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Some Results On Quaternion 3-Space

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Abstract: In this paper, the set **J′**=**H**(Q_4,Jγ) of 4 by 4 matrices, with entries in a quaternion F-algebra Q, that are symmetric with respect to the canonical involution Jγ is studied. **J′** is also the special Jordan matrix algebra and some results related to points and lines of the quaternion 3-space **P(J′)** defined by the algebra are introduced. Finally, by taking dual ring **Q**:= Q+Qε (ε∉Q, $\varepsilon^2=0$) instead of Q, the obtained results are carried to a more general state.

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***Keywords*:** Special Jordan matrix algebra, quaternion, quaternion 3-space.

Kuaterniyon 3-Uzay Üzerine Bazı Sonuçlar

Özet: Bu makalede, girdileri bir Q kuaterniyon F-cebirinden alınan ve Jγ kanonik involusyonuna göre simetrik olan 4×4 boyutlu matrislerin oluşturduğu **J′**=**H**(Q_4,Jγ) kümesi ile çalışılmıştır. Bu **J′** kümesi aynı zamanda bir özel Jordan matris cebiridir ve bu cebir ile tanımlanan **P(J′)** kuaterniyon 3-uzayın noktalar ve doğruları ile ilgili bazı sonuçlar sunulmuştur. Son olarak, Q yerine **Q**:= Q+Qε (ε∉Q, $\varepsilon^2=0$) dual halkası alınarak elde edilen sonuçlar daha genel bir duruma taşınmıştır.

***Anahtar Kelimeler*:** Özel Jordan matris cebiri, kuaterniyon, kuaterniyon 3-uzay

1. INTRODUCTION and PRELIMINARIES

In [5], Faulkner deals with **J**=**H**($\mathbf{O}_3$,Jγ), the set of 3 by 3 matrices with entries in an octonion algebra **O** defined over a field F, that are symmetric with respect to the canonical involution $J\gamma: X \to \gamma^{-1}\,\overline{X}^t\,\gamma$ where the γ_i are non-zero elements of F and $\gamma:=\mathrm{diag}\{\gamma_1,\gamma_2,\gamma_3\}$. Hence, any element X of **J** is of the form

$$X=\begin{pmatrix} \alpha_1 & \gamma_2 a_3 & \gamma_3\bar{a}_2 \\ \gamma_1\bar{a}_3 & \alpha_2 & \gamma_3 a_1 \\ \gamma_1 a_2 & \gamma_2\bar{a}_1 & \alpha_3 \end{pmatrix} \quad \text{for } \alpha_i\in F \text{ and } a_i\in\mathbf{O}.$$

If it is defined a cubic form N such that N(X):=detX, a quadratic mapping $X\to X^{\#}$:=adjoint of X, and a basepoint $C:=I_3$ on **J** are defined, then the triple (**J**,N,C) is a quadratic (exceptional) Jordan algebra

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under the operator $U_X Y = T(X,Y)X - 2(X^{\#} \times Y)$ [9]. Then, for $X = \begin{pmatrix} \alpha_1 & \gamma_2 a_3 & \gamma_3 \overline{a}_2 \\ \gamma_1 \overline{a}_3 & \alpha_2 & \gamma_3 a_1 \\ \gamma_1 a_2 & \gamma_2 \overline{a}_1 & \alpha_3 \end{pmatrix}$ and $Y = \begin{pmatrix} \beta_1 & \gamma_2 b_3 & \gamma_3 \overline{b}_2 \\ \gamma_1 \overline{b}_3 & \beta_2 & \gamma_3 b_1 \\ \gamma_1 b_2 & \gamma_2 \overline{b}_1 & \beta_3 \end{pmatrix} \in \mathbf{J}$, we can give the similar results to those given in [6, 9]:

$$N(X) = \alpha_1\alpha_2\alpha_3 - \alpha_1\gamma_2\gamma_3 n(a_1) - \alpha_2\gamma_3\gamma_1 n(a_2) - \alpha_3\gamma_1\gamma_2 n(a_3) + \gamma_1\gamma_2\gamma_3 2t((a_1a_2)a_3),$$

$$X^{\#} = (X_{ij})_{3\times3} \text{ for } X_{ii} = \alpha_j\alpha_k - \gamma_j\gamma_k n(a_i),\ X_{ij} = \gamma_i \gamma_k a_i a_j - \gamma_i\alpha_k \overline{a}_k \text{ and } X_{ji} = \overline{X_{ij}},$$

$$X \times Y = (z_{ij})_{3\times3} \text{ for } \begin{cases} z_{ii} = (1/2)\left[\alpha_j\beta_k + \beta_j\alpha_k - 2\gamma_j\gamma_k n(a_i, b_i)\right] \\ z_{ii} = (1/2)\left(\gamma_j\left[\gamma_k\overline{(a_i b_j + b_i a_j)} - (\alpha_k b_k + \beta_k a_k)\right]\right) \end{cases},\ z_{ji} = \overline{z}_{ij},$$

$$T(X,Y) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + 2\gamma_2\gamma_3 n(a_1,b_1) + 2\gamma_3\gamma_1 n(a_2,b_2) + 2\gamma_1\gamma_2 n(a_3,b_3),$$

where (i,j,k) is a cyclic permutation of (1,2,3), n (defined by $n(x) := x\overline{x}$) is the norm (quadratic) form over **O**, t (defined by $t(x) := (1/2)(x + \overline{x})$) is the trace (linear) form over **O** and finally n(x,y) (defined by $n(x,y) := (1/2)[n(x+y) - n(x) - n(y)]$) is symmetric bilinear norm w.r.t. n.

Let Π denote the set of elements of rank 1 in **J**. Then,

$\Pi = \{X \mid X \in \mathbf{J} - \{0\} \text{ and } X \times X = X^{\#} = 0\}$.

Note that, if $X \in \Pi$ and α is a non-zero element in F, then $\alpha X \in \Pi$. For $X \in \Pi$, let X_* and X^* be two copies of the set $\{\alpha X \mid \alpha \in F - \{0\}\}$.

Now, we are ready to give the definition of an octonion plane **P(J)** from [5, Chapter 3].

The octonion plane **P(J)** = (**P**,**L**,|,$\sqcup$) consists of the incidence structure (**P**, **L**, |) (points, lines, and incidence), and the connection relation is defined as follows:

P = $\{X_* \mid X \in \Pi\}$, **L** = $\{X^* \mid X \in \Pi\}$,

$X_* | Y^*$, X_* is on Y^*, if $V_{Y,X} = 0$, that is, $V_{Y,X} =: \{1XY\} = \{X1Y\} = \{XY1\} = X \cdot Y = 0$ where $X \cdot Y = (1/2)(XY + YX)$ (Jordan multiplication),

$X_* \sqcup Y_*$, X_* is connected to Y_* if $X \times Y = 0$,

$X^* \sqcup Y^*$, X^* is connected to Y^* if $X \times Y = 0$,

$X_* \sqcup Y^*$, X_* is connected (or near) to Y^* if $T(X,Y) = 0$.

In [7, Chapter III.2, Theorem 1], Jacobson showed that the fact that $(\mathbf{D}_n, J\gamma)$ is a Jordan algebra implies that **D** is associative if n≥4 but alternative with its symmetric elements in the nucleus if n=3. Therefore, in the case of n≥4 we can study with the elements of a quaternion algebra, which is associative (but not commutative) and moreover the Jordan matrix algebra $(\mathbf{D}_n, J\gamma)$ is necessarily special (that is, not exceptional) since **D** is associative [7, p.138].

Let F be a field and let $Q=\{r_0+r_1i_1+r_2i_2+r_3i_3 \mid r_i \in F\}$ be a quaternion division F-algebra. From now on, we assume that the characteristic of F is different from 2. We denote the anti-involution over Q by j ($j(x):=x$), the norm (quadratic) form over Q by n ($n(x):=x\bar{x}\in F$), and the trace (linear) form over Q by t ($t(x):=(1/2)(x+\bar{x})\in F$). In this case, $x=r_0-r_1i_1-r_2i_2-r_3i_3$, $n(x) = r_0^2-c_1r_1^2-c_2r_2^2+c_1c_2r_3^2$ where c_1, c_2 are non-zero elements in the multiplication table [8, p.448] and $t(x)=r_0$ for any $x=r_0+r_1i_1+r_2i_2+r_3i_3\in Q$. For example, for $F = \mathbb{R}$ and $c_1=c_2=-1$ we have Hamilton's quaternion (division) algebra and so we reach the result: $n(x) = r_0^2+r_1^2+r_2^2+r_3^2=0 \Leftrightarrow x=0$.

J′ = **H**(Q_4,$J\gamma$), the set of 4 by 4 matrices, with entries in an quaternion division F-algebra, that are symmetric with respect to the canonical involution $J\gamma: X\to\gamma^{-1}\bar{X}^t\gamma$ where the γ_i are non-zero elements of F and $\gamma:=\mathrm{diag}\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$. Hence, any element X of **J′** is of the form

$$X=\left[x_{ij}\right]=\begin{pmatrix} \alpha_1 & \gamma_2 a_{12} & \gamma_3\bar{a}_{13} & \gamma_4 a_{14} \\ \gamma_1\bar{a}_{12} & \alpha_2 & \gamma_3 a_{23} & \gamma_4\bar{a}_{24} \\ \gamma_1 a_{13} & \gamma_2\bar{a}_{23} & \alpha_3 & \gamma_4 a_{34} \\ \gamma_1\bar{a}_{14} & \gamma_2 a_{24} & \gamma_3\bar{a}_{34} & \alpha_4 \end{pmatrix} \text{ for } \alpha_i\in F \text{ and } a_{ij}\in Q.$$

If we take a quartic (fourth degree) form N such that $N(X):=\det X$, a cubic mapping $X\to X^{\#}:=$adjoint of X, and a basepoint $C:=I_4$ on **J**, then: it is clear that

$$T(X,Y) = \alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3+\alpha_4\beta_4+2\gamma_1\gamma_2 n(a_{12},b_{12})+2\gamma_1\gamma_3 n(a_{13},b_{13})+2\gamma_1\gamma_4 n(a_{14},b_{14})$$
$$+2\gamma_2\gamma_3 n(a_{23},b_{23})+2\gamma_2\gamma_4 n(a_{24},b_{24})+2\gamma_3\gamma_4 n(a_{34},b_{34}),$$

as $T(X,Y):=T(X\cdot Y) = \mathrm{trace}(X\cdot Y)$. Besides, note that $X\times Y$ must be equal to $(1/6)[(X+Y)^{\#}-X^{\#}-Y^{\#}]$ and specifically, $X\times X = X^{\#}$ as in the case of n = 3.

Now, from [1], we can give some informations about the quaternion (but, not dual) 3-space **P(J′)**=(**P**,**L**,|,⊔) where **J′** is the 28-dimensional special Jordan matrix algebra. Then, the set of points **P** consists of the following four classes (which we call as points of types 1,2,3 and 4, respectively):

$$\{ P_1 = \begin{pmatrix} 1 & \gamma_1^{-1}\gamma_2\bar{x}_2 & \gamma_1^{-1}\gamma_3\bar{x}_3 & \gamma_1^{-1}\gamma_4\bar{x}_4 \\ x_2 & \gamma_1^{-1}\gamma_2 n(x_2) & \gamma_1^{-1}\gamma_3 x_2\bar{x}_3 & \gamma_1^{-1}\gamma_4 x_2\bar{x}_4 \\ x_3 & \gamma_1^{-1}\gamma_2 x_3\bar{x}_2 & \gamma_1^{-1}\gamma_3 n(x_3) & \gamma_1^{-1}\gamma_4 x_3\bar{x}_4 \\ x_4 & \gamma_1^{-1}\gamma_2 x_4\bar{x}_2 & \gamma_1^{-1}\gamma_3 x_4\bar{x}_3 & \gamma_1^{-1}\gamma_4 n(x_4) \end{pmatrix} =: \begin{pmatrix} 1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^t \mid x_i\in Q\}\cup$$

$$\{ P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma_2^{-1}\gamma_3\bar{x}_3 & \gamma_2^{-1}\gamma_4\bar{x}_4 \\ 0 & x_3 & \gamma_2^{-1}\gamma_3 n(x_3) & \gamma_2^{-1}\gamma_4 x_3\bar{x}_4 \\ 0 & x_4 & \gamma_2^{-1}\gamma_3 x_4\bar{x}_3 & \gamma_2^{-1}\gamma_4 n(x_4) \end{pmatrix} =: \begin{pmatrix} 0 \\ 1 \\ x_3 \\ x_4 \end{pmatrix}^t \mid x_3,x_4\in Q\}\cup$$

$$\{ P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \gamma_3^{-1}\gamma_4\bar{x}_4 \\ 0 & 0 & x_4 & \gamma_3^{-1}\gamma_4 n(x_4) \end{pmatrix} =: \begin{pmatrix} 0 \\ 0 \\ 1 \\ x_4 \end{pmatrix}^t \mid x_4\in Q\}\cup\{ P_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =: \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^t \},$$

the set of lines **L** consists of the following four classes (which we call as lines of types 1,2,3 and 4, respectively):

$$\{ l_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^t \} \cup$$

$$\{ l_2 = \begin{bmatrix} \gamma_1^{-1}{}_1\gamma_2 n(m_1) & -\gamma_1^{-1}\gamma_2\bar{m}_1 & 0 & 0 \\ -m_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: \begin{bmatrix} m_1 \\ 1 \\ 0 \\ 0 \end{bmatrix}^t \mid m_1 \in Q \} \cup$$

$$\{ l_3 = \begin{bmatrix} \gamma_1^{-1}\gamma_3 n(m_1) & \gamma_1^{-1}\gamma_3\bar{m}_1 m_2 & -\gamma_1^{-1}\gamma_3\bar{m}_1 & 0 \\ \gamma_2^{-1}\gamma_3\bar{m}_2 m_1 & \gamma_2^{-1}\gamma_3 n(m_2) & -\gamma_2^{-1}\gamma_3\bar{m}_2 & 0 \\ -m_1 & -m_2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =: \begin{bmatrix} m_1 \\ m_2 \\ 1 \\ 0 \end{bmatrix}^t \mid m_1, m_2 \in Q \} \cup$$

$$\{ l_4 = \begin{bmatrix} \gamma_1^{-1}\gamma_4 n(m_1) & \gamma_1^{-1}\gamma_4\bar{m}_1 m_2 & \gamma_1^{-1}\gamma_4\bar{m}_1 m_3 & -\gamma_1^{-1}\gamma_4\bar{m}_1 \\ \gamma_2^{-1}\gamma_4\bar{m}_2 m_1 & \gamma_2^{-1}\gamma_4 n(m_2) & \gamma_2^{-1}\gamma_4\bar{m}_2 m_3 & -\gamma_2^{-1}\gamma_4\bar{m}_2 \\ \gamma_3^{-1}\gamma_4\bar{m}_3 m_1 & \gamma_3^{-1}\gamma_4\bar{m}_3 m_2 & \gamma_3^{-1}\gamma_4 n(m_3) & -\gamma_3^{-1}\gamma_4\bar{m}_3 \\ -m_1 & -m_2 & -m_3 & 1 \end{bmatrix} =: \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ 1 \end{bmatrix}^t \mid m_i \in Q \}.$$

The incidence relation |, equivalent to $X \cdot Y = 0$, is defined as follows:

$[1,0,0,0] = \{(0,1,y_3,y_4) \mid y_3,y_4 \in Q\} \cup \{(0,0,1,z_4) \mid z_4 \in Q\} \cup \{(0,0,0,1)\}$,

$[l_1,1,0,0] = \{(1,l_1,x_3,x_4) \mid x_3,x_4 \in Q\} \cup \{(0,0,1,z_4) \mid z_4 \in Q\} \cup \{(0,0,0,1)\}$,

$[m_1,m_2,1,0] = \{(1,x_2,m_1+m_2x_2,x_4) \mid x_2,x_4 \in Q\} \cup \{(0,1,m_2,y_4) \mid y_4 \in Q\} \cup \{(0,0,0,1)\}$,

$[n_1,n_2,n_3,1] = \{(1,x_2,x_3,n_1+n_2x_2+n_3x_3,) \mid x_2,x_3 \in Q\} \cup \{(0,1,y_3,n_2+n_3y_3,) \mid y_3 \in Q\} \cup$

$\{(0,0,1,n_3)\}$.

Finally by the relation equivalent to the connection relation $\sqcup$ given by $X \times Y = 0$ in the case $n = 3$ (see [2] for this equivalence), we can define the connection relation $\sqcup$ in this space as follows:

$(x_1,x_2,x_3,x_4) \sqcup (y_1,y_2,y_3,y_4) \Leftrightarrow x_i - y_i = 0$ for i=1,2,3,4,

$[k_1,k_2,k_3,k_4] \sqcup [n_1,n_2,n_3,n_4] \Leftrightarrow k_i - n_i = 0$ for i=1,2,3,4.

Besides, from types of points on lines, it is clear that a point and a line of same type is not connected (near). Moreover, the result is equivalent to $T(X,Y) \neq 0$ for a point (or line) X and a line Y (or point), respectively. In the other cases, we say that they are connected (near).

2. THE MAIN RESULTS

Now, we will investigate the intersection points of lines in the space **P(J′)**.

First we take the different types of lines:

[1,0,0,0]∩[r,1,0,0] = {(0,0,1,z) | z∈Q}∪{(0,0,0,1)},

[1,0,0,0]∩[r,s,1,0] = {(0,1,s,z) | z∈Q}∪{(0,0,0,1)},

[1,0,0,0]∩[r,s,t,1] = {(0,1,y,s+ty) | y∈Q}∪{(0,0,1,t)},

[m,1,0,0]∩[r,s,1,0] = {(1,m,r+sm,z) | z∈Q}∪{(0,0,0,1)},

[m,1,0,0]∩[r,s,t,1] = {(1,m,y,r+sm+ty) | y∈Q}∪{(0,0,1,t)},

[m,n,1,0]∩[r,s,t,1] = {(1,x,m+nx,r+sx+t(m+nx) | x∈Q}∪{(0,1,n,s+tn)}.

Now we examine the same types of lines:

First we take lines [m,n,p,1] and [r,s,t,1] of the fourth types. We can determine the intersection points of these lines in three cases as follows:

i. If n-s ≠ 0, p-t = 0, then the intersection points are

$\{(1,-(n-s)^{-1}(m-r),y,m-n(n-s)^{-1}(m-r)+py) \mid y\in Q\}\cup\{(0,0,1,p=t)\}$.

ii. If p-t ≠ 0, then the intersection points are

$\{(1,x,-(p-t)^{-1}((m-r)+(n-s)x),m+nx-p(p-t)^{-1}((m-r)+(n-s)x)) \mid x\in Q\}$

$\cup\{(0,1,-(p-t)^{-1}(n-s),n-p(p-t)^{-1}(n-s))\}$.

iii. If n-s = 0, p-t = 0 and m-r ≠ 0, then the intersection points are

{(0,1,v,n+pv) | v∈Q}∪{(0,0,1,p)}.

Now we take lines [m,n,1,0] and [r,s,1,0] of the third types. We can determine the intersection points of these lines in two cases as follows:

i. If m-r ≠ 0 and n-s = 0, then the intersection points are

{(0,1,n=s,w) | w∈Q}∪{(0,0,0,1)}.

ii. If n-s ≠ 0, then the intersection points are

$\{(1,-(n-s)^{-1}(m-r),m-n(n-s)^{-1}(m-r),z) \mid z\in Q\}\cup\{(0,0,0,1)\}$.

Finally we take [m,1,0,0] and [r,1,0,0] of the second types lines. In this case, If m-r ≠ 0, then the intersection points are

{(0,0,1,w) | w∈Q}∪{(0,0,0,1)}.

Now, conversely, we would like to determine the lines joining these two points, according to the two points in the space **P(J′)** are of different and of the same types.

First let us examine the case that the two points are of different types:

$(1,x,y,z)\cup(0,1,v,w) = \{[z-(w-tv)x-ty,w-tv,t,1] \mid t\in Q\}\cup\{[y-vx,v,1,0]\}$,

$(1,x,y,z)\cup(0,0,1,w) = \{[z-sx-wy,s,w,1] \mid s\in Q\}\cup\{[x,1,0,0]\}$,

$(1,x,y,z)\cup(0,0,0,1) = \{[y-sx,s,1,0] \mid s\in Q\}\cup\{[x,1,0,0]\}$,

$(0,1,y,z)\cup(0,0,1,w) = \{[r,z-wy,w,1] \mid r\in Q\}\cup\{[1,0,0,0]\}$,

$(0,1,y,z)\cup(0,0,0,1) = \{[r,y,1,0] \mid r\in Q\}\cup\{[1,0,0,0]\}$,

$(0,0,1,z)\cup(0,0,0,1) = \{[r,1,0,0] \mid r\in Q\}\cup\{[1,0,0,0]\}$.

Now we can examine the other case. Let the two points be of the same type:

If we take points (1,x,y,z) and (1,u,v,w) of the first type, then we complete this examination in the following three cases:

i. If $x-u \neq 0$, then the lines joining these points are

$\{[z-((z-w)+c(v-y))(x-u)^{-1}x-cy,(z-w)(x-u)^{-1}+c(v-y)(x-u)^{-1},c,1] \mid c\in Q\}$

$\cup\{[y-(y-v)(x-u)^{-1}x,(y-v)(x-u)^{-1},1,0]\}$.

ii. If $x-u = 0$, $y-v = 0$ and $z-w \neq 0$, then the lines joining these points are

$\{[y-sx,s,1,0] \mid s\in Q\}\cup\{[x,1,0,0]\}$.

iii. If $x-u = 0$ and $y-v \neq 0$, then the lines joining these points are

$\{[z-bx-(z-w)(y-v)^{-1}y,b,(z-w)(y-v)^{-1},1] \mid b\in Q\}\cup\{[x=u,1,0,0]\}$.

If we take points (0,1,y,z) and (0,1,v,w) of the second type, then we can determine the lines joining these points in the following two cases:

i. If $y-v \neq 0$, then the lines joining these points are

$\{[a,z-(z-w)(y-v)^{-1}y,(z-w)(y-v)^{-1},1] \mid a\in Q\}\cup\{[1,0,0,0]\}$.

ii. If $y-v = 0$ and $z-w \neq 0$, then the lines joining these points are

$\{[r,y=v,1,0] \mid r\in Q\}\cup\{[1,0,0,0]\}$.

Finally, if we take points (0,0,1,z) and (0,0,1,w) of the third type then we obtain the lines joining these points in the following one case:

i. If $z-w \neq 0$, then the lines joining these points are

$\{[r,1,0,0] \mid r\in Q\}\cup\{[1,0,0,0]\}$.

Now, we would like to carry the results over the dual ring $\mathbf{Q}:=Q+Q\varepsilon$, $\varepsilon\notin Q$ and $\varepsilon^2=0$ with the maximal ideal $\mathbf{I}=Q\varepsilon$ (of non-units). Note that $\mathbf{Q}$ does not have to be a local ring with the maximal ideal $\mathbf{I}$ in the case $\varepsilon^2=k\in F-\{0\}$. For, in this case, the inverse of any $x=a+b\varepsilon\in\mathbf{Q}$ would be $x^{-1}=b^{-1}[a(ab^{-1}a-kb)^{-1}]+(kb-ab^{-1}a)^{-1}\varepsilon$. As for the case we study, that is, for k=0, it is clear that $(a+b\varepsilon)^{-1}=a^{-1}-a^{-1}ba^{-1}\varepsilon$ (we know that

a^{-1} exists for all a∈Q-{0}). Therefore, the non-unit elements of **Q** consist of the maximal ideal **I**=Qε. For more detailed information about **Q** it can be seen to [3,4]. So, we can find the intersection points of any two lines and the lines joining any two points in the space **P(J″)** where **J″**=**H**(**Q**$_4$,Jγ). By similar calculations, first we take the different types of lines:

$$[1,m,n,k]\cap[q,1,s,t]$$

$$= \{((1-mq)^{-1}(ms+n+(mt+k)z),q(1-mq)^{-1}(ms+n+(mt+k)z)+s+tz,1,z) \mid z\in\mathbf{Q}\}$$

$$\cup\{((1-mq)^{-1}((ms+n)w+mt+k),q(1-mq)^{-1}((ms+n)w+mt+k)+sw+t,w,1) \mid w\in\mathbf{I}\}$$

where m,n,k,s,t∈**I**,

$$[1,m,n,k]\cap[q,s,1,t]$$

$$= \{(m+n(1-qn)^{-1}(qm+s+(qk+t)z)+kz,1,(1-qn)^{-1}(qm+s+(qk+t)z),z) \mid z\in\mathbf{Q}\}$$

$$\cup\{(mv+n(1-qn)^{-1}((qm+s)v+qk+t)+k,v,(1-qn)^{-1}((qm+s)v+qk+t),1) \mid v\in\mathbf{I}\}$$

where m,n,k,t∈**I**,

$$[1,m,n,k]\cap[q,s,t,1]$$

$$= \{((1-kq)^{-1}(m+ks+(n+kt)y),1,y,q(1-kq)^{-1}(m+ks+(n+kt)y)+s+ty) \mid y\in\mathbf{Q}\}$$

$$\cup\{((1-kq)^{-1}(n+kt+(m+ks)v),v,1,q(1-kq)^{-1}(n+kt+(m+ks)v)+sv+t) \mid v\in\mathbf{I}\}$$

where m,n,k∈**I**,

$$[m,1,n,k]\cap[q,s,1,t]$$

$$= \{(1,m+n(1-sn)^{-1}((q+sm)+(sk+t)z)+kz,(1-sn)^{-1}((q+sm)+(sk+t)z),z) \mid z\in\mathbf{Q}\}$$

$$\cup\{(u,(1-ns)^{-1}((m+nq)u+nt+k),qu+s(1-ns)^{-1}((m+nq)u+nt+k)+t,1) \mid u\in\mathbf{I}\}$$

where n,k,t∈**I**,

$$[m,1,n,k]\cap[q,s,t,1]$$

$$= \{(1,(1-ks)^{-1}((m+kq)+(n+kt)y),y,q+s(1-ks)^{-1}((m+kq)+(n+kt)y)+ty) \mid y\in\mathbf{Q}\}$$

$$\cup\{(u,(1-ks)^{-1}((n+kt)+(m+kq)u),1,qu+s(1-ks)^{-1}((n+kt)+(m+kq)u)+t) \mid u\in\mathbf{I}\}$$

where n,k∈**I**,

$$[m,n,1,k]\cap[q,s,t,1]$$

$$= \{(1,x,(1-kt)^{-1}((m+kq)+(n+ks)x),q+sx+t(1-kt)^{-1}((m+kq)+(n+ks)x)) \mid x\in\mathbf{Q}\}$$

$$\cup\{(u,1,(1-kt)^{-1}((n+ks)+(m+kq)u),qu+s+t(1-kt)^{-1}((n+ks)+(m+kq)u)) \mid u\in\mathbf{I}\}$$

where k∈**I**.

Now we examine the same types of lines:

First we take lines [m,n,k,1] and [q,s,t,1] of the fourth types. We can determine the intersection points of these lines in three cases as follows:

i. If $n\text{-}s\notin\mathbf{I}$, $k\text{-}t\in\mathbf{I}$, then the intersection points are

$$\{(1,-(n-s)^{-1}(m-q+(k-t)y),y,m-n(n-s)^{-1}(m-q+(k-t)y)+ky) \mid y\in\mathbf{Q}\}$$

$$\cup\{(a,-(n-s)^{-1}(k-t+(m-q)a),1,ma-n(n-s)^{-1}(k-t+(m-q)a)+k) \mid a\in\mathbf{I}\}.$$

ii. If $k\text{-}t\notin\mathbf{I}$, then the intersection points are

$$\{(1,x,-(k-t)^{-1}(m-q+(n-s)x),m+nx-k(k-t)^{-1}(m-q+(n-s)x)) \mid x\in\mathbf{Q}\}$$

$$\cup\{(u,1,-(k-t)^{-1}((m-q)u+n-s),mu+n-k(k-t)^{-1}((m-q)u+n-s)) \mid u\in\mathbf{I}\}.$$

iii. If $n\text{-}s\in\mathbf{I}$, $k\text{-}t\in\mathbf{I}$ and $m\text{-}q\notin\mathbf{I}$, then the intersection points are

$$\{(-(m-q)^{-1}(n-s+(k-t)v),1,v,-m(m-q)^{-1}(n-s+(k-t)v)+n+kv) \mid v\in\mathbf{Q}\}$$

$$\cup\{(-(m-q)^{-1}((n-s)b+k-t),b,1,-m(m-q)^{-1}((n-s)b+k-t)+nb+k) \mid b\in\mathbf{I}\}.$$

Now we take lines [m,n,1,k] and [q,s,1,t] of the third types, where $k,t\in\mathbf{I}$. We can determine the intersection points of these lines in two cases as follows:

i. If $m\text{-}q\notin\mathbf{I}$ and $n\text{-}s\in\mathbf{I}$, then the intersection points are

$$\{(-(m-q)^{-1}(n-s+(k-t)w),1,-m(m-q)^{-1}(n-s+(k-t)w)+n+kw,w) \mid w\in\mathbf{Q}\}$$

$$\cup\{(-(m-q)^{-1}((n-s)b+k-t),b,-m(m-q)^{-1}((n-s)b+k-t)+nb+k,1) \mid b\in\mathbf{I}\}.$$

ii. If $n\text{-}s\notin\mathbf{I}$, then the intersection points are

$$\{(1,-(n-s)^{-1}(m-q+(k-t)z),m-n(n-s)^{-1}(m-q+(k-t)z)+kz,z) \mid z\in\mathbf{Q}\}$$

$$\cup\{(a,-(n-s)^{-1}(k-t+(m-q)a),ma-n(n-s)^{-1}(k-t+(m-q)a)+k,1) \mid a\in\mathbf{I}\}.$$

Finally we take [m,1,n,k] and [q,1,s,t] of the second types lines, where $n,k,s,t\in\mathbf{I}$. In this case, if $m\text{-}q\notin\mathbf{I}$ then the intersection points are

$$\{(-(m-q)^{-1}(n-s+(k-t)w),-m(m-q)^{-1}(n-s+(k-t)w)+n+kw,1,w) \mid w\in\mathbf{Q}\}$$

$$\cup\{(-(m-q)^{-1}((n-s)c+k-t),-m(m-q)^{-1}((n-s)c+k-t)+nc+k,c,1) \mid c\in\mathbf{I}\}.$$

Now, conversely, we would like to determine the lines joining these two points, according to the two points in the 3-space are of different and of the same types.

First let us examine the case that the two points are of different types:

$$(1,x,y,z) \cup (u,1,v,w)$$

$$= \{[z-(w-zu-t(v-yu))(1-xu)^{-1}x-ty,(w-zu-t(v-yu))(1-xu)^{-1},t,1] \mid t\in\mathbf{Q}\}$$

$$\cup\{[y-(v-yu-k(w-zu))(1-xu)^{-1}x-kz,(v-yu-k(w-zu))(1-xu)^{-1},1,k] \mid k\in\mathbf{I}\}$$

where $u\in\mathbf{I}$,

$(1,x,y,z) \cup (u,v,1,w)$

$= \{[z-sx-(w-zu-s(v-xu))(1-yu)^{-1}y,s,(w-zu-s(v-xu))(1-yu)^{-1},1] \mid s\in\mathbf{Q}\}$

$\cup\{[x-(v-xu-k(w-zu))(1-yu)^{-1}y-kz,1,(v-xu-k(w-zu))(1-yu)^{-1},k] \mid k\in\mathbf{I}\}$

where $u,v\in\mathbf{I}$,

$(1,x,y,z) \cup (u,v,w,1)$

$= \{[y-sx-(w-yu-s(v-xu))(1-zu)^{-1}z,s,1,(w-yu-s(v-xu))(1-zu)^{-1}] \mid s\in\mathbf{Q}\}$

$\cup\{[x-ny-(v-xu-n(w-yu))(1-zu)^{-1}z,1,n,(v-xu-n(w-yu))(1-zu)^{-1}] \mid n\in\mathbf{I}\}$

where $u,v,w\in\mathbf{I}$,

$(x,1,y,z) \cup (u,v,1,w)$

$= \{[q,z-qx-(w-zv-q(u-xv))(1-yv)^{-1}y,(w-zv-q(u-xv))(1-yv)^{-1},1] \mid q\in\mathbf{Q}\}$

$\cup\{[1,x-(u-xv-k(w-zv))(1-yv)^{-1}y-kz,(u-xv-k(w-zv))(1-yv)^{-1},k] \mid k\in\mathbf{I}\}$

where $x,u,v\in\mathbf{I}$,

$(x,1,y,z) \cup (u,v,w,1)$

$= \{[q,y-qx-(w-yv-q(u-xv))(1-zv)^{-1}z,1,(w-yv-q(u-xv))(1-zv)^{-1}] \mid q\in\mathbf{Q}\}$

$\cup\{[1,x-ny-(u-xv-n(w-yv))(1-zv)^{-1}z,n,(u-xv-n(w-yv))(1-zv)^{-1}] \mid n\in\mathbf{I}\}$

where $x,u,v,w\in\mathbf{I}$,

$(x,y,1,z) \cup (u,v,w,1)$

$= \{[q,1,y-qx-(v-yw-q(u-xw))(1-zw)^{-1}z,(v-yw-q(u-xw))(1-zw)^{-1}] \mid q\in\mathbf{Q}\}$

$\cup\{[1,m,x-my-(u-xw-m(v-yw))(1-zw)^{-1}z,(u-xw-m(v-yw))(1-zw)^{-1}] \mid m\in\mathbf{I}\}$

where $x,y,u,v,w\in\mathbf{I}$.

Now we can examine the other case. Let the two points be of the same type:

If we take points $(1,x,y,z)$ and $(1,u,v,w)$ of the first type, then we complete this examination in the following three cases:

i. If $x-u\notin\mathbf{I}$, then the lines joining these points are

$\{[z-(z-w-c(y-v))(x-u)^{-1}x-cy,(z-w-c(y-v))(x-u)^{-1},c,1] \mid c\in\mathbf{Q}\}$

$\cup\{[y-(y-v-t(z-w))(x-u)^{-1}x-tz,(y-v-t(z-w))(x-u)^{-1},1,t] \mid t\in\mathbf{I}\}$.

ii. If $x-u\in\mathbf{I}$, $y-v\in\mathbf{I}$ and $z-w\notin\mathbf{I}$, then the lines joining these points are

$\{[y-sx-(y-v-s(x-u))(z-w)^{-1}z,s,1,(y-v-s(x-u))(z-w)^{-1}] \mid s\in\mathbf{Q}\}$

$\cup\{[x-ny-(x-u-n(y-v))(z-w)^{-1}z,1,n,(x-u-n(y-v))(z-w)^{-1}] \mid n\in\mathbf{I}\}$.

iii. If $x-u\in\mathbf{I}$, $y-v\notin\mathbf{I}$, then the lines joining these points are

$$\{[z-bx-(z-w-b(u-x))(y-v)^{-1}y,b,(z-w-b(u-x))(y-v)^{-1},1] \mid b\in\mathbf{Q}\}$$

$$\cup\{[x-(x-u-k(z-w))(y-v)^{-1}y-kz,1,(x-u-k(z-w))(y-v)^{-1},k] \mid k\in\mathbf{I}\}.$$

If we take points $(x,1,y,z)$ and $(u,1,v,w)$ of the second type, where $x,u\in\mathbf{I}$, then we can determine the lines joining these points in the following two cases:

i. If $y-v\notin\mathbf{I}$, then the lines joining these points are

$$\{[a,z-ax-(z-w-a(x-u))(y-v)^{-1}y,(z-w-a(x-u))(y-v)^{-1},1] \mid a\in\mathbf{Q}\}$$

$$\cup\{[1,x-(x-u-k(z-w))(y-v)^{-1}y-kz,(x-u-k(z-w))(y-v)^{-1},k] \mid k\in\mathbf{I}\}.$$

ii. If $y-v\in\mathbf{I}$ and $z-w\notin\mathbf{I}$, then the lines joining these points are

$$\{[q,y-qx-(y-v-q(x-u))(z-w)^{-1}z,1,(y-v-q(x-u))(z-w)^{-1}] \mid q\in\mathbf{Q}\}$$

$$\cup\{[1,x-ny-(x-u-n(y-v))(z-w)^{-1}z,n,(x-u-n(y-v))(z-w)^{-1}] \mid n\in\mathbf{I}\}.$$

Finally, if we take points $(x,y,1,z)$ and $(u,v,1,w)$ of the third type, where $x,y,u,v\in\mathbf{I}$, then we obtain the lines joining these points in the following one case:

i. If $z-w\notin\mathbf{I}$, then the lines joining these points are

$$\{[q,1,y-qx-(y-v-q(x-u))(z-w)^{-1}z,(y-v-q(x-u))(z-w)^{-1}] \mid q\in\mathbf{Q}\}$$

$$\cup\{[1,m,x-my-(x-u-m(y-v))(z-w)^{-1}z,(x-u-m(y-v))(z-w)^{-1}] \mid m\in\mathbf{I}\}.$$

So, we have completed the examination related to find the intersection points of any two lines and the lines joining any two points in the 3-space $\mathbf{P(J'')}$. Note that the previous results are obtained if we choose $\varepsilon=0$ (in this case, $\mathbf{I}=\{0\}$).

Finally, we would like to make an evaluation between quaternion 2-space (plane) and the quaternion 3-space. If we follow the way which is similar to the construction given for 3-space, we have the following for the 2-space $(\mathbf{P},\mathbf{L},|,\sqcup)$:

The set of points is

$$\mathbf{P} = \{(1,x_2,x_3) \mid x_2,x_3\in\mathbf{Q}\}\cup\{(x_1,1,x_3) \mid x_1\in\mathbf{I},x_3\in\mathbf{Q}\}\cup\{(x_1,x_2,1) \mid x_1,x_2\in\mathbf{I}\}.$$

The set of lines is

$$\mathbf{L} = \{[1,m_2,m_3] \mid m_2,m_3\in\mathbf{I}\}\cup\{[m_1,1,m_3,] \mid m_1\in\mathbf{Q},m_3\in\mathbf{I}\}\cup\{[m_1,m_2,1] \mid m_1,m_2\in\mathbf{Q}\}.$$

The incidence relation " | " is

$$[1,m_2,m_3]=\{(m_2+m_3y_3,1,y_3) \mid y_3\in\mathbf{Q}\}\cup\{(m_2z_2+m_3,z_2,1) \mid z_2\in\mathbf{I}\},$$

$$[m_1,1,m_3]=\{(1,m_1+m_3y_3,y_3) \mid y_3\in\mathbf{Q}\}\cup\{(z_1,m_1z_1+m_3,1) \mid z_1\in\mathbf{I}\},$$

$$[m_1,m_2,1]=\{(1,y_2,m_1+m_2y_2) \mid y_2\in\mathbf{Q}\}\cup\{(z_1,1,m_1z_1+m_2) \mid z_1\in\mathbf{I}\}.$$

The connection relation "$\sqcup$" is

$P = (x_1,x_2,x_3) \sqcup (y_1,y_2,y_3) = Q \Leftrightarrow x_i-y_i \in \mathbf{I}\ (i=1,2,3), \forall P,Q \in \mathbf{P};$

$g = [m_1,m_2,m_3] \sqcup [p_1,p_2,p_3] = h \Leftrightarrow m_i-p_i \in \mathbf{I}\ (i=1,2,3)), \forall g,h \in \mathbf{L}.$

The 2-space is isomorphic to the projective Klingenberg plane given in [3,4]. In a projective Klingenberg plane, it is well known that two non-connected (non-neighbour in [3,4]) lines meet at a unique point. However, this situation is different in 3-space as two lines with this propery meet at least at two points (see the results at pages 7 and 8). This means that 2-space and 3-space are different from each other.

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