\frac{\tau_0}{\eta}} e^{-\frac{\tau_0}{\mu_0}}}{2N(\eta)} \left[-N_{12}(\eta)\boldsymbol{\Phi}_{\mathbf{1}}^{\boldsymbol{T}}(\eta,-\mu_0) + N_{11}(\eta)\boldsymbol{\Phi}_{\mathbf{2}}^{\boldsymbol{T}}(\eta,-\mu_0)\right]\boldsymbol{F} \\
& - \frac{e^{-\frac{\tau_0}{\eta}}}{N(\eta)} \sum_{\alpha=1}^{N} [-N_{12}(\eta)\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}^{\mathbf{1}}(\eta) + N_{11}(\eta)\boldsymbol{\Upsilon}_{\boldsymbol{\alpha}}^{\mathbf{2}}(\eta)]\boldsymbol{b}_{\boldsymbol{\alpha}}.
\end{aligned} \tag{17d}
$$

Now, multiplying the surface intensities in Eqs.(12a-b) by μ^{m+1}, integrating over $\mu \,\epsilon\, (0,1)$ and using Eqs.(13-17), we obtain

$$\sum_{\alpha=0}^{N}\left\{\int_0^1 \mu^{m+1}\mathbf{K}(\mu)d\mu + \frac{1}{N(\nu_0)}\boldsymbol{A}_m(\eta_0)\mathbf{Y}_\alpha(\eta_0) + \int_0^1 \frac{\boldsymbol{A}_m^1(\eta)[N_{22}(\eta)\mathbf{Y}_\alpha^1(\eta) - N_{12}(\eta)\mathbf{Y}_\alpha^2(\eta)]}{N(\eta)}d\eta\right.$$

$$\left.+\int_0^1 \frac{\boldsymbol{A}_m^2(\eta)[-N_{12}(\eta)\mathbf{Y}_\alpha^1(\eta) + N_{11}(\eta)\mathbf{Y}_\alpha^2(\eta)]}{N(\eta)}d\eta\right\}\boldsymbol{a}_\alpha$$

$$+\sum_{\alpha=0}^{N}\left\{\frac{e^{-\frac{\tau_0}{\eta_0}}}{N(\eta_0)}\boldsymbol{B}_m(\eta_0)\left[\mathbf{Y}_\alpha(\eta_0) - \lambda_0\boldsymbol{B}_0^T(\eta_0)\boldsymbol{D}\int_0^1 \mu'^{\,\alpha+1}\mathbf{K}(\mu')\,d\mu'\right]\right.$$

$$+\int_0^1 e^{-\frac{\tau_0}{\eta}}\,\frac{\boldsymbol{B}_m^1(\eta)\,[N_{22}(\eta)\mathbf{Y}_\alpha^1(\eta) - N_{12}(\eta)\mathbf{Y}_\alpha^2(\eta)]}{N(\eta)}d\eta$$

$$+\int_0^1 e^{-\frac{\tau_0}{\eta}}\,\frac{\mathbf{B}_m^2(\eta)\,[-N_{12}(\eta)\mathbf{Y}_\alpha^1(\eta) + N_{11}(\eta)\mathbf{Y}_\alpha^2(\eta)]}{N(\eta)}d\eta$$

$$-\lambda_0\int_0^1 e^{-\frac{\tau_0}{\eta}}\,\frac{[N_{22}(\eta)\boldsymbol{B}_m^1(\eta) - N_{12}(\eta)\boldsymbol{B}_m^2(\eta)]}{N(\eta)}\boldsymbol{B}_0^{1\,T}(\eta)\boldsymbol{D}\int_0^1 \mu'^{\,\alpha+1}\mathbf{K}(\mu')\,d\mu'\,d\eta$$

$$\left.-\lambda_0\int_0^1 e^{-\frac{\tau_0}{\eta}}\,\frac{[-N_{12}(\eta)\boldsymbol{B}_m^1(\eta) + N_{11}(\eta)\boldsymbol{B}_m^2(\eta)]}{N(\eta)}\boldsymbol{B}_0^{2\,T}(\eta)\boldsymbol{D}\int_0^1 \mu'^{\,\alpha+1}\mathbf{K}(\mu')\,d\mu'\,d\eta\right\}\boldsymbol{b}_\alpha \tag{18}$$

$$=\frac{\mu_0}{2\,N(\eta_0)}\boldsymbol{\Phi}^T(\eta_0,\mu_0)\boldsymbol{F}\boldsymbol{A}_m(\eta_0) + \mu_0\frac{e^{-\frac{\tau_0}{\eta_0}}e^{-\frac{\tau_0}{\mu_0}}}{2N(\eta_0)}\left[\lambda_0\,\boldsymbol{B}_0^T(\eta_0)\boldsymbol{D} - \boldsymbol{\Phi}^T(\eta_0,-\mu_0)\right]\boldsymbol{F}\,\boldsymbol{B}_m(\eta_0)$$

$$+\frac{\lambda_0\mu_0\,e^{-\frac{\tau_0}{\mu_0}}}{2}\int_0^1\frac{e^{-\frac{\tau_0}{\eta}}}{N(\eta)}\left[N_{22}(\eta)\mathbf{B}_0^{1\,T}(\eta) - N_{12}(\eta)\mathbf{B}_0^{2\,T}(\eta)\right]\boldsymbol{D}\boldsymbol{F}\boldsymbol{B}_m^1(\eta)\,d\eta$$

$$-\frac{\mu_0 e^{-\frac{\tau_0}{\mu_0}}}{2}\int_0^1\frac{e^{-\frac{\tau_0}{\eta}}}{N(\eta)}[N_{22}(\eta)\boldsymbol{\Phi}_1^T(\eta,-\mu_0) - N_{12}(\eta)\boldsymbol{\Phi}_2^T(\eta,-\mu_0)]\boldsymbol{F}\,\boldsymbol{B}_m^1(\eta)\,d\eta$$

$$+\frac{\mu_0}{2}\int_0^1\frac{1}{N(\eta)}[N_{22}(\eta)\boldsymbol{\Phi}_1^T(\eta,\mu_0) - N_{12}(\eta)\boldsymbol{\Phi}_2^T(\eta,\mu_0)]\boldsymbol{F}\,\boldsymbol{A}_m^1(\eta)\,d\eta$$

$$+\frac{\mu_0}{2}\int_0^1\frac{1}{N(\eta)}[-N_{21}(\eta)\boldsymbol{\Phi}_1^T(\eta,\mu_0) + N_{11}(\eta)\boldsymbol{\Phi}_2^T(\eta,\mu_0)]\boldsymbol{F}\,\boldsymbol{A}_m^2(\eta)\,d\eta$$

$$+\frac{\lambda_0\mu_0 e^{-\frac{\tau_0}{\mu_0}}}{2}\int_0^1\frac{e^{-\frac{\tau_0}{\eta}}}{N(\eta)}\left[-N_{12}(\eta)\mathbf{B}_0^{1\,T}(\eta) + N_{11}(\eta)\mathbf{B}_0^{2\,T}(\eta)\right]\boldsymbol{D}\boldsymbol{F}\,\boldsymbol{B}_m^2(\eta)\,d\eta$$

$$-\frac{\mu_0\,e^{-\frac{\tau_0}{\mu_0}}}{2}\int_0^1\frac{e^{-\frac{\tau_0}{\eta}}}{N(\eta)}[-N_{12}(\eta)\boldsymbol{\Phi}_1^T(\eta,-\mu_0) + N_{11}(\eta)\boldsymbol{\Phi}_2^T(\eta,-\mu_0)]\boldsymbol{F}\,\boldsymbol{B}_m^2(\eta)\,d\eta$$

$$
\begin{aligned}
&\sum_{\alpha=0}^{N}\left\{\frac{e^{-\frac{\tau_0}{\eta_0}}}{N(\eta_0)}\boldsymbol{B}_m(\eta_0)\boldsymbol{\Upsilon}_\alpha(\eta_0)+\int_0^1 e^{-\frac{\tau_0}{\eta}}\,\frac{\boldsymbol{B}_m^1(\eta)[N_{22}(\eta)\boldsymbol{\Upsilon}_\alpha^1(\eta)-N_{12}(\eta)\boldsymbol{\Upsilon}_\alpha^2(\eta)]}{N(\eta)}d\eta\right.\\
&\qquad\left.+\int_0^1 e^{-\frac{\tau_0}{\eta}}\,\frac{\boldsymbol{B}_m^2(\eta)[-N_{12}(\eta)\boldsymbol{\Upsilon}_\alpha^1(\eta)+N_{11}(\eta)\boldsymbol{\Upsilon}_\alpha^2(\eta)]}{N(\eta)}d\eta\right\}\boldsymbol{a}_\alpha\\
&\qquad+\sum_{\alpha=0}^{N}\left\{\int_0^1 \mu^{m+1}\mathbf{K}(\mu)d\mu+\frac{\boldsymbol{A}_m(\eta_0)}{N(\eta_0)}\left[\boldsymbol{\Upsilon}_\alpha(\eta_0)-\lambda_0\boldsymbol{B}_0^T(\eta_0)\boldsymbol{D}\int_0^1 \mu'^{\,\alpha+1}\mathbf{K}(\mu')\,d\mu'\right]\right.\\
&\qquad+\int_0^1 \frac{\boldsymbol{A}_m^1(\eta)[N_{22}(\eta)\boldsymbol{\Upsilon}_\alpha^1(\eta)-N_{12}(\eta)\boldsymbol{\Upsilon}_\alpha^2(\eta)]}{N(\eta)}d\eta\\
&\qquad+\int_0^1 \frac{\boldsymbol{A}_m^2(\eta)[-N_{12}(\eta)\boldsymbol{\Upsilon}_\alpha^1(\eta)+N_{11}(\eta)\boldsymbol{\Upsilon}_\alpha^2(\eta)]}{N(\eta)}d\eta\\
&\qquad-\lambda_0\int_0^1 \frac{[N_{22}(\eta)\boldsymbol{A}_m^1(\eta)-N_{12}(\eta)\boldsymbol{A}_m^2(\eta)]}{N(\eta)}\boldsymbol{B}_0^{1\,T}(\eta)\boldsymbol{D}\int_0^1 \mu'^{\,\alpha+1}\mathbf{K}(\mu')\,d\mu'\,d\eta\\
&\qquad\left.-\lambda_0\int_0^1 \frac{[-N_{12}(\eta)\boldsymbol{A}_m^1(\eta)+N_{11}(\eta)\boldsymbol{A}_m^2(\eta)]}{N(\eta)}\boldsymbol{B}_0^{2\,T}(\eta)\boldsymbol{D}\int_0^1 \mu'^{\,\alpha+1}\mathbf{K}(\mu')\,d\mu'\,d\eta\right\}\boldsymbol{b}_\alpha\\
&\qquad=-\frac{\mu_0^{m+1}}{2}e^{-\frac{\tau_0}{\mu_0}}\boldsymbol{F}+\frac{\mu_0 e^{-\frac{\tau_0}{\eta_0}}}{2\,N(\eta_0)}\boldsymbol{\Phi}^T(\eta_0,\mu_0)\boldsymbol{F}\boldsymbol{B}_m(\eta_0) \qquad (19)\\
&\qquad+\mu_0\frac{e^{-\frac{\tau_0}{\mu_0}}}{2N(\eta_0)}\left[\lambda_0\,\boldsymbol{B}_0^T(\eta_0)\boldsymbol{D}-\boldsymbol{\Phi}^T(\eta_0,-\mu_0)\right]\boldsymbol{F}\,\boldsymbol{A}_m(\eta_0)\\
&\qquad+\frac{\lambda_0\mu_0\,e^{-\frac{\tau_0}{\mu_0}}}{2}\int_0^1\frac{1}{N(\eta)}\left[N_{22}(\eta)\mathbf{B}_0^{1\,T}(\eta)-N_{12}(\eta)\mathbf{B}_0^{2\,T}(\eta)\right]\boldsymbol{D}\boldsymbol{F}\boldsymbol{A}_m^1(\eta)\,d\eta\\
&\qquad-\frac{\mu_0 e^{-\frac{\tau_0}{\mu_0}}}{2}\int_0^1\frac{1}{N(\eta)}[N_{22}(\eta)\boldsymbol{\Phi}_1^T(\eta,-\mu_0)-N_{12}(\eta)\boldsymbol{\Phi}_2^T(\eta,-\mu_0)]\boldsymbol{F}\,\boldsymbol{A}_m^1(\eta)\,d\eta\\
&\qquad+\frac{\mu_0}{2}\int_0^1\frac{e^{-\frac{\tau_0}{\eta}}}{N(\eta)}[N_{22}(\eta)\boldsymbol{\Phi}_1^T(\eta,\mu_0)-N_{12}(\eta)\boldsymbol{\Phi}_2^T(\eta,\mu_0)]\boldsymbol{F}\,\boldsymbol{B}_m^1(\eta)\,d\eta\\
&\qquad+\frac{\mu_0}{2}\int_0^1\frac{e^{-\frac{\tau_0}{\eta}}}{N(\eta)}[-N_{21}(\eta)\boldsymbol{\Phi}_1^T(\eta,\mu_0)+N_{11}(\eta)\boldsymbol{\Phi}_2^T(\eta,\mu_0)]\boldsymbol{F}\,\boldsymbol{B}_m^2(\eta)\,d\eta\\
&\qquad+\frac{\lambda_0\mu_0 e^{-\frac{\tau_0}{\mu_0}}}{2}\int_0^1\frac{1}{N(\eta)}\left[-N_{12}(\eta)\mathbf{B}_0^{1\,T}(\eta)+N_{11}(\eta)\mathbf{B}_0^{2\,T}(\eta)\right]\boldsymbol{D}\boldsymbol{F}\,\boldsymbol{A}_m^2(\eta)\,d\eta\\
&\qquad-\frac{\mu_0\,e^{-\frac{\tau_0}{\mu_0}}}{2}\int_0^1\frac{1}{N(\eta)}[-N_{12}(\eta)\boldsymbol{\Phi}_1^T(\eta,-\mu_0)+N_{11}(\eta)\boldsymbol{\Phi}_2^T(\eta,-\mu_0)]\boldsymbol{F}\,\boldsymbol{A}_m^2(\eta)\,d\eta
\end{aligned}
$$

3. NUMERICAL RESULTS

The unknown constants $\boldsymbol{a}_{\boldsymbol{\alpha}}$ and $\boldsymbol{b}_{\boldsymbol{\alpha}}$ can be obtained from Eqs.(18,19) and then we can calculate the albedo [3]

$$A^* = \frac{1}{\mu_0}\int_0^1 \mu\,[I_\ell(0,-\mu) + I_r(0,-\mu)\,]\,d\mu \tag{20}$$

and the transmission factor

$$B^* = \frac{1}{\mu_0}\int_0^1 \mu\,[I_\ell(\tau_0,\mu) + I_r(\tau_0,\mu)\,]\,d\mu\,. \tag{21}$$

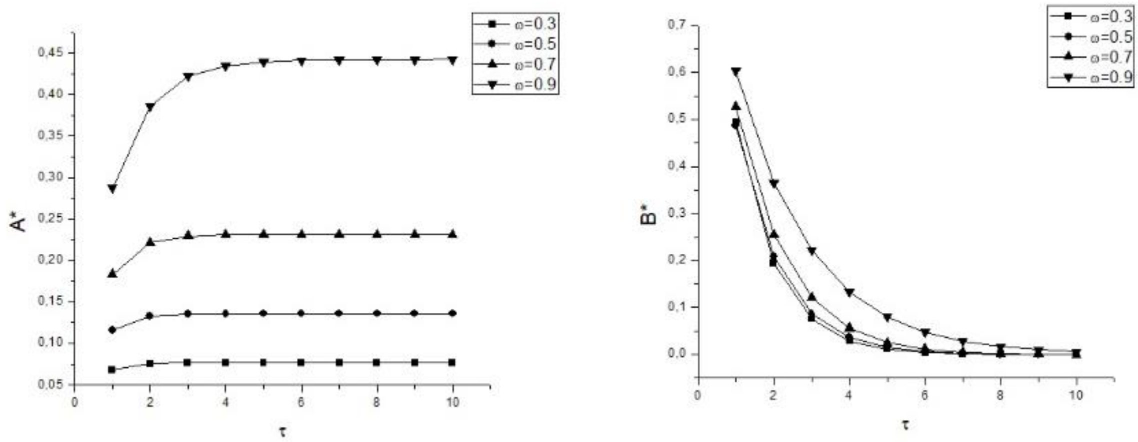

Figure 1. Albedo and transmission factors versus slab thickness for $c = 1$, $\lambda_0 = 0$ and different values of ω

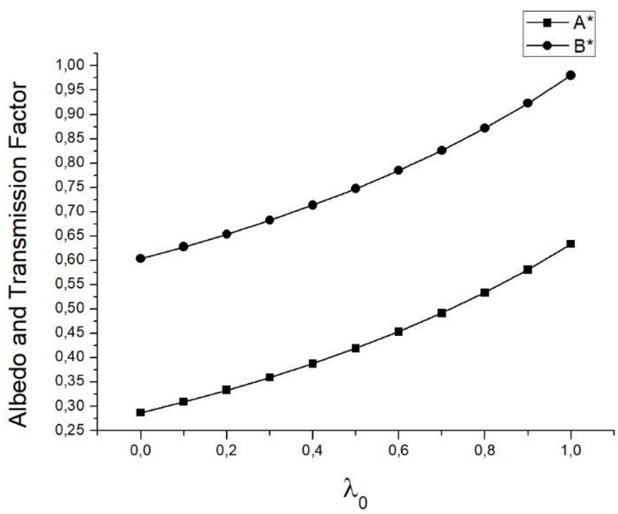

Figure 2. Albedo and transmission factors versus reflection coefficients for $\omega = 0.9$, $c = 1$ and $\tau = 1$

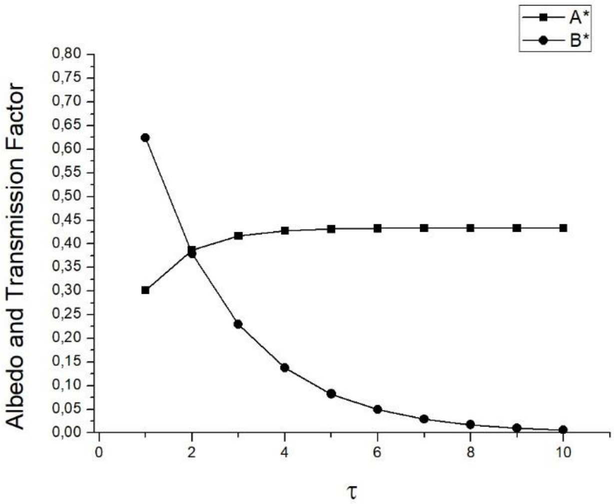

Figure 3. Albedo and transmission factors versus slab thickness for $\omega = 0.9$, $c = 0.8$ and $\lambda_0 = 0.1$

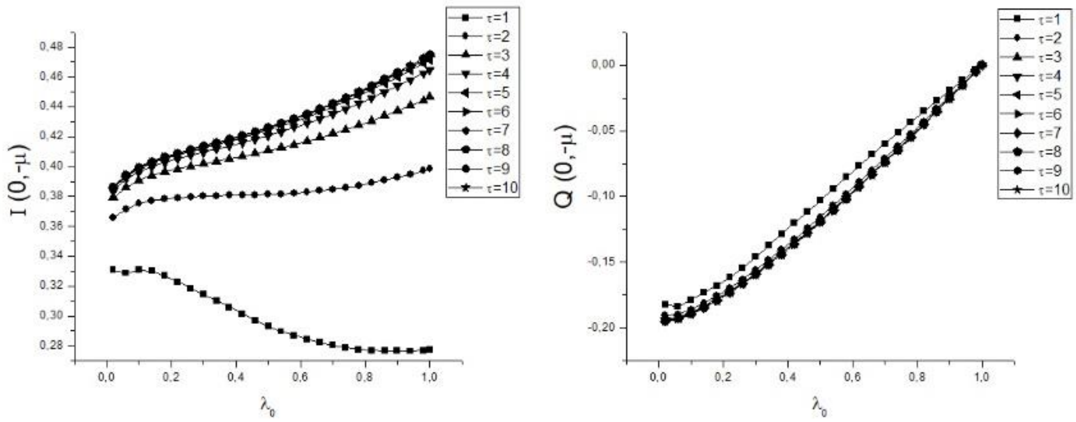

Figure 4. The behavior of the two Stokes parameters according to the optical variable for $\omega = 0.9$, $c = 1$ and different values of λ_0

4. CONCLUSIONS

The vector equation of transfer is solved by using H_N method for a combination of Rayleigh and isotropic scattering case in a finite plane-parallel atmosphere. Albedo and transmission factors are obtained for different values of optical variable and reflection coefficient. These values are agreement with Refs.[3,4]. In Figure 1, the behavior of albedo and transmission factors are given according to optical variable for general Rayleigh scattering case and $\lambda_0 = 0$. The effect of reflection coefficient to the albedo and transmission factors can be seen in Figure 2 for $\tau = 1$ and general Rayleigh scattering case. Albedo and transmission factors depending on optical variable are obtained for general mixture of Rayleigh- and isotropic-scattering for $\lambda_0 = 0.1$ and given in Figure 3. Figure 4 shows the two-Stokes parameters according to reflection coefficient for $\omega = 0.9$, $c = 1$.

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