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Non-Isotropic Potential Theoretic Inequality

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Abstract: In this paper, the new weighted inequalities were derived by β-distance which is similar to the given inequality for the potential operator defined in [1]. The results presented here would provide extensions of those given in earlier works.

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İzotropik Olmayan Potansiyel Teorik Eşitsizlik

Özet: Bu yazıda, [1] 'de tanımlanan potansiyel operatör için verilen eşitsizliğe benzer β-mesafesi ile türetilen yeni ağırlıklı eşitsizlikler elde edilmiştir. Burada sunulan sonuçlar daha önceki çalışmalarda ki verilenleri destekler.

Anahtar Kelimeler: Adams iz eşitsizliği, Stummel sınıfı, Morrey uzayları, izotropik olmayan mesafe.

1. INTRODUCTION

The following inequality has been obtained by D. Adams [1];

Let V be a non negative function in the Morrey space $L_{1,\lambda}(\mathbb{R}^n), \lambda > n-p$.

For $\forall u \in C_0^\infty(\mathbb{R}^n), q = p\dfrac{\lambda}{n-p}, 1 < p < n$, the following inequality is valid;

$$\left(\int_{\mathbb{R}^n} |u(x)|^q V(x)dx\right)^{\frac{1}{q}} \le C(p,\lambda,n)\|V\|^{\frac{1}{q}}_{L_{1,\lambda}(\mathbb{R}^n)}\|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{1}$$

where $L_{1,\lambda}(\mathbb{R}^n)$ is Morrey space.

Morrey spaces , $L_{p,\lambda}$, were introduced by Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations [7]. Later, Morrey spaces found important applications to Navier Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients and potential theory. An exposition of the Morrey spaces can be found in the book [5].

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Morrey spaces were widely studied during last decades, including the study of classical operators of harmonic analysis such as maximal, singular and potential operators.

Definition 1.1 *Let* $1 \le p < \infty$, $0 \le \lambda \le n$. *We define the Morrey space* , $L_{p,\lambda}\left(\mathbb{R}^n\right)$, *as the set of finite normed locally integrable functions*

$$\|f\|_{L_{p,\lambda}} := \sup_{x \in \mathbb{R}^n, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))}, \tag{2}$$

Note that if $p = 1$, $L_{1,\lambda}\left(\mathbb{R}^n\right)$ Morrey space is defined as follows;

$$\begin{aligned} L_{1,\lambda}\left(\mathbb{R}^n\right) &= \left\{ f \in L_1^{loc}\left(\mathbb{R}^n\right) : \|f\|_{L_{1,\lambda}\left(\mathbb{R}^n\right)} \right. \\ &\equiv \left. \sup_{x \in \mathbb{R}^n, r>0} \frac{1}{r^\lambda} \int_{|x-y|<r} |f(y)| dy < +\infty \right\}, 0 < \lambda < n. \end{aligned} \tag{3}$$

According to the definition of $L_{p,\lambda}$, the parameter p describes the local integrability, while λ describes the global integrability. Unlike $L_{p,\lambda}$ with $p > 1$, it is not the case that we can characterize $L_{1,\lambda}$ in terms of the Littlewood-Paley decomposition. For this reason, the singular integral operators like the Riesz transforms are not bounded on $L_{1,\lambda}$. Nevertheless , this space can be compared with other function spaces. This is what we do in the present paper.

This paper aims at using β-distance establish an imbedding similar to (1), assuming more general hypotheses on the function V .

Firstly, we define a non isotropic distance or β-distance in n dimensional Euclidean space $\mathbb{R}^n$.

It is well known that the families of integral operators with positive kernels have many applications in different problems, in the theory of differantial equation, harmonic analysis etc. Integral operators depending on difference between the variables have principal aplications. For multidimensional case, this type of kernels are functions of euclidean distance between two points.

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_n), \beta_k \ge \frac{1}{2}$, $k = 1,2,\ldots,n$ and $|\beta| = \beta_1 + \beta_2 + \ldots + \beta_n$. For $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$

$$|x-y|_\beta := (|x_1 - y_1|^{\frac{1}{\beta_1}} + |x_2 - y_2|^{\frac{1}{\beta_2}} + \ldots + |x_n - y_n|^{\frac{1}{\beta_n}})^{\frac{|\beta|}{n}},$$

is the non-isotropic distance or β-distance x and y, given in [2], ([10]−[13]), [17].

For any positive t, it is easy to see that this distance has the following properties of homogeneity

,

$$\left(\left|t^{\beta_1}x_1\right|^{\frac{1}{\beta_1}}+\ldots+\left|t^{\beta_n}x_n\right|^{\frac{1}{\beta_n}}\right)^{\frac{|\beta|}{n}}=t^{\frac{|\beta|}{n}}|x|_\beta,\ t>0. \tag{4}$$

This equality gives us that the non-isotropic β-distance is the order of a homogeneous function $\frac{|\beta|}{n}$. Thus the non-isotropic β-distance has the following properties:

$$1. |x|_\beta = 0 \Leftrightarrow x=\theta, \theta=(0,0,\ldots,0).$$

$$2. \left|t^\beta x\right|_\beta = |t|^{\frac{|\beta|}{n}}|x|_\beta.$$

$$3. |x+y|_\beta \le k(|x|_\beta+|y|_\beta),$$

where $k=2^{\left(1+\frac{1}{\beta_{\min}}\right)^{\frac{|\beta|}{n}}}$, $\beta_{\min}=\min\{\beta_1,\beta_2,\ldots,\beta_n\}$.

Here we consider β-spherical coordinates by the following formulas:

$$\begin{aligned}
x_1 &= (\rho\cos\varphi_1)^{2\beta_1}.\\
x_2 &= (\rho\sin\varphi_1\cos\varphi_2)^{2\beta_2}.\\
&\vdots\\
x_{n-1} &= (\rho\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\cos\varphi_{n-1})^{2\beta_{n-1}}\\
x_n &= (\rho\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\sin\varphi_{n-1})^{2\beta_n}
\end{aligned} \tag{5}$$

where $0\le\varphi_1,\varphi_2,\ldots,\varphi_{n-2}\le\pi$ and $0\le\varphi_{n-1}\le 2\pi$.

By using β-spherical coordinates, we get that $|x|_\beta=\rho^{\frac{2|\beta|}{n}}$.

Firstly, we will define the $\beta-$ball $B_\beta(x,r)$ generated by the $\beta-$distance. For a positive r and any $x\in\mathbb{R}^n$, the open β-ball with radius r and a center x as

$$B_\beta(x,r)=\{\sigma:|x-y|_\beta<r\}$$

2. PRELIMINARY RESULTS

In this section, we introduce Morrey space $L^\beta_{1,\lambda}(\mathbb{R}^n)$ and S^β_p, we give some results relating them. The Stummel class S_p was introduced by Ragusa and Zamboni [8]. This class is a class of functions related to local behavior of mapping by generalized fractional integral operators and the generalized

Morrey spaces are classes of functions related to local behavior of Hardy-Littlewood maximal function.Now, we introduce S_p class depending on β -distance as follows.

Definition 2.1 *Let* $1 < p < n$,

$$S_p^\beta = \left\{ f \in L_1^{loc}\left(\mathbb{R}^n\right): \sup_{x\in\mathbb{R}^n} \int_{|x-y|_\beta < r} \frac{|f(y)|}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}}} dy = \eta_\beta(r) \to 0 \, for r \to 0 \right\}. \quad (6)$$

$L_{1,\lambda}^\beta\left(\mathbb{R}^n\right)$ Morrey space is defined as follows.

Definition 2.2 *Morrey space* $L_{1,\lambda}^\beta\left(\mathbb{R}^n\right)$ *generated by* β *-distance;*

$$\begin{aligned} L_{1,\lambda}^\beta\left(\mathbb{R}^n\right) &= \left\{ f \in L_1^{loc}\left(\mathbb{R}^n\right): \| f \|_{L_{1,\lambda}^\beta\left(\mathbb{R}^n\right)} \right. \\ &\left. \equiv \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{r^{\frac{2|\beta|}{n}\lambda}} \int_{|x-y|_\beta < r} |f(y)| dy < +\infty \right\}, 0 < \lambda < n \end{aligned}$$

where $\lambda > n - p$.

The next lemma gives a relation between the space S_p^β and $L_{1,\lambda}^\beta$.

Lemma 2.1 *If* V *belongs to* $L_{1,\lambda}^\beta\left(\mathbb{R}^n\right)$*, then* V *belongs to* S_p^β*, and*

$$\int_{|x-y|_\beta < r} \frac{|V(y)|}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}}} dy \le C(n, p, \lambda, \beta) r^{(\lambda-(n-p))\frac{2|\beta|}{n}} \|V\|_{L_{1,\lambda}^\beta\left(\mathbb{R}^n\right)},$$

where $(n-p)\frac{2|\beta|}{n} < \lambda \frac{2|\beta|}{n} < n \frac{2|\beta|}{n}$.

Conversely, if V *belongs to* S_p^β *and* $\eta_\beta(r): r^{\alpha\frac{2|\beta|}{n}}$*, then* V *belongs to* $L_{1,(n-p+\alpha)\frac{2|\beta|}{n}}^\beta\left(\mathbb{R}^n\right)$.

Proof. About the first part, we have

$$\int_{|x-y|_\beta<r}\frac{|V(y)|}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}}}dy$$

$$=\sum_{k=0}^{+\infty}\int_{\frac{r}{2^{k+1}}\le|x-y|_\beta<\frac{r}{2^k}}\frac{|V(y)|}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}}}dy$$

$$\le\sum_{k=0}^{+\infty}\left(\frac{2^{k+1}}{r}\right)^{(n-p)\frac{2|\beta|}{n}}\int_{|x-y|_\beta<\frac{r}{2^k}}|V(y)|dy$$

$$\le 2^{(n-p)\frac{2|\beta|}{n}}\sum_{k=0}^{+\infty}\left(\frac{2^k}{r}\right)^{(n-p)\frac{2|\beta|}{n}}\left(\frac{r}{2^k}\right)^{\frac{2|\beta|}{n}\lambda}\sup_{r>0}\frac{1}{\left(\frac{r}{2^k}\right)^{\frac{2|\beta|}{n}\lambda}}\int_{|x-y|_\beta<\frac{r}{2^k}}|V(y)|dy$$

$$\le 2^{(n-p)\frac{2|\beta|}{n}}r^{(\lambda-(n-p))\frac{2|\beta|}{n}}\sum_{k=0}^{+\infty}2^{k\frac{2|\beta|}{n}((n-p)-\lambda)}\|V\|_{L_{1,\lambda}^{\beta}(\mathbb{R}^n)}$$

$$=r^{(\lambda-(n-p))\frac{2|\beta|}{n}}C(n,p,\lambda,\beta)\|V\|_{L_{1,\lambda}^{\beta}(\mathbb{R}^n)}.$$

The second part is obvious, indeed

$$\int_{|x-y|_\beta<r}|V(y)|dy\le r^{(n-p)\frac{2|\beta|}{n}}\int_{|x-y|_\beta<r}\frac{|V(y)|}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}}}dy\le Cr^{(n-p+\alpha)\frac{2|\beta|}{n}}.$$

Lemma 2.2 *Let* $V\in S_p^\beta$. *Then there exists a positive constant* $C_d=C_d(n)$ *such that*

$$\eta_\beta(r)\le C_d\eta_\beta\left(\frac{r}{2}\right),\ r>0.$$

Proof. Let $m=m(n)\in\mathbb{N}$, $x_1,\ldots,x_{m(n)}\in B_\beta(x_0,r)$ such that

$$B_\beta(x_0,r)\subseteq\bigcup_{j=1}^m B_\beta\left(x_j,\frac{r}{2}\right).$$

We have

$$\int_{|x_0-y|_\beta<r}\frac{|V(y)|}{|x_0-y|_\beta^{(n-p)\frac{2|\beta|}{n}}}dy\le\sum_{j=1}^m\int_{|x_j-y|_\beta<\frac{r}{2}}\frac{|V(y)|}{|x_0-y|_\beta^{(n-p)\frac{2|\beta|}{n}}}dy=\sum_{j=1}^m I_j$$

and

$$I_j = \int_{|x_0-y|_\beta \geq |x_j-y|_\beta, |x_j-y|_\beta < \frac{r}{2}} \frac{|V(y)|}{|x_0-y|_\beta^{(n-p)\frac{2|\beta|}{n}}} dy$$

$$+ \int_{|x_0-y|_\beta < |x_j-y|_\beta < \frac{r}{2}} \frac{|V(y)|}{|x_0-y|_\beta^{(n-p)\frac{2|\beta|}{n}}} dy$$

$$= A_j + B_j.$$

Since

$$A_j \leq \int_{|x_j-y|_\beta < \frac{r}{2}} \frac{|V(y)|}{|x_j-y|_\beta^{(n-p)\frac{2|\beta|}{n}}} dy \leq \eta_\beta(\frac{r}{2})$$

$$B_j \leq \int_{|x_0-y|_\beta < \frac{r}{2}} \frac{|V(y)|}{|x_0-y|_\beta^{(n-p)\frac{2|\beta|}{n}}} dy \leq \eta_\beta(\frac{r}{2})$$

then, we get the conclusion.

The following definition gives a generalization of S_p^β.

Definition 2.3 *Let* $\varphi :]0,+\infty[\to]0,+\infty[$ *be a non-decreasing continuous function with* $\lim_{t\to 0} \varphi(t) = 0$. *We say that* $V : \mathbb{R}^n \to \mathbb{R}$ *belongs to the class* $S_{p,\varphi}^\beta$ *if and only if there exists a non decreasing function* $\xi_\beta :]0,+\infty[\to]0,+\infty[$ *with* $\lim_{r\to 0} \xi_\beta(r) = 0$ *such that*

$$\sup_{x\in\mathbb{R}^n} \int_{|x-y|_\beta < r} \frac{|V(y)|}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}} \varphi(|x-y|_\beta)} dy \leq \xi_\beta(r), 1 < p < n. \qquad (7)$$

In order to show that a function $V \in S_p^\beta$ belongs to an appropriate $S_{p,\varphi}^\beta$, we give the following lemma.

Lemma 2.3 *Let* $V \in S_p^\beta$ *such that* $\exists \gamma \in]0,1[: \int_0^1 t^{-1} \eta_\beta^{1-\gamma}(t) dt < +\infty$, *where* $\eta_\beta(t)$ *is the Stummel modulus generated by* β*-distance of* V. *Then* $V \in S_{p,\eta_\beta^\gamma}^\beta$ *and*

$$\int_{|x-y|_\beta < r} \frac{|V(y)|}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}} \eta_\beta^\gamma(|x-y|_\beta)} dy \leq \mu_\beta(r), \qquad (8)$$

where

$$\mu_\beta(r)=\frac{2}{C}\int_0^r t^{-1}\eta_\beta^{1-\gamma}(t)dt.$$

Proof. Using Lemma 2.2, we can obtain

$$\int_{|x-y|_\beta<r}\frac{|V(y)|}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}}\eta_\beta^\gamma\left(|x-y|_\beta\right)}dy=\sum_{k=0}^{+\infty}\int_{\frac{r}{2^{k+1}}\le|x-y|_\beta<\frac{r}{2^k}}\frac{|V(y)|}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}}\eta_\beta^\gamma\left(|x-y|_\beta\right)}dy$$

$$\le\sum_{k=0}^{+\infty}\eta_\beta\left(\frac{r}{2^k}\right)\left[\eta_\beta\left(\frac{r}{2^{k+1}}\right)\right]^{-\gamma}$$

$$\le C^{-\gamma}\sum_{k=0}^{+\infty}\left[\eta_\beta\left(\frac{r}{2^k}\right)\right]^{1-\gamma}.$$

The last series converges observing that

$$\int_0^r t^{-1}\eta_\beta^{1-\gamma}(t)dt=\sum_{k=0}^{+\infty}\int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}}t^{-1}\eta_\beta^{1-\gamma}(t)dt$$

$$\ge\sum_{k=0}^{+\infty}\left[\eta_\beta\left(\frac{r}{2^{k+1}}\right)\right]^{1-\gamma}\frac{2^k}{r}\int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}}dt$$

$$\ge\frac{1}{2}C^{1-\gamma}\sum_{k=0}^{+\infty}\left[\eta_\beta\left(\frac{r}{2^k}\right)\right]^{1-\gamma}.$$

3. MAIN RESULTS

In this section ,under the more general hypotheses for function V , we will obtain embeddings like (1) using β -distance.

Firstly, we need the following definitions:

Let f and h be measurable functions such that $f\in L_1^{loc}(\mathbb{R}^n)$ and $h\ge 0$, we set the fractional integral generated by β -distance of order p as

$$I_p^\beta(f)(x)=\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}}}dy, \tag{9}$$

and we get generalized fractional integral generated by β -distance;

$$I^{\beta}_{p,h}(f)(x)=\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|_{\beta}^{(n-p)\frac{2|\beta|}{n}}h\left(|x-y|_{\beta}\right)}dy. \tag{10}$$

The important properties of the fractional integrals, their generalizations were studied by many authors. We refer to papers $[6]-[13]$, [17].

Theorem 3.1 *Let* $V\in S^{\beta}_{p,\varphi}$ *with* $\varphi(t)$ *and* $\xi_{\beta}(t)$ *as in Definition 2.3. Then, for any* $\sigma\in\,]0,1[$, *there exists a non-decreasing positive function* $G(t)$ *such that*

$$\int_{B_{\beta}(y,r)}G\left(\frac{I^{\beta}_{p,\varphi^{\sigma}}\left(f^{p}\right)}{\|f\|_{p}^{p}}\right)V(x)dx\le\xi_{\beta}(r) \tag{11}$$

for all $f\in C_0^{\infty}(\mathbb{R}^n)$ *,where* $B_{\beta}(.,r)$ *is* $\beta-$ball *with radius* r *containing the support of* f. *Also*

$$\lim_{t\to\infty}\frac{G(t)}{t}=+\infty. \tag{12}$$

Proof. For $\varepsilon>0$ and $0<\sigma<1$, we obtain,

$$\begin{aligned}I^{\beta}_{p,\varphi^{\sigma}}\left(f^{p}\right)(x) &=\int_{|x-y|_{\beta}\le\varepsilon}\frac{|f(y)|^{p}\varphi\left(|x-y|_{\beta}\right)}{|x-y|_{\beta}^{(n-p)\frac{2|\beta|}{n}}\varphi^{\sigma}\left(|x-y|_{\beta}\right)\varphi\left(|x-y|_{\beta}\right)}dy\\ &+\int_{|x-y|_{\beta}>\varepsilon}\frac{|f(y)|^{p}}{|x-y|_{\beta}^{(n-p)\frac{2|\beta|}{n}}\varphi^{\sigma}\left(|x-y|_{\beta}\right)}dy\\ &\le\varphi^{1-\sigma}(\varepsilon)I^{\beta}_{p,\varphi}\left(f^{p}\right)+\frac{1}{\varepsilon^{(n-p)\frac{2|\beta|}{n}}\varphi^{\sigma}(\varepsilon)}\|f\|_{p}^{p}.\end{aligned} \tag{13}$$

Letting $\varepsilon^{(n-p)\frac{2|\beta|}{n}}\varphi(\varepsilon)=\Phi(\varepsilon)$, we choose

$$\varepsilon=\Phi^{-1}\left(\frac{\|f\|_{p}^{p}}{I^{\beta}_{p,\varphi}\left(f^{p}\right)}\right),$$

a choice which makes the two terms on the right hand side of (13) equal.

From (13), we obtain

$$\frac{I^{\beta}_{p,\varphi^{\sigma}} f^{p}(x)}{\| f \|_p^p} \le \frac{2}{\left[\Phi^{-1}\left(\frac{\|f\|_p^p}{I^{\beta}_{p,\varphi}(f^p)}\right)\right]^{(n-p)\frac{2|\beta|}{n}} \varphi^{\sigma}\left[\Phi^{-1}\left(\frac{\|f\|_p^p}{I^{\beta}_{p,\varphi}(f^p)}\right)\right]}$$

If

$$\psi(t) = \frac{2}{\left[\Phi^{-1}(\frac{1}{t})\right]^{(n-p)\frac{2|\beta|}{n}} \varphi^{\sigma}\left[\Phi^{-1}\left(\frac{1}{t}\right)\right]}$$

and

$$G(t) = \psi^{-1}(t),$$

we have

$$G\left(\frac{I^{\beta}_{p,\varphi^{\sigma}}(f^p)}{\| f \|_p^p}\right) \le \frac{I^{\beta}_{p,\varphi}(f^p)}{\| f \|_p^p}.$$

Finally, using Fubini's theorem

$$\int_{B_\beta(y,r)} G\left(\frac{I^{\beta}_{p,\varphi^{\sigma}}(f^p)(x)}{\|f\|_p^p}\right) | V(x) | \, dx$$

$$\le \frac{1}{\|f\|_p^p} \int_{B_\beta(y,r)} I^{\beta}_{p,\varphi}(f^p)(x) | V(x) | \, dx$$

$$= \frac{1}{\|f\|_p^p} \int_{B_\beta(y,r)} \left(\int_{\mathbb{R}^n} \frac{| f(y) |^p}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}} \varphi(|x-y|_\beta)} dy \right) |V(x)| dx$$

$$= \frac{1}{\|f\|_p^p} \int_{\mathbb{R}^n} \left(\int_{B_\beta(y,r)} \frac{| V(x) |}{|x-y|_\beta^{(n-p)\frac{2|\beta|}{n}} \varphi(|x-y|_\beta)} dx \right) |f(y)|^p dy \le \xi_\beta(r).$$

So (11) was obtained.

(12) is easily seen to be equivalent to

$$\lim_{s\to 0}\frac{[\Phi^{-1}(s)]^{(n-p)\frac{2|\beta|}{n}}\varphi^{\sigma}[\Phi^{-1}(s)]}{s}=+\infty. \tag{14}$$

Choosing $H(t)=t^{(n-p)\frac{2|\beta|}{n}}\varphi^{\sigma}(t)$, (14) can be rewritten as

$$\lim_{s\to 0}\frac{H(\Phi^{-1}(s))}{s}=+\infty. \tag{15}$$

Since $\lim_{s\to 0}\frac{\Phi(s)}{H(s)}=\lim_{s\to 0}\varphi^{1-\sigma}(s)=0$ we obtain (15).

Lemma 3.1 *Let* $h:]0,+\infty[\to]0,+\infty[$ *such that* $\int_0^1\frac{[h(t)]^{p'/p}}{t}dt<+\infty$ $(p':\frac{1}{p'}+\frac{1}{p}=1)$. *Then*

$$I_1^{\beta}(f)\le C(n,p,diam(sptf),h)\left[I_{p,h}(f^p)\right]^{\frac{1}{p}}$$

for all $f\in C_0^{\infty}(\mathbb{R}^n)$.

Proof. Using the Hölder inequality, we get

$$\begin{aligned}I_1^{\beta}(f)&=\int_{\mathbb{R}^n}\frac{|f(y)|\,h^{\frac{1}{p}}(|x-y|_{\beta})}{|x-y|_{\beta}^{n-1}h^{\frac{1}{p}}(|x-y|_{\beta})}dy\\&\le\left[I_{p,h}(f^p)\right]^{\frac{1}{p}}\left(\int_{B_{\beta}(y,r)}\frac{h^{\frac{p'}{p}}(|x-y|_{\beta})}{|x-y|_{\beta}^{n}}dy\right)^{\frac{1}{p'}}\end{aligned}$$

where $B_{\beta}(y,r)\supseteq sptf$.

Corollary 3.1 *Under the hypotheses of Theorem 1 and for all* $u\in C_0^{\infty}(\mathbb{R}^n)$, *letting* $\int_0^1\frac{[\varphi(t)]^{\frac{\sigma p'}{p}}}{t}dt<+\infty$, *we get*

$$\int_{B_{\beta}(y,r)}G\left(\frac{|u|^p}{\|\nabla u\|_p^p}\right)V(x)dx\le C(n,p,diam(sptu),\varphi)\xi_{\beta}(r), \tag{16}$$

where $B_{\beta}(y,r)\supseteq sptu$.

Proof. Using Lemma 2.1 and Theorem 3.1, we have the following inequality

$$|u| \le C(n) I_1^{\beta}(|\nabla u|).$$

Remark 3.1 *If we choose the function* $\varphi^{\sigma}(t)$ *with a more general non-decreasing function* $\delta :]0,+\infty[\to]0,+\infty[$ *such that*

$$\lim_{t\to 0} \delta(t) = 0, \quad \lim_{t\to 0} \frac{\varphi(t)}{\delta(t)} = 0,$$

$\frac{\varphi(t)}{\delta(t)}$ is non-decreasing, where $\varphi(t)$ is as in Definition 2.3, the previous results are also valid.

Proposition 3.1 *Let* $V \in S_p^{\beta}, V \ge 0, \sigma \in]0,1[$, $\gamma = \dfrac{1}{\frac{\sigma p'}{p} + 1}$ *and assume that*

$$\int_0^1 \frac{[\eta_\beta(t)]^{1-\gamma}}{t} dt < +\infty \tag{17}$$

Then

$$V \in S_{p,\eta_\beta^{\gamma}} \tag{18}$$

and for every $u \in C_0^{\infty}(\mathbb{R}^n)$, *there exists* a *non decreasing positive function* $G(t)$ *such that*

$$\int_{B_\beta(y,r)} G\left(\frac{|u|^p}{\|\nabla u\|_p^p}\right) V(x) dx \le C(n, p, \eta_\beta) \mu_\beta(r) \tag{19}$$

where $B_\beta(y,r) \supseteq \operatorname{spt} u$ and

$$\mu_\beta(r) = \frac{2}{C} \int_0^r t^{-1} \eta_\beta^{1-\gamma}(t) dt. \tag{20}$$

Now we give an example of a function $f \in S_p^{\beta}$, $f \ge 0$. But for $\lambda > n-2$, we choose $f \notin L_{1,\lambda}^{\beta}$.

Example 3.1 *Let* $\chi_B(y)$ *be the characteristic function of* B *and*

$$f(x) = \frac{1}{|x|_\beta^2 \left|\log|x|_\beta\right|^6} \chi_B(x),$$

where $B_\beta(0,\delta)$ the $\beta-$ball centered in 0 and radius $\delta = e^{-3}$. We obtain that the function

$$\eta_\beta(r)=\sup_{x\in\mathbb{R}^n}\int_{|x-y|_\beta<r}\frac{f(y)}{|x-y|_\beta^{n-2}}dy,$$

is such that

(i) $\lim_{r\to 0}\eta_\beta(r)=0$

(ii) $\int_0^r \frac{\eta_\beta^{1/4}(\rho)}{\rho}d\rho<+\infty.$

Proof. For $x\in\mathbb{R}^n$ and $r>0$, we obtain

$$\int_{|x-y|_\beta<r}\frac{1}{|y|_\beta^2|x-y|_\beta^{n-2}\left|\log|y|_\beta\right|^6}\chi_B(y)dy$$

$$=\int_{|y|_\beta<|x-y|_\beta<r}\frac{1}{|y|_\beta^2|x-y|_\beta^{n-2}\left|\log|y|_\beta\right|^6}\chi_B(y)dy$$
$$+\int_{\{|x-y|_\beta<r\}\cap\{|x-y|_\beta<|y|_\beta<\delta\}}\frac{1}{|y|_\beta^2|x-y|_\beta^{n-2}\left|\log|y|_\beta\right|^6}\chi_B(y)dy=A_1+A_2.$$

For A_1, letting $\sigma=\min(r,\delta)$

$$A_1=\int_{|y|_\beta<|x-y|_\beta<r}\frac{1}{|y|_\beta^2|x-y|_\beta^{n-2}\left|\log|y|_\beta\right|^6}\chi_B(y)dy$$

$$\le\int_{|y|_\beta<r}\frac{1}{|y|_\beta^n\left|\log|y|_\beta\right|^6}\chi_B(y)dy=C(n)\frac{1}{(-\log\sigma)^5},$$

and for A_2, considering that the function $\frac{1}{t^2(-\log t)^6}$ is decreasing in $]0,e^{-3}[$, we obtain ;

$$A_2=\int_{\{|x-y|_\beta<r\}\cap\{|x-y|_\beta<|y|_\beta<\delta\}}\frac{1}{|y|_\beta^2|x-y|_\beta^{n-2}\left|\log|y|_\beta\right|^6}\chi_B(y)dy$$

$$=\int_{\{|x-y|_\beta<r\}\cap\{|x-y|_\beta<|y|_\beta<\delta\}}\frac{1}{|y|_\beta^2|x-y|_\beta^{n-2}\left|\log|y|_\beta\right|^6}dy$$

$$\le\int_{\{|z|_\beta<r\}\cap\{|z|_\beta<\delta\}}\frac{dz}{|z|_\beta^n(-\log|z|_\beta)^6}=C(n)\frac{1}{(-\log\sigma)^5}.$$

Then we have

$$\eta_\beta(r) \le L(r) \equiv 2C(n)\frac{1}{(-\log\sigma)^5}.$$

Because $\lim_{r\to 0} L(r) = 0$ we get (i).

Only considering $r < \delta$,

$$\int_0^r \frac{\eta_\beta^{\frac{1}{4}}(\rho)}{\rho}d\rho \le \int_0^r \frac{L^{\frac{1}{4}}(\rho)}{\rho}d\rho$$

$$= (2C(n))^{\frac{1}{4}}\int_0^r \frac{(-\log\rho)^{-\frac{5}{4}}}{\rho}d\rho$$

$$= (2C(n))^{\frac{1}{4}}\frac{4}{(-\log r)^{\frac{1}{4}}} < +\infty.$$

So, we have proved (ii).

Now, for $\lambda > n-2$, we prove that the function $f \notin L^{\beta}_{1,\lambda}$.

For $\varepsilon > 0, \lambda = n-2+\varepsilon$, the following quantity is unbounded.

$$\frac{1}{r^{n-2+\varepsilon}}\int_{B_\beta(0,r)} \frac{\chi_B(y)}{|y|_\beta^2 \left|\log|y|_\beta\right|^6}dy = \frac{C(n)}{r^{n-2+\varepsilon}}\int_0^r \frac{\rho^{n-1}}{\rho^2(-\log\rho)^6}d\rho$$

$$> \frac{C(n)}{2^{n-2}r^\varepsilon}\int_{\frac{r}{2}}^r \frac{d\rho}{(-\log\rho)^6\rho}$$

$$= \frac{1}{5}\frac{C(n)}{2^{n-2}r^\varepsilon}\left[\frac{1}{(-\log r)^5} - \frac{1}{(-\log(\frac{1}{2}r))^5}\right].$$

Remark 3.2 *Throughout this study, if we choose* $\beta_1 = \beta_2 = ... = \beta_n = \frac{1}{2}$, *then we have the conclusions of [8].*

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