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New Generalized Inequalities for Functions of Bounded Variation

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Abstract. In this paper, firstly we obtain some generalized trapezoid and midpoint type inequalities for functions of bounded variation using two new generalized identities for Riemann-Stieltjes integrals. Then quadrature formula is also provided.

***Keywords*:** Functions of bounded variation, Ostrowski type inequalities, Riemann-Stieltjes integrals.

Sınırlı Varyasyonlu Fonksiyonlar için Yeni Genelleşmiş Eşitsizlikler

Özet. Bu makalede ilk olarak Riemann-Stieltjes integrallleri için genelleşmiş yeni iki eşitlik kullanılarak sınırlı varyasyonlu fonksiyonlar için yamuk (trapezoid) ve orta nokta (midpoint) tipli bazı genelleşmiş eşitsizlikler elde edilmiştir. Daha sonra karesel formül de sağlanmıştır.

***Anahtar Kelimeler*:** Sınırlı varyasyonlu fonksiyon, Ostrowski tipli eşitsizlikler, Riemann-Stieltjes integralleri.

1. INTRODUCTION

the differentiable mappings.

Theorem 1. Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is the best possible.

Ostrowski inequality has applications in numerical integration, probability and optimization theory, stochastic, statistics, information and integral operator theory. During the past few years, many authors have studied on Ostrowski type inequalities for functions of bounded variation, see for example ([1]-[14], [16]-[18]). Until now, a large number of research papers and books have been written on Ostrowski inequalities and their numerous applications.

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Definition 1. Let $P: a=x_0<x_1<...<x_n=b$ be any partition of $[a,b]$ and let $\Delta f(x_i)=f(x_{i+1})-f(x_i)$. Then $f(x)$ is said to be of bounded variation if the sum

$$\sum_{i=1}^{m}|\Delta f(x_i)|$$

is bounded for all such partitions. Let f be of bounded variation on $[a,b]$, and $\sum(P)$ denotes the sum $\sum_{i=1}^{n}|\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number

$$V_f(a,b):=\sup\left\{\sum(P) : P\in P([a,b])\right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denote the family of partitions of $[a,b]$.

In [12], Dragomir proved the following Ostrowski type inequalities for functions of bounded variation:

Theorem 2. Let $f : [a,b]\to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]V_f(a,b) \tag{1.2}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

Dragomir gave the following trapezoid inequality and midpoint inequality in [9] and [10], respectively:

Theorem 3. Let $f : [a,b]\to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequality

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)dt\right|\leq\frac{1}{2}(b-a)V_f(a,b). \tag{1.3}$$

The constant $\frac{1}{2}$ is the best possible.

Theorem 4. Let $f : [a,b]\to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequality

$$\left|(b-a)f\left(\frac{a+b}{2}\right)-\int_a^b f(t)dt\right|\leq\frac{1}{2}(b-a)V_f(a,b). \tag{1.4}$$

The constant $\frac{1}{2}$ is the best possible.

We introduce the notation $I_n : a=x_0<x_1<...<x_n=b$ for a division of the interval $[a,b]$ with $h_i := x_{i+1}-x_i$ and $v(h)=\max\{h_i : i=0,1,...,n-1\}$. Then we have

$$\int_a^b f(t)dt = A_T(f,I_n) + R_T(f,I_n) \tag{1.5}$$

where

$$A_T(f,I_n) := \sum_{i=0}^{n} \frac{f(x_i)+f(x_{i+1})}{2} h_i \tag{1.6}$$

and the remainder term satisfies

$$|R_T(f,I_n)| \le \frac{1}{2} v(h) V_f(a,b). \tag{1.7}$$

Similarly, we have

$$\int_a^b f(t)dt = A_M(f,I_n) + R_M(f,I_n) \tag{1.8}$$

where

$$A_T(f,I_n) := \sum_{i=0}^{n} f\left(\frac{x_i+x_{i+1}}{2}\right) h_i \tag{1.9}$$

and the remainder term satisfies

$$|R_M(f,I_n)| \le \frac{1}{2} v(h) V_f(a,b). \tag{1.10}$$

In this work, we obtain some new generalized trapezoid and midpoint type integral inequalities for functions of bounded variation by using the new kernel which is given by Tseng and Hwang in [19]. Then we give some applications for our results.

2. GENERALIZED TRAPEZOID AND MIDPOINT INEQUALITIES

Throughout this paper, let $a \le c < d \le b$ in $\mathbb{R}$ with $a+b=c+d$.

Now, we give our main results:

Theorem 5. Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then, we have the following generalized inequality

$$\left|\int_a^b f(t)dt - \left\{\left(\frac{a+b}{2} - c\right)\left[f(c) + f(d)\right] + (c-a)\left[f(a) + f(b)\right]\right\}\right|$$
$$\leq \max\left\{(c-a), \left(\frac{a+b}{2} - c\right), \left(d - \frac{a+b}{2}\right), (b-d)\right\} V_f(a,b). \tag{2.1}$$

Proof. Consider the kernel $P_1(x)$ as follows:

$$P_1(x) = \begin{cases} c - x, & x \in [a,c) \\ \frac{a+b}{2} - x, & x \in [c,d) \\ d - x, & x \in [d,b]. \end{cases}$$

Integration by parts gives us

$$\int_a^b P_1(x)df(x)$$
$$= \int_a^b f(t)dt - \left\{\left(\frac{a+b}{2} - c\right)\left[f(c) + f(d)\right] + (c-a)\left[f(a) + f(b)\right]\right\}. \tag{2.2}$$

It is well known that if $g, \ f : [a,b] \to \mathrm{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exists and

$$\left|\int_a^b g(t)df(t)\right| \leq \sup_{t\in[a,b]} |g(t)| V_f(a,b). \tag{2.3}$$

On the other hand, by using (2.3), we get

$$\left|\int_a^b P_1(x)df(x)\right|$$
$$\leq \left|\int_a^c (c-x)df(x)\right| + \left|\int_c^d \left(\frac{a+b}{2} - x\right)df(x)\right| + \left|\int_d^b (d-x)df(x)\right|$$

$$\leq \sup_{x\in[a,c]}|c-x|V_f(a,c)+\sup_{x\in[c,d]}\left|\frac{a+b}{2}-x\right|V_f(c,d)+\sup_{x\in[d,b]}|d-x|V_f(d,b)$$

$$=(c-a)V_f(a,c)+\max\left\{\left(\frac{a+b}{2}-c\right),\left(d-\frac{a+b}{2}\right)\right\}V_f(c,d)+(b-d)V_f(d,b)$$

$$\leq \max\left\{(c-a),\left(\frac{a+b}{2}-c\right),\left(d-\frac{a+b}{2}\right),(b-d)\right\}V_f(a,b).$$

This completes the proof.

Remark 1. If we choose $c=a$ and $d=b$ in Theorem 5, then the inequality (2.1) reduces to the trapezoid inequality (1.3).

Corollary 1. Under the assumption of Theorem 5, let $c=(1-\lambda)a+\lambda b$ and $d=\lambda a+(1-\lambda)b$ with $0\leq\lambda<\frac{1}{2}$, then we have the following inequality

$$\left|\int_a^b f(t)dt-\left\{(b-a)\left(\frac{1}{2}-\lambda\right)\left[f((1-\lambda)a+\lambda b)+f(\lambda a+(1-\lambda)b)\right]\right.\right.$$

$$\left.\left.+\lambda(b-a)[f(a)+f(b)]\right\}\right| \tag{2.4}$$

$$\leq (b-a)\left[\frac{1}{4}+\left|\lambda-\frac{1}{4}\right|\right]V_f(a,b).$$

Remark 2. If we choose $\lambda=0$ in Corollary 1, then the inequality (2.4) reduces to the trapezoid inequality (1.3).

Corollary 2. If we choose $\lambda=\frac{1}{3}$ in Corollary 1, we have the inequality

$$\left|\int_a^b f(t)dt-\frac{b-a}{6}\left\{f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)+2[f(a)+f(b)]\right\}\right|$$

$$\leq \frac{1}{3}(b-a)V_f(a,b).$$

Corollary 3. If we choose $\lambda=\frac{1}{4}$ in Corollary 1, we have the inequality

$$\left|\int_a^b f(t)dt - \frac{b-a}{4}\left\{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f(a)+f(b)\right\}\right|$$

$$\leq \frac{1}{4}(b-a)V_f(a,b).$$

Corollary 4. Under the assumption of Theorem 5, suppose that $f \in C^1[a,b]$. Then we have

$$\left|\int_a^b f(t)dt - \left\{\left(\frac{a+b}{2}-c\right)[f(c)+f(d)]+(c-a)[f(a)+f(b)]\right\}\right|$$

$$\leq \max\left\{(c-a),\left(\frac{a+b}{2}-c\right),\left(d-\frac{a+b}{2}\right),(b-d)\right\}\|f'\|_1,$$

where $\|.\|_1$ is the L_1-norm defined by

$$\|f'\|_1 := \int_a^b f'(t)dt.$$

Corollary 5. Under the assumption of Theorem 5, let $f : [a,b] \to \mathrm{R}$ be a Lipschitzian with the constant $L > 0$. Then

$$\left|\int_a^b f(t)dt - \left\{\left(\frac{a+b}{2}-c\right)[f(c)+f(d)]+(c-a)[f(a)+f(b)]\right\}\right|$$

$$\leq \max\left\{(c-a),\left(\frac{a+b}{2}-c\right),\left(d-\frac{a+b}{2}\right),(b-d)\right\}(b-a)L.$$

Theorem 6. Let $f : [a,b] \to \mathrm{R}$ be a mapping of bounded variation on $[a,b]$. Then, we have the following generalized inequality

$$\left|\int_a^b f(t)dt - \left\{(d-c)f\left(\frac{a+b}{2}\right)+(c-a)[f(a)+f(b)]\right\}\right| \tag{2.5}$$

$$\leq \max\left\{(c-a),\left(\frac{a+b}{2}-c\right),\left(d-\frac{a+b}{2}\right),(b-d)\right\}V_f(a,b).$$

Proof. Integration by parts gives us

$$\int_a^b P_2(x)df(x)$$
$$=\int_a^b f(t)dt-\left\{(d-c)f\left(\frac{a+b}{2}\right)+(c-a)[f(a)+f(b)]\right\}.$$

where the kernel $P_2(x)$ is defined by

$$P_2(x)=\begin{cases} a-x, & x\in[a,c) \\ c-x, & x\in\left[c,\frac{a+b}{2}\right) \\ d-x, & x\in\left[\frac{a+b}{2},d\right) \\ b-x, & x\in[d,b]. \end{cases}$$

Using the inequalities (2.3), we have

$$\left|\int_a^b P_2(x)df(x)\right|$$

$$\le\left|\int_a^c (a-x)df(x)\right|+\left|\int_c^{\frac{a+b}{2}}(c-x)df(x)\right|+\left|\int_{\frac{a+b}{2}}^d (d-x)df(x)\right|+\left|\int_d^b (b-x)df(x)\right|$$
$$\le\sup_{x\in[a,c]}|a-x|\,V_f(a,c)+\sup_{x\in\left[c,\frac{a+b}{2}\right]}|c-x|\,V_f\left(c,\frac{a+b}{2}\right)$$
$$+\sup_{x\in\left[\frac{a+b}{2},d\right]}|d-x|V_f\left(\frac{a+b}{2},d\right)+\sup_{x\in[b,d]}|b-x|V_f(d,b)$$
$$=(c-a)V_f(a,c)+\left(\frac{a+b}{2}-c\right)V_f\left(c,\frac{a+b}{2}\right)+\left(d-\frac{a+b}{2}\right)V_f\left(\frac{a+b}{2},d\right)+(b-d)V_f(d,b))$$
$$\le\max\left\{(c-a),\left(\frac{a+b}{2}-c\right),\left(d-\frac{a+b}{2}\right),(b-d)\right\}V_f(a,b).$$

Thus the proof is completed.

Remark 3. If we choose $c=a$ and $d=b$ in Theorem 6, then the inequality (2.5) reduces to the midpoint inequality (1.4).

Corollary 6. Under the assumption of Theorem 6, let $c=(1-\lambda)a+\lambda b$ and $d=\lambda a+(1-\lambda)b$ with $0\le\lambda<\frac{1}{2}$, then we have the following inequality

$$\left|\int_a^b f(t)dt - \{\lambda(b-a)[f((1-\lambda)a+\lambda b)+f(\lambda a+(1-\lambda)b)]\right.$$

$$\left.+(b-a)(1-2\lambda)f\left(\frac{a+b}{2}\right)\}\right|$$

$$\leq (b-a)\left[\frac{1}{4}+\left|\lambda-\frac{1}{4}\right|\right]V_f(a,b).$$

Remark 4. If we choose $\lambda=0$ in Corollary 6, then the inequality (2.6) reduces to the midpoint inequality (1.4).

Corollary 7. If we choose $\lambda=\frac{1}{3}$ in Corollary 6, we have the inequality

$$\left|\int_a^b f(t)dt - \frac{b-a}{3}\left\{f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)+f\left(\frac{a+b}{2}\right)\right\}\right|$$

$$\leq \frac{1}{3}(b-a)V_f(a,b).$$

Corollary 8. If we choose $\lambda=\frac{1}{4}$ in Corollary 6, we have the inequality

$$\left|\int_a^b f(t)dt - \frac{b-a}{4}\left\{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+2f\left(\frac{a+b}{2}\right)\right\}\right|$$

$$\leq \frac{1}{4}(b-a)V_f(a,b).$$

Corollary 9. Under the assumption of Theorem 6, suppose that $f\in C^1[a,b]$. Then we have

$$\left|\int_a^b f(t)dt - \left\{(d-c)f\left(\frac{a+b}{2}\right)+(c-a)[f(a)+f(b)]\right\}\right|$$

$$\leq \max\left\{(c-a),\left(\frac{a+b}{2}-c\right),\left(d-\frac{a+b}{2}\right),(b-d)\right\}\|f'\|_1.$$

Corollary 10. Under the assumption of Theorem 6, let $f:[a,b]\to \mathrm{R}$ be a Lipschitzian with the constant $L>0$. Then

$$\left|\int_a^b f(t)dt-\left\{(d-c)f\left(\frac{a+b}{2}\right)+(c-a)[f(a)+f(b)]\right\}\right|$$

$$\leq \max\left\{(c-a),\left(\frac{a+b}{2}-c\right),\left(d-\frac{a+b}{2}\right),(b-d)\right\}(b-a)L.$$

3. APPLICATION TO QUADRATURE FORMULA

Now we introduce the intermediate points c_i and d_i, $x_i \leq c_i < d_i \leq x_{i+1}$, $(i=0,1,\ldots,n-1)$ in the division $I_n : a=x_0<x_1<\ldots<x_n=b$. Let $h_i := x_{i+1}-x_i$ and $v(h)=\max\{h_i : i=0,1,\ldots,n-1\}$ and define the sum

$$A_T(f,I_n,c_i,d_i) \tag{3.1}$$
$$:=\sum_{i=0}^{n}\left\{\left(\frac{x_i+x_{i+1}}{2}-c_i\right)[f(c_i)+f(d_i)]+(c_i-x_i)[f(x_i)+f(x_{i+1})]\right\}.$$

Then the following theorem holds:

Theorem 7. Let f be as in Theorem 5. Then

$$\int_a^b f(t)dt = A_T(f,I_n,c_i,d_i)+R_T(f,I_n,c_i,d_i) \tag{3.2}$$

where $A_T(f,I_n,c_i,d_i)$ is defined as above and the remainder term $R(f,I_n)$ satisfies

$$|R_T(f,I_n,c_i,d_i)| \tag{3.3}$$
$$\leq \max_{i\in\{0,1,\ldots,n-1\}}\left[\max\left\{(c_i-x_i),\left(\frac{x_i+x_{i+1}}{2}-c_i\right),\left(d_i-\frac{x_i+x_{i+1}}{2}\right),(x_{i+1}-d_i)\right\}\right]V_f(a,b).$$

Proof. Applying Theorem 5 with the interval $[x_i,x_{i+1}]$ $(i=0,1,\ldots,n-1)$, we have

$$\left|\int_{x_i}^{x_{i+1}} f(t)dt-\left\{\left(\frac{x_i+x_{i+1}}{2}-c_i\right)[f(c_i)+f(d_i)]+(c_i-x_i)[f(x_i)+f(x_{i+1})]\right\}\right| \tag{3.4}$$
$$\leq \max\left\{(c_i-x_i),\left(\frac{x_i+x_{i+1}}{2}-c_i\right),\left(d_i-\frac{x_i+x_{i+1}}{2}\right),(x_{i+1}-d_i)\right\}V_f(x_i,x_{i+1}).$$

for all $i\in\{0,1,...,n-1\}$. Summing the inequality (3.4) over i from 0 to $n-1$ and using the generalized triangle inequality, we have

$$
\begin{aligned}
&\left|R_T(f,I_n,c_i,d_i)\right| \\
&\leq \sum_{i=0}^{n}\max\left\{(c_i-x_i),\left(\frac{x_i+x_{i+1}}{2}-c_i\right),\left(d_i-\frac{x_i+x_{i+1}}{2}\right),(x_{i+1}-d_i)\right\}V_f(x_i,x_{i+1}) \\
&\leq \max_{i\in\{0,1,...,n-1\}}\left[\max\left\{(c_i-x_i),\left(\frac{x_i+x_{i+1}}{2}-c_i\right),\left(d_i-\frac{x_i+x_{i+1}}{2}\right),(x_{i+1}-d_i)\right\}\right]\sum_{i=0}^{n}V_f(x_i,x_{i+1}) \\
&= \max_{i\in\{0,1,...,n-1\}}\left[\max\left\{(c_i-x_i),\left(\frac{x_i+x_{i+1}}{2}-c_i\right),\left(d_i-\frac{x_i+x_{i+1}}{2}\right),(x_{i+1}-d_i)\right\}\right]V_f(a,b)
\end{aligned}
$$

which completes the proof.

Remark 5. If we choose $c_i=x_i$ and $d_i=x_{i+1}$ in Theorem 7, then we have (1.5) with (1.6) and (1.7).

By using Theorem 6 and following similar steps of Theorem 5, we have the following theorem.

Theorem 8. Let f be as in Theorem 6. Then

$$\int_a^b f(t)dt = A_M(f,I_n,c_i,d_i)+R_M(f,I_n,c_i,d_i) \tag{3.5}$$

where $A_M(f,I_n,c_i,d_i)$ is defined as

$$A_M(f,I_n,c_i,d_i):=\sum_{i=0}^{n}\left\{(d_i-c_i)f\left(\frac{x_i+x_{i+1}}{2}\right)+(c_i-x_i)[f(x_i)+f(x_{i+1})]\right\}. \tag{3.6}$$

and the remainder term $R_M(f,I_n,c_i,d_i)$ satisfies

$$
\begin{aligned}
&\left|R_M(f,I_n,c_i,d_i)\right| \\
&\leq \max_{i\in\{0,1,...,n-1\}}\left[\max\left\{(c_i-x_i),\left(\frac{x_i+x_{i+1}}{2}-c_i\right),\left(d_i-\frac{x_i+x_{i+1}}{2}\right),(x_{i+1}-d_i)\right\}\right]\bigvee_a^b(f).
\end{aligned}
$$

Remark 6. If we choose $c_i=x_i$ and $d_i=x_{i+1}$ in Theorem 8, then we get (1.10) with (1.8) and (1.9).

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