Cumhuriyet Sci. J., 46(2) (2025) 410-423
DOI: https://doi.org/10.17776/csj.1656241

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Digraph Groups Without Leaves Whose Arc Count Exceeds Their Vertex Count by One

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Research Article

History
Received: 12/03/2025
Accepted: 11/06/2025

ABSTRACT

This paper investigates a particular class of digraph groups that are defined by non-empty presentations. Each relation is expressed in the form R(x,y), where x and y are distinct generators, and R(·,·) is based on a fixed cyclically reduced word R(a,b) involving both a and b. A directed graph is constructed for each such presentation, where vertices correspond to generators and edges represent the relations. In previous research, Cihan identified 35 families of digraphs satisfying |V(Γ)|=|A(Γ)|-1, of which 11 do not contain leaves. This paper demonstrates that, with two exception families, the rank of the associated groups is either 1 or 2.

Keywords: Digraph group, Pride group, Finite cyclic, Rank, Presentations.

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Introduction

This paper focuses on a distinct category of finite connected digraphs and their corresponding group presentations, in which each relator is structured as $R(x,y)$, where x and y are distinct generators and $R(\cdot,\cdot)$ is determined by some fixed cyclically reduced word $R(a,b)$ in the free group generated by a and b that involves both a and b. Such groups have previously been analyzed in the paper by Cuno and Williams [1].

A fundamental aspect of this study is the construction of a group presentation from a digraph. Let Λ be a finite *digraph* with a set of vertices $V(\Lambda)$ and a set of directed arcs $A(\Lambda)$. Each vertex $v \in V(\Lambda)$ correspond to the generators x_v while each arc $(u,v) \in A(\Lambda)$ correspond to the relators $R(x_u,x_v)$. Consequently, the group $G_\Lambda(R)$ is defined by the presentation.

$$P_\Lambda(R) = \langle\, x_v \ \big(v \in V(\Lambda)\big) \,\big|\, R(x_u,x_v) \ \big((u,v) \in A(\Lambda)\big) \,\rangle.$$

A digraph group is defined as a group that is isomorphic to $G_\Lambda(R)$ for some Λ and R [1]. In 2020, the terminology of digraph groups was first introduced by Cuno and Williams [1]. However, the study of such groups predates this terminology, as several previously explored group classes fall into the category of digraph groups, even though they were not explicitly classified as such. In particular, Cihan and Williams also examined the Johnson and Mennicke digraph groups in [2].

Consider the free group with basis $x_0, \dots, x_{n-1}$ and let w be a word in the free group, where $n > 0$. The shift, denoted by θ, is the free group automorphism mapping $x_i \mapsto x_{i+1}$, with subscripts $\bmod n$. Then is called a *cyclic presentation*, and we write $G_n(w)$ for the corresponding cyclically presented group [3].

If w involves exactly two generators then $G_n(w)$ is a digraph group by setting Λ to be a directed n-cycle, i.e. $V(\Lambda) = \{1,2,\dots,n\}$ and $A(\Lambda) = \{(1,2),(2,3),\dots,(n,1)\}$.

Our focus will be on determining when digraph groups are finite. It is well known that if a group is defined by a presentation with more generators than relators, it must be infinite, which can be verified by abelianizing the groups [3-4]. Therefore, we will concentrate on cases where the number of relators is greater than or equal to the number of generators ($|V| \leqslant |A|$). The first case we consider is balanced presentations, where the number of generators and relators is equal. Cuno and Williams [1] studied digraph groups $G_\Gamma(R)$ under the condition $|V(\Gamma)| = |A(\Gamma)|$, with the additional assumption that the undirected graph is triangle-free (i.e., $\mathrm{girth}(\Gamma) \geqslant 4$). In most cases, they showed that the corresponding group $G_\Gamma(R)$ is either a finite cyclic group or infinite. Building on this, Cihan [5] identified 35 digraph families satisfying $|V(\Gamma)| = |A(\Gamma)| - 1$, of which 11 of them do not contain a leaf. In this paper, we aim to characterize finite cyclic digraph groups within the case $|V(\Gamma)| = |A(\Gamma)| - 1$, specifically for digraphs without leaves. Before formally defining these digraph classes, we first construct graphs that meet the conditions outlined in [5].

Under these circumstances $(i), (ii), (iv), (v)$ in [5], there are possible 11 digraph families without leaf as indicated in Figure 1 [5].

We will begin by presenting the classes of digraphs in Figure 1 and stating the main theorem. Following this, we will provide some remarks and lemmas that will be referenced throughout the paper. Next, we will focus on proving whether the corresponding groups are finite cyclic groups. For five of these digraph families, we demonstrate that $G_\Lambda(R)$ is a finite and cyclic group (i.e., $\mathrm{rank}(G_\Lambda(R)) = 1$), and in these cases, we will determine the group's order. In four of the families, we establish that the rank of $G_\Lambda(R)$ is either 1 or 2.

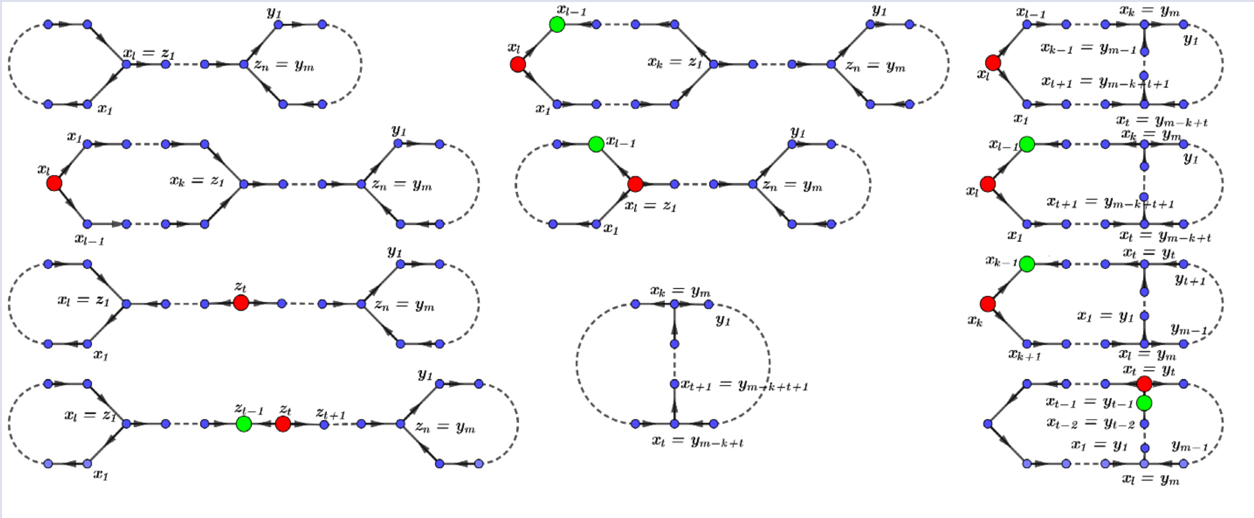

Figure 1. Classes of digraphs without leaf referred to in the statement of Main Theorem

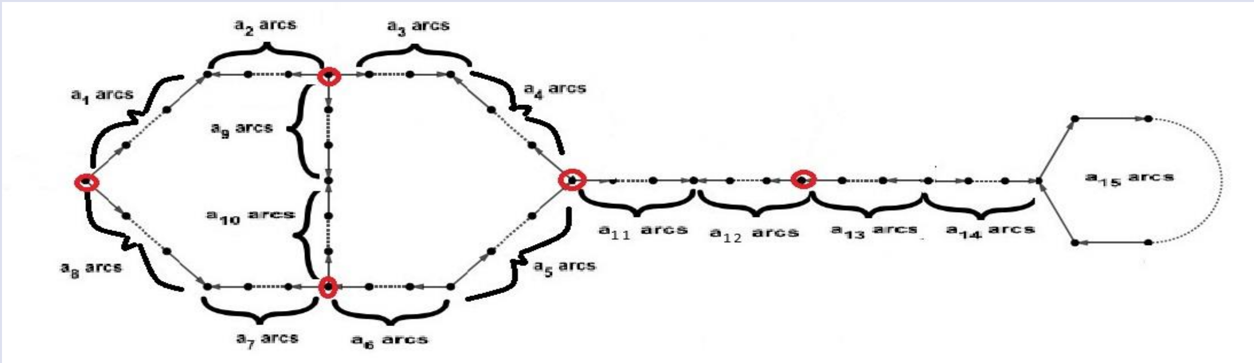

Figure 2. The digraph family that covers all possible digraphs of Main Theorem

Main Theorem . Let Λ be a non-empty finite digraph without leaf that the number of generators is equal to the number of relators minus one ($|V| = |A| - 1$) whose underlying undirected graph has girth n $(n \geqslant 4)$ and let $R(a,b)$ be a cyclically reduced word that involves both a and b with exponent sums α and $-\beta$ in a and b, respectively where $|\alpha| \geqslant 2$, $|\beta| \geqslant 2|$. If $G_\Lambda(R)$ is finite, then $\alpha \neq 0$, $\beta \neq 0$, $(\alpha,\beta) = 1$, $\alpha^n - \beta^n \neq 0$, $a^\alpha = b^\beta$ in $K = \langle\, a,b \mid R(a,b)\,\rangle$, $G_\Lambda(R)$ is non-trivial, and Λ is the digraph in Figure 1.

It is important to note that these digraph families can also be constructed using alternative notations. For instance, the digraph (i) can be represented by the sequence $a_1, a_3, a_6, a_7, a_{11}, a_{14}, a_{15}$, but it is equally valid to construct it as $a_1, a_7, a_9, a_{11}, a_{15}$. The primary objective in constructing these digraph families is to systematically integrate the designated circle with an appropriate arc, when applicable, as illustrated in Figure 2.

	The possible arcs a_i in Figure 2	in which case
(i)	a_1, a_3, a_6, a_7, a_{11}, a_{14}, a_{15}	$G_\Lambda(R) \cong \left|\alpha^{(l,m)} - \beta^{(l,m)}\right|$
(ii)	a_1, a_3, a_5, a_8, a_{11}, a_{14}, a_{15} ($l<2k$) ($l>2k$) ($l=2k$)	$G_\Lambda(R) \cong \left|\alpha^{\min\{l-k,\lvert n+l-k-1\rvert\}}(\alpha(m,2k-l) - \beta(m,2k-l)\right|$ $G_\Lambda(R) \cong \left|\alpha^{\min\{k,\lvert n+l-k-1\rvert\}}(\alpha(m,l-2k) - \beta(m,l-2k)\right|$ $G_\Lambda(R) \cong \left|\alpha^{n+l-k-1}(\alpha^m - \beta^m)\right|$
(iii)	a_1, a_3, a_6, a_7, a_{12}, a_{14}, a_{15}	$G_\Lambda(R) \cong \left|\alpha^{\min\{\lvert n-1\rvert,\lvert t-1\rvert\}}(\alpha(l,m) - \beta(l,m)\right|$
(iv)	a_1, a_3, a_6, a_7, a_{11}, a_{14}, a_{15}, (a_{12} = 1)	$rank(G_\Lambda(R)) \in \{1, 2\}$,
(v)	a_4, a_5, a_6, a_{11}, a_{14}, $a_{15\mathbb{Z}}$, (a_1 =1)	$rank(G_\Lambda(R)) \in \{1, 2\}$,
(vi)	a_1, a_6, a_7, a_{11}, a_{14}, a_{15}, (a_4 = 1)	$rank(G_\Lambda(R)) \in \{1, 2\}$,
(vii)	a_2, a_3, a_6, a_8, a_{10}	$G_\Lambda(R) \cong \left|\alpha^{(l,m)} - \beta^{(l,m)}\right|$

(viii)	a_1, a_3, a_6, a_8, a_{10}	$G_\Lambda(R)\left\|\alpha^{\min\{k,\|l-k\|\}}(\alpha(m,1-2k)-\beta(m,1-2k)\right\|$
(ix)	a_2, a_3, a_6, a_8, a_{10}, (a_1 = 1)	$rank(G_\Lambda(R)) \in \{1, 2\}$,
(x)	a_2, a_4, a_5, a_8, a_{10}, (a_1 = 1)	?
(xi)	a_2, a_3, a_6, a_8, a_{10}, (a_9 = 1)	?

A Brief Overview Before Proving The Main Theorem

We now state a reflection principle , a convention, partially introduced by Pride in [6], Lemma 2.4 is a specialisation of a result due to Pride, which was stated without proof in [6] and the proof was stated in [1] and the Lemma 2.5 proved by Cuno & Williams in [1] as we use them frequently throughout the paper. It is also important to understand why we have these conditions in our theorem by the readers.

Remark 2.1 [1]. We will occasionally make use of a reflection principle: if Λ is any digraph and $\mathrm{R}(\mathrm{a},\mathrm{b})$ is any word, then we may consider the digraph Λ' that is obtained from Λ by reversing the direction of each arc and the word $\mathrm{R}'(\mathrm{a},\mathrm{b})$ that is obtained from $\mathrm{R}(\mathrm{a},\mathrm{b})$ by interchanging a and b and further replacing every letter by its inverse so that also α and β are interchanged (without any change of sign). Then, by definition, $G_\Lambda(R) \cong G_{\Lambda'}(R')$.

Convention 1 [1] We use α and $-\beta$ to represent the exponent sums of a and b in a cyclically reduced word $\mathrm{R}(\mathrm{a},\mathrm{b})$, respectively, and K is used to indicate a group defined by the presentation $\langle\, \mathrm{a},\mathrm{b} \mid \mathrm{R}(\mathrm{a},\mathrm{b})\,\rangle$. As far as cyclic permutation is considered, the word R has the form $a^{\alpha_1}b^{\beta_1}\cdots a^{\alpha_t}b^{\beta_t}$ with $t \geqslant 1$ and $\alpha_i,\beta_i \in \mathbb{Z}\backslash\{0\}$ $(1 \leqslant i \leqslant t)$.

The following property is defined by Pride in [6]: If no non-empty word of the form $a^k b^{-\ell} (k, \ell \in)$ is equal to the identity in that group, then a two-generator group with generators a and b is said to have *Property* W_1 *(with respect to a and b)*. Under the hypothesis that the girth of the underlying undirected graph of Λ is at least 4.

Corollary 2.2. [6] Let Λ be a non-empty finite digraph whose underlying undirected graph has $girth\,(G) \geqslant 4$ and let $R(a,b)$ be as in convention 1. If K has Property W_1, then $G_\Lambda(R)$ is infinite.

It is therefore important to study groups that do not have Property W_1.

Proposition 2.3. [6] If there exist $k,\ell \in \mathbb{Z}\backslash\{0\}$ with $a^k = b^\ell$ in K, then $\alpha \neq 0$, $\beta \neq 0$, and $a^\alpha = b^\beta$ in K.

Therefore, K does not have Property W_1 if and only if $\alpha \neq 0$, $\beta \neq 0$, and $a^\alpha = b^\beta$ in K.

Lemma 2.4. [1, 6] Let Λ be a non-empty finite digraph whose underlying undirected graph has girth at least 4 and let $R(a,b)$ be a cyclically reduced word that involves both a and b. Let $R(a,b)$ be as in convention 1 and $|\alpha| \geqslant 2$ and $|\beta| \geqslant 2$. If $G_\Lambda(R)$ is finite, then $(\alpha,\beta)=1$ and Λ has at most one source and at most one sink.

We will now state Lemma 2.5 (a),(b) proved by Cuno and Williams [1] and we add $(c),(d)$. It enables us to simplify the presentations that arise. Therefore, it is stated here for later use without further explanation throughout the paper.

Lemma 2.5. [1] Let $R(a,b)$ be a word such that $a^\alpha = b^\beta$ in K and let G be a group defined by a presentation $\langle\, \mathcal{X} \mid \mathcal{R}\,\rangle$. Further suppose that there are distinct generators $x_i, x_j \in \mathcal{X}$ such that $R(x_i,x_j) \in \mathcal{R}$. Then the following hold:

(a) If $x_i^\gamma \in \mathcal{R}$ for some $\gamma \in \mathbb{Z}$ with $(\alpha,\gamma)=1$, then every $p \in \mathbb{Z}$ with $p\alpha \equiv 1 \pmod{\gamma}$ yields a new presentation $\langle\, \mathcal{X} \setminus \{x_i\} \mid \mathcal{S}\,\rangle$ of G. The relators $\mathcal{S}$ are obtained from $\mathcal{R}$ by removing $R(x_i,x_j)$ and x_i^γ, replacing all remaining occurrences of x_i by $x_j^{p\beta}$, and adjoining $x_j^{\beta\gamma}$.

(b) If $x_j^\gamma \in \mathcal{R}$ for some $\gamma \in \mathbb{Z}$ with $(\beta,\gamma)=1$, then every $p \in \mathbb{Z}$ with $p\beta \equiv 1 \pmod{\gamma}$ yields a new presentation $\langle\, \mathcal{X} \setminus \{x_j\} \mid \mathcal{S}\,\rangle$ of G. The relators $\mathcal{S}$ are obtained from $\mathcal{R}$ by removing $R(x_i,x_j)$ and x_j^γ, replacing all remaining occurrences of x_j by $x_i^{p\alpha}$, and adjoining $x_i^{\alpha\gamma}$.

(c) If $x_i^\gamma \in \mathcal{R}$ for some $\gamma \in$ with $(\alpha,\gamma)=1$ then every $p \in \mathbb{Z}$ with $p\alpha \equiv 1 \pmod{\gamma}$ yields a new presentation $\langle\, \mathcal{X} \mid \mathcal{S}\,\rangle$ of G where $\mathcal{S} = \mathcal{R} \cup \{x_i x_j^{-p\beta}, x_j^{\beta\gamma}\}$.

(d) If $x_j^\gamma \in \mathcal{R}$ for some $\gamma \in$ with $(\beta,\gamma)=1$ then every $p \in$ with $p\beta \equiv 1 \pmod{\gamma}$ yields a new presentation $\langle\, \mathcal{X} \mid \mathcal{S}\,\rangle$ of G where $\mathcal{S} = \mathcal{R} \cup \{x_j x_i^{-p\alpha}, x_i^{\alpha\gamma}\}$.

If Λ is a directed n-cycle $(n \geqslant 4)$ and $R(a,b)$ is a cyclically reduced word that involves both a and b, then $G_\Lambda(R)$ can never be finite of rank 3 or trivial [1]. We now give precise statement of Theorem 2.6 that forms the cornerstone of this paper. The following Theorem 2.6 was stated without proof in [6], a proof was given in [7] and a different proof was given in [4].

Theorem 2.6. [1, 6, 7] Let $R(a,b)$ be as in Convention 1. Further suppose that $(\alpha,\beta)=1$ and $a^\alpha = b^\beta$ in K. If $\Lambda = \Lambda(n)$, where $\Lambda(n)$ is directed n −cycle $(n \geqslant 2)$, then $G_\Lambda(R) \cong \mathbb{Z}_{|\alpha^n-\beta^n|}$.

The Theorem 2.6 is generalized from cyclic presentations to balanced presentations (i.e. $|V|=|A|$) in [4]. We extend the theorem from balanced presentations to $|V|=|A|-1$ without leaf in this paper.

Lemma 2.7. [1] Let $\mathrm{R}(\mathrm{a},\mathrm{b})$ be as in Theorem 1. Further suppose that $(\alpha,\beta)=1$ and $a^\alpha = b^\beta$ in K. Then the following hold:

(a) If $\Lambda = \Lambda(n;\overset{m}{\rightarrow}))$ $(n \geqslant 2,\ m \geqslant 1)$, then $G_\Lambda(R) \cong \mathbb{Z}_{|\beta^m(\alpha^n-\beta^n)|}$.

(b) If $\Lambda = \Lambda(n;\overset{m}{\leftarrow})$ $(n \geqslant 2,\ m \geqslant 1)$, then $G_\Lambda(R) \cong \mathbb{Z}_{|\alpha^m(\alpha^n-\beta^n)|}$.

In many of our digraphs Γ there will be a configuration of the form $\Lambda(n;\overset{m}{\rightarrow})$ or $\Lambda(n;\overset{m}{\leftarrow})$; Lemma 2.7 allows us to replace this sub-digraph with a vertex x_v and adding a corresponding relator x_v^P to the presentation. To assist the reader in (i) we will explain this reduction in detail, then in later we will use this technique without further explanation.

We now show that if l, k are vertices of a directed circuit (see Figure 3) then the generator x_k can be written in terms of generator x_l. Then we will use this relation in our presentations. We set $\gamma = \alpha^l - \beta^l$ and $\zeta = \beta(p\alpha - 1)$, where $p\alpha \equiv 1 \pmod{\gamma}$.

Lemma 2.8. Suppose that Λ_1 is the circuit in Figure 3 and k, l are vertices of Λ_1 and suppose $(\alpha, \beta) = 1$. Let p, q be integers such that $p\alpha + q\beta = 1$. Then $x_k = x_l^{p^{l-k}\beta^{l-k}}$.

Proof. In this case, we have the presentation of $G_{\Lambda_1}(R)$ for Figure 3,

$$\left\langle x_1, x_2, \dots, x_k \dots, x_l \;\middle|\; \begin{matrix} x_1^{\gamma}, R(x_1,x_2), R(x_2,x_3), \dots, R(x_{k-1},x_k), \\ R(x_k,x_{k+1}) \dots, R(x_{l-1},x_l), R(x_l,x_1) \end{matrix} \right\rangle.$$

Note that $p\alpha \equiv 1 \pmod{\gamma}$. We continue simplifying this presentation by Lemma 2.5 (a),

$$\begin{aligned}
G_{\Lambda_1}(R) &= \left\langle x_2, \dots, x_k \dots, x_l \;\middle|\; x_2^{\beta\gamma}, R(x_2,x_3), \dots, R(x_{k-1},x_k), R(x_k,x_{k+1}) \dots, R(x_{l-1},x_l), R\left(x_l, x_2^{p\beta}\right) \right\rangle \\
&= \left\langle x_3, \dots, x_k \dots, x_l \;\middle|\; x_3^{\beta^2\gamma}, R(x_3,x_4), \dots, R(x_{k-1},x_k), R(x_k,x_{k+1}) \dots, R(x_{l-1},x_l), R(x_l, x_3^{p^2\beta^2}) \right\rangle \\
&= \cdots \\
&= \left\langle x_k, x_{k+1}, \dots, x_l \;\middle|\; x_k^{\beta^{k-1}\gamma}, R(x_k,x_{k+1}), R(x_{k+1},x_{k+2}), \dots, R(x_{l-1},x_l), R(x_l, x_k^{p^{k-1}\beta^{k-1}}) \right\rangle \\
&= \left\langle x_k, x_{k+1}, \dots, x_l \;\middle|\; x_k^{\beta^{k-1}\gamma}, x_k = x_{k+1}^{p\beta}, R(x_k,x_{k+1}), R(x_{k+1},x_{k+2}), \dots, R(x_{l-1},x_l), R(x_l, x_k^{p^{k-1}\beta^{k-1}}) \right\rangle \\
&= \left\langle x_k, x_{k+1}, \dots, x_l \;\middle|\; x_k^{\beta^{k-1}\gamma}, x_k = x_{k+1}^{p\beta}, R(x_{k+1}^{p\beta},x_{k+1}), R(x_{k+1},x_{k+2}), \dots, R(x_{l-1},x_l), R(x_l, x_{k+1}^{p^k\beta^k}) \right\rangle \\
&= \left\langle x_k, x_{k+1}, \dots, x_l \;\middle|\; x_k^{\beta^{k-1}\gamma}, x_k = x_{k+1}^{p\beta}, x_{k+1}^{\beta(p\alpha-1)}, R(x_{k+1},x_{k+2}), \dots, R(x_{l-1},x_l), R(x_l, x_{k+1}^{p^k\beta^k}) \right\rangle \\
&= \left\langle x_k, x_{k+2}, \dots, x_l \;\middle|\; x_k^{\beta^{k-1}\gamma}, x_k = x_{k+2}^{p^2\beta^2}, x_{k+2}^{\beta\zeta}, R(x_{k+2},x_{k+3}), \dots, R(x_{l-1},x_l), R(x_l, x_{k+2}^{p^{k+1}\beta^{k+1}}) \right\rangle \\
&= \left\langle x_k, x_{k+3}, \dots, x_l \;\middle|\; x_k^{\beta^{k-1}\gamma}, x_k = x_{k+3}^{p^3\beta^3}, x_{k+3}^{\beta^2\zeta}, R(x_{k+3},x_{k+4}), \dots, R(x_{l-1},x_l), R(x_l, x_{k+3}^{p^{k+2}\beta^{k+2}}) \right\rangle \\
&= \cdots \\
&= \left\langle x_k, x_l \;\middle|\; x_k^{\beta^{k-1}\gamma}, x_k = x_l^{p^{l-k}\beta^{l-k}}, x_l^{\beta^{l-k-1}\zeta}, R(x_l, x_l^{p^{l-1}\beta^{l-1}}) \right\rangle \\
&= \left\langle x_k, x_l \;\middle|\; x_k^{\beta^{k-1}\gamma}, x_k = x_l^{p^{l-k}\beta^{l-k}}, x_l^{\beta^{l-k-1}\zeta}, x_l^{p^{l-1}\beta^{l-1}-\alpha} \right\rangle \\
&= \left\langle x_k, x_l \;\middle|\; x_k^{\beta^{k-1}\gamma}, x_k = x_l^{p^{l-k}\beta^{l-k}}, x_l^{\gamma} \right\rangle. \text{ Hence } x_k = x_l^{p^{l-k}\beta^{l-k}}
\end{aligned}$$

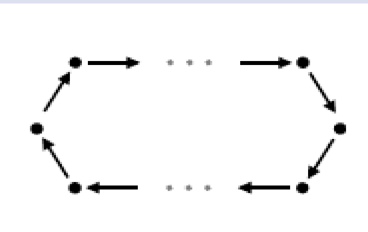

Figure 1. Λ_1: set up a relation between x_k and x_l.

Remark 2.9. Suppose $(\alpha, \beta) = 1$, $l, m \geqslant 1$ and let $\gamma = \alpha^l - \beta^l, \eta = \alpha^m - \beta^m$. Then $(\alpha, \gamma) = (\beta, \eta) = 1$.

Proving The Main theorem

Recall that we can always suppose that $\alpha \neq 0$, $\beta \neq 0$, $|\alpha| \geqslant 2$, $|\beta| \geqslant 2$, $(\alpha, \beta) = 1$,and $a^\alpha = b^\beta$ in K. Otherwise, the group K has Property W_1 and thus $G_\Lambda(R)$ is infinite by Corollary 2.2 and Proposition 2.3.

(i) $a_1, a_3, a_6, a_7, a_{11}, a_{14}, a_{15}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n \end{matrix} \;\middle|\; \begin{matrix} R(x_1,x_2), R(x_2,x_3), \dots, R(x_l,x_1), \\ R(y_1,y_2), R(y_2,y_3), \dots, R(y_m,y_1), \\ R(z_1,z_2), R(z_2,z_3), \dots, R(z_{n-1},z_n), \\ x_l = z_1, z_n = y_m \end{matrix} \right\rangle$$

$a^\alpha = b^\beta$ in $K = \langle\, a, b \mid R(a,b) \,\rangle$ by Proposition 2.3, thus we get

$$x_1^{\alpha^l} = x_2^{\alpha^{l-1}\beta} = x_3^{\alpha^{l-2}\beta^2} = \cdots = x_l^{\alpha\beta^{l-1}} = x_1^{\beta^l}.$$
$$y_1^{\alpha^m} = y_2^{\alpha^{m-1}\beta} = y_3^{\alpha^{m-2}\beta^2} = \cdots = y_m^{\alpha\beta^{m-1}} = y_1^{\beta^m}.$$

We set $\gamma = \alpha^l - \beta^l$ and $\eta = \alpha^m - \beta^m$ obtain that $x_1^\gamma = 1$, and $y_1^\eta = 1$ in $G_\Gamma(R)$. Adjoining the relator x_1^γ and y_1^η yield

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n \end{matrix} \;\middle|\; \begin{matrix} x_1^\gamma, R(x_1,x_2), R(x_2,x_3), \dots, R(x_l,x_1), \\ y_1^\eta, R(y_1,y_2), R(y_2,y_3), \dots, R(y_m,y_1), \\ R(z_1,z_2), R(z_2,z_3), \dots, R(z_{n-1},z_n), \\ x_l = z_1, z_n = y_m \end{matrix} \right\rangle$$

Applying precisely the same transformations as in the proof of Theorem 2.6, we get

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n \end{matrix} \;\middle|\; \begin{matrix} x_2^{\beta\gamma}, R(x_2,x_3), \dots, R(x_l,x_2^{p\beta}), \\ y_2^{\beta\eta}, R(y_2,y_3), \dots, R(y_m,y_2^{p\beta}), \\ R(z_1,z_2), R(z_2,z_3), \dots, R(z_{n-1},z_n), \\ x_l = z_1, z_n = y_m \end{matrix} \right\rangle$$

Simplifying this presentation in that way, what remains is

$$G_\Gamma(R) = \left\langle \begin{matrix} x_l, y_m, \\ z_1, \dots, z_n \end{matrix} \;\middle|\; \begin{matrix} x_l^\gamma \\ y_m^\eta \\ R(z_1,z_2), R(z_2,z_3), \dots, R(z_{n-1},z_n), \\ x_l = z_1, z_n = y_m \end{matrix} \right\rangle$$
$$= \left\langle \begin{matrix} x_l, y_m, \\ z_1, \dots, z_n \end{matrix} \;\middle|\; \begin{matrix} x_l^\gamma \\ z_n^\eta, R(z_1,z_2), R(z_2,z_3), \dots, R(z_{n-1},z_n), \\ x_l = z_1 \end{matrix} \right\rangle.$$

Since $(\beta, \eta) = 1$ by Remark 2.9 and an iterated application of Lemma 2.5 (b) for the relation inside the box yields

$$\begin{aligned} G_\Gamma(R) &= \langle\, x_l, z_1 \mid x_l^\gamma, z_1^{\alpha^{n-1}\eta}, x_l = z_1 \,\rangle \\ &= \langle\, x_l \mid x_l^\gamma, x_l^{\alpha^{n-1}\eta} \,\rangle \\ &= \langle\, x_l \mid x_l^{((\alpha^l-\beta^l),\alpha^{n-1}(\alpha^m-\beta^m))} \,\rangle \\ &= \langle\, x_l \mid x_l^{((\alpha^l-\beta^l),(\alpha^m-\beta^m))} \,\rangle \\ &= \langle\, x_l \mid x_l^{\alpha^{(l,m)}-\beta^{(l,m)}} \,\rangle. \end{aligned}$$

So $G_\Gamma(R)$ is finite cyclic of order $\alpha^{(l,m)} - \beta^{(l,m)}$.

(ii) $\boldsymbol{a_1, a_3, a_5, a_8, a_{11}, a_{14}, a_{15}}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n \end{matrix} \;\middle|\; \begin{matrix} R(x_l,x_1), R(x_1,x_2), R(x_2,x_3), \dots, R(x_{k-1},x_k), \\ R(x_l,x_{l-1}), R(x_{l-1},x_{l-2}), \dots, R(x_{k+1},x_k), \\ R(y_1,y_2), R(y_2,y_3), \dots, R(y_m,y_1), \\ R(z_1,z_2), R(z_2,z_3), \dots, R(z_{n-1},z_n), \\ x_k = z_1, z_n = y_m \end{matrix} \right\rangle.$$

We set as $\gamma = \alpha^m - \beta^m$ and after applying precisely the same transformations as in the proof of Lemma 2.5 (b) for the relation inside the box yields

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \dots, x_l, \\ y_m, \\ z_1, \dots, z_n \end{matrix} \;\middle|\; \begin{matrix} R(x_l,x_1), R(x_1,x_2), R(x_2,x_3), \dots, R(x_{k-1},x_k), \\ R(x_l,x_{l-1}), R(x_{l-1},x_{l-2}), \dots, R(x_{k+1},x_k), \\ y_m^\gamma, \\ R(z_1,z_2), R(z_2,z_3), \dots, R(z_{n-1},z_n), \\ x_k = z_1, z_n = y_m \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} x_1, \dots, x_l, \\ z_1, \dots, z_n \end{matrix} \middle| \begin{matrix} R(x_l, x_1), R(x_1, x_2), R(x_2, x_3), \dots, R(x_{k-1}, x_k), \\ R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_k), \\ z_n^{\gamma}, R(z_1, z_2), R(z_2, z_3), \dots, R(z_{n-1}, z_n), \\ x_k = z_1 \end{matrix} \right\rangle.$$

Since $(\beta, \gamma) = 1$ and (see Remark 2.9), an iterated application of Lemma 2.5 (b) yields

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \dots, x_l, \\ z_1 \end{matrix} \middle| \begin{matrix} R(x_l, x_1), R(x_1, x_2), R(x_2, x_3), \dots, R(x_{k-1}, x_k), \\ R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_k), \\ z_1^{\alpha^{n-1}\gamma}, x_k = z_1 \end{matrix} \right\rangle$$

$$= \left\langle x_1, \dots, x_k, \dots, x_l \middle| \begin{matrix} R(x_l, x_1), R(x_1, x_2), R(x_2, x_3), \dots, \\ R(x_{k-1}, x_k), \\ x_k^{\alpha^{n-1}\gamma}, \\ R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_k) \end{matrix} \right\rangle.$$

Since $(\beta, \alpha^{n-1}\gamma) = (\beta, \gamma) = 1$ (see Remark 2.9) there exists integers p, q such that $p\beta + q\gamma = 1$ and hence $p\beta = 1 \pmod{\gamma}$, an iterated application of Lemma 2.5 (b) yields

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \dots, x_{k-1}, \\ x_{k+1}, \dots, x_l \end{matrix} \middle| \begin{matrix} R(x_l, x_1), R(x_1, x_2), R(x_2, x_3), \dots, R(x_{k-1}, x_{k+1}^{p\alpha}), \\ x_{k+1}^{\alpha^n \gamma}, R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+2}, x_{k+1}) \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} x_1, \dots, x_{k-1}, \\ x_{k+2}, \dots, x_l \end{matrix} \middle| \begin{matrix} R(x_l, x_1), R(x_1, x_2), R(x_2, x_3), \dots, R(x_{k-1}, x_{k+2}^{p\alpha^2}), \\ x_{k+2}^{\alpha^{n+1}\gamma}, R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+3}, x_{k+2}) \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} x_1, \dots, x_{k-1}, \\ x_l \end{matrix} \middle| \begin{matrix} x_l^{\alpha^{n+l-k-1}\gamma}, \\ R(x_l, x_1), R(x_1, x_2), R(x_2, x_3), \dots, R(x_{k-1}, x_l^{(p\alpha)^{l-k}}) \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} x_1, \dots, x_{k-1}, \\ x_l \end{matrix} \middle| \begin{matrix} x_l^{\alpha^{n+l-k-1}\gamma}, \\ R(x_l, x_1), R(x_1, x_2), R(x_2, x_3), \dots, R(x_{k-1}, x_l^{(p\alpha)^{l-k}}), \\ x_i = x_l^{(p\alpha)^i} (1 \leqslant i \leqslant k-1) \text{ by Lemma 2.5 (b)} \end{matrix} \right\rangle$$

$$= \left\langle x_l \middle| \begin{matrix} x_l^{\alpha^{n+l-k-1}\gamma}, R(x_l, x_l^{p\alpha}), R(x_l^{p\alpha}, x_l^{(p\alpha)^2}), R(x_l^{(p\alpha)^2}, x_l^{(p\alpha)^3}), \dots, \\ R(x_l^{(p\alpha)^{k-1}}, x_l^{(p\alpha)^{l-k}}) \end{matrix} \right\rangle$$

$$= \left\langle x_l \middle| \begin{matrix} x_l^{\alpha^{n+l-k-1}\gamma}, x_l^{\alpha - \beta p\alpha}, x_l^{p\alpha^2 - \beta(p\alpha)^2}, \dots, x_l^{\alpha(p\alpha)^{k-2} - \beta(p\alpha)^{k-1}}, \\ x_l^{\alpha(p\alpha)^{k-1} - \beta(p\alpha)^{l-k}} \end{matrix} \right\rangle.$$

We can remove redundant relators $x_l^{\alpha - \beta p\alpha}, x_l^{p\alpha^2 - \beta(p\alpha)^2}, \dots, x_l^{\alpha(p\alpha)^{k-2} - \beta(p\alpha)^{k-1}}$ since $p\beta \equiv 1 \bmod \gamma$. Thus, we get

$$\begin{aligned} G_\Gamma(R) &= \langle\, x_l \mid x_l^{\alpha^{n+l-k-1}\gamma}, x_l^{p^{k-1}\alpha^k - p^{l-k}\alpha^{l-k}\beta} \,\rangle \\ &= \langle\, x_l \mid x_l^{(\alpha^{n+l-k-1}\gamma, p^{k-1}\alpha^k - p^{l-k}\alpha^{l-k}\beta)} \,\rangle \\ &= \langle\, x_l \mid x_l^d \,\rangle, \text{where } d = (\alpha^{n+l-k-1}\gamma, p^{k-1}\alpha^k - p^{l-k}\alpha^{l-k}\beta). \end{aligned}$$

$$\begin{aligned} d &= (\alpha^{n+l-k-1}\gamma, p^{k-1}\alpha^k - p^{l-k}\alpha^{l-k}\beta) \\ &= (\alpha^{n+l-k-1}\gamma, (p\beta)p^{k-1}\alpha^k - p^{l-k}\alpha^{l-k}\beta) \text{ since } p\beta \equiv 1 \pmod{\gamma} \\ &= (\alpha^{n+l-k-1}\gamma, \beta(p^k\alpha^k - p^{l-k}\alpha^{l-k})) \\ &= (\alpha^{n+l-k-1}\gamma, p^k\alpha^k - p^{l-k}\alpha^{l-k}) \text{ since } (\beta, \alpha\gamma) = 1 \\ &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^k - (p\alpha)^{l-k}). \end{aligned}$$

After that supposing $k < l - k$ and continue to simplify the equation above, we get

$$\begin{aligned} d &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^k(1-(p\alpha)^{l-2k})) \text{ since } p\beta \equiv 1 \ (\text{mod } \gamma) \\ &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^k((p\beta)^{l-2k}-(p\alpha)^{l-2k})) \text{ since } p\beta \equiv 1 \ (\text{mod } \gamma) \\ &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^k(p^{l-2k}(\beta^{l-2k}-\alpha^{l-2k}))) \\ &= (\alpha^{n+l-k-1}\gamma, p^{l-k}\alpha^k(\beta^{l-2k}-\alpha^{l-2k})) \\ &= (\alpha^{min\{k,n+l-k-1\}}(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)}). \end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{\min\{k,|n+l-k-1|\}}(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})$.

Now supposing $k > l-k$ and simplifying the equation, we get

$$\begin{aligned} d &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^{l-k}((p\alpha)^{2k-l}-1)) \text{ since } p\beta \equiv 1 \ (\text{mod } \gamma) \\ &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^{l-k}((p\alpha)^{2k-l}-(p\beta)^{2k-l})) \text{ since } p\beta \equiv 1 \ (\text{mod } \gamma) \\ &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^{l-k}(p^{2k-l}(\beta^{2k-l}-\alpha^{2k-l}))) \\ &= (\alpha^{n+l-k-1}\gamma, p^{k}\alpha^{l-k}(\beta^{2k-l}-\alpha^{2k-l})) \\ &= (\alpha^{min\{l-k,n+l-k-1\}}(\alpha^{(m,2k-l)}-\beta^{(m,2k-l)}). \end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{\min\{l-k,|n+l-k-1|\}}(\alpha^{(m,2k-l)}-\beta^{(m,2k-l)})$.
Now, supposing $k = l-k$, we get

$$x_l^{p^{k-1}\alpha^k - p^{l-k}\alpha^{l-k}\beta} = x_l^{(p\alpha)^k-(p\alpha)^{l-k}} = x_l^{(p\alpha)^k-(p\alpha)^k} = x^0 = 1.$$

Thus, we can remove redundant relators from the presentation. Hence, we get $G_\Gamma(R) = \langle\, x_l \mid x_l^{\alpha^{n+l-k-1}\gamma}\,\rangle$. Therefore, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{n+l-k-1}(\alpha^m-\beta^m)$.

(iii) $a_1, a_3, a_6, a_7, a_{12}, a_{14}, a_{15}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1,\dots,x_l, \\ y_1,\dots,y_m, \\ z_1,\dots,z_n \end{matrix} \;\middle|\; \begin{matrix} x_1^\gamma, R(x_1,x_2), R(x_2,x_3),\dots,R(x_l,x_1), \\ y_1^\eta, R(y_1,y_2), R(y_2,y_3),\dots,R(y_m,y_1), \\ R(z_t,z_{t-1}), R(z_{t-1},z_{t-2}),\dots,R(z_2,z_1), \\ R(z_t,z_{t+1}), R(z_{t+1},z_{t+2}),\dots,R(z_{n-1},z_n), \\ x_l = z_1, y_m = z_n \end{matrix} \right\rangle.$$

We set $\gamma = \alpha^l - \beta^l$ and $\eta = \alpha^m - \beta^m$, and apply precisely the same transformations as in the proof of Theorem 2.6. Then, what remains is

$$\begin{aligned} G_\Gamma(R) &= \left\langle \begin{matrix} x_l, y_m, \\ z_1,\dots,z_n \end{matrix} \;\middle|\; \begin{matrix} x_l^\gamma, y_m^\eta, \\ R(z_t,z_{t-1}), R(z_{t-1},z_{t-2}),\dots,R(z_2,z_1), \\ R(z_t,z_{t+1}), R(z_{t+1},z_{t+2}),\dots,R(z_{n-1},z_n), \\ x_l = z_1, y_m = z_n \end{matrix} \right\rangle \\ &= \left\langle z_1,\dots,z_n \;\middle|\; \begin{matrix} z_1^\gamma, R(z_t,z_{t-1}), R(z_{t-1},z_{t-2}),\dots,R(z_2,z_1), \\ z_n^\eta, R(z_t,z_{t+1}), R(z_{t+1},z_{t+2}),\dots,R(z_{n-1},z_n) \end{matrix} \right\rangle. \end{aligned}$$

Since $(\beta,\gamma) = 1$ and $(\beta,\eta) = 1$ (see Remark 2.9) and an iterated application of Lemma 2.5 (b) yields

$$\begin{aligned} G_\Gamma(R) &= \left\langle z_2,\dots,z_{n-1} \;\middle|\; \begin{matrix} z_2^{\alpha\gamma}, R(z_t,z_{t-1}), R(z_{t-1},z_{t-2}),\dots,R(z_3,z_2), \\ z_{n-1}^{\alpha\eta}, R(z_t,z_{t+1}), R(z_{t+1},z_{t+2}),\dots,R(z_{n-2},z_{n-1}) \end{matrix} \right\rangle \\ &= \left\langle z_t \;\middle|\; z_t^{\alpha^{t-1}\gamma}, z_t^{\alpha^{n-t}\eta} \right\rangle \\ &= \left\langle z_t \;\middle|\; z_t^{\alpha^{\min\{n-t,t-1\}}(\alpha^{(l,m)}-\beta^{(l,m)})} \right\rangle. \end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{\min\{n-t,t-1\}}(\alpha^{(l,m)}-\beta^{(l,m)})$.

(iv) $a_1, a_3, a_6, a_7, a_{11}, a_{14}, a_{15}, (a_{12} = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n \end{matrix} \middle| \begin{matrix} R(x_1,x_2), R(x_2,x_3), \ldots, R(x_l,x_1), \\ R(z_1,z_2), R(z_2,z_3), \ldots, R(z_{t-2},z_{t-1}), \\ x_l = z_1, \\ R(y_1,y_2), R(y_2,y_3), \ldots, R(y_m,y_1), \\ R(z_t,z_{t+1}), R(z_{t+1},z_{t+2}), \ldots, R(z_{n-1},z_n), \\ y_m = z_n, \\ R(z_t,z_{t-1}) \end{matrix} \right\rangle$$

We set $\gamma = \alpha^l - \beta^l$ and $\eta = \alpha^m - \beta^m$, and apply precisely the same transformations as we have in the form $\Gamma(n;\overset{m}{\rightarrow})$ for the first box and $\Gamma(n;\overset{m}{\leftarrow})$ for the second box, by Lemma 2.7, we get

$$G_\Gamma(R) = \langle z_{t-1}, z_t \mid z_{t-1}^{\beta^{t-2}\gamma}, z_t^{\alpha^{n-t}\eta}, R(z_t,z_{t-1}) \rangle.$$

After we get this presentation, we cannot eliminate z_{t-1} or z_t from the presentation. It is because we are not able to apply Lemma 2.5 further since $(\alpha^{n-t}\eta, \alpha) \neq 1$ and $(\beta^{t-2}\gamma, \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(v) $a_4, a_5, a_6, a_{11}, a_{14}, a_{15}, (a_1 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n \end{matrix} \middle| \begin{matrix} R(x_l,x_{l-1}), \\ R(x_k,x_{k+1}), \ldots, R(x_{l-2},x_{l-1}), \\ R(x_l,x_1), R(x_1,x_2), \ldots, R(x_{k-1},x_k), \\ R(y_1,y_2), R(y_2,y_3), \ldots, R(y_m,y_1), \\ R(z_1,z_2), R(z_2,z_3), \ldots, R(z_{n-1},z_n), \\ x_k = z_1, z_n = y_m \end{matrix} \right\rangle.$$

We set $\gamma = \alpha^m - \beta^m$ and apply precisely the same transformations as in the proof of (ii) to obtain that

$$G_\Gamma(R) = \left\langle x_1, \ldots, x_l \middle| \begin{matrix} R(x_l,x_{l-1}), \\ R(x_k,x_{k+1}), \ldots, R(x_{l-2},x_{l-1}), \\ x_k^{\alpha^{n-1}\gamma}, R(x_l,x_1), R(x_1,x_2), \ldots, R(x_{k-1},x_k) \end{matrix} \right\rangle.$$

Since $(\beta, \alpha^{n-1}\gamma) = 1$ (see Remark 2.9) and an iterated application of Lemma 2.5 (b) yields

$$G_\Gamma(R) = \left\langle x_k, x_{k+1}, \ldots, x_l \middle| \begin{matrix} R(x_l,x_{l-1}), \\ R(x_k,x_{k+1}), \ldots, R(x_{l-2},x_{l-1}), \\ x_l^{\alpha^{n+k-1}\gamma} \end{matrix} \right\rangle.$$

Adjoin the relations $x_i = x_{l-1}^{(p\beta)^{l-1-i}}$ for $k+1 \leqslant i \leqslant l-1$, where $p \in \mathbb{Z}$, to the presentation so we get

$$\begin{aligned} G_\Gamma(R) &= \left\langle x_k, \ldots, x_{l-1}, x_l \middle| \begin{matrix} R(x_l,x_{l-1}), \\ R(x_k,x_{k+1}), \ldots, R(x_{l-2},x_{l-1}), \\ x_l^{\alpha^{n+k-1}\gamma}, x_i = x_{l-1}^{(p\beta)^{l-1-i}} \; for \; k+1 \leqslant i \leqslant l-1 \end{matrix} \right\rangle \\ &= \left\langle x_{l-1}, x_l \middle| \begin{matrix} R(x_l,x_{l-1}), \\ R(x_{l-1}^{(p\beta)^{l-k-1}}, x_{l-1}^{(p\beta)^{l-k-2}}), \ldots, R(x_{l-1}^{p\beta}, x_{l-1}), \\ x_l^{\alpha^{n+k-1}\gamma}, \end{matrix} \right\rangle \\ &= \left\langle x_{l-1}, x_l \middle| x_l^{\alpha^{n+k-1}\gamma}, R(x_l,x_{l-1}) \right\rangle. \end{aligned}$$

After we get this presentation, we cannot eliminate x_{l-1} or x_l from the presentation. It is because we are not able to apply Lemma 2.5 further since $(\alpha^{n+k-1}\gamma, \alpha) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(vi) $\boldsymbol{a_1, a_6, a_7, a_{11}, a_{14}, a_{15}, (a_4 = 1)}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n \end{matrix} \middle| \begin{matrix} R(x_l, x_{l-1}), \\ R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-1}, x_k), \\ y_1^\eta, R(y_1, y_2), R(y_2, y_3), \dots, R(y_m, y_1), \\ R(z_1, z_2), R(z_2, z_3), \dots, R(z_{n-1}, z_n), \\ x_l = z_1, y_m = z_n \end{matrix} \right\rangle.$$

We set $\gamma = \alpha^m - \beta^m$ and apply precisely the same transformations as in the proof (v) for the relation inside the box to obtain that

$$G_\Gamma(R) = \left\langle x_1, \dots, x_l \middle| \begin{matrix} R(x_l, x_{l-1}), \\ R(x_l, x_1), R(x_1, x_2), \dots, R(x_{l-2}, x_{l-1}), \\ x_l^{\alpha^{n-1}\gamma} \end{matrix} \right\rangle.$$

Now, adjoin these relations $x_l = x_1^{p\beta}, x_i = x_{l-1}^{(p\beta)^{l-1-i}}$ for $1 \leqslant i \leqslant l-1$, where $p \in \mathbb{Z}$, to the presentation and we get

$$\begin{aligned} G_\Gamma(R) &= \left\langle x_1, \dots, x_l \middle| \begin{matrix} R(x_l, x_{l-1}), \\ R(x_l, x_1), R(x_1, x_2), \dots, R(x_{l-2}, x_{l-1}), \\ x_l^{\alpha^{n-1}\gamma}, \\ x_l = x_1^{p\beta}, x_i = x_{l-1}^{(p\beta)^{l-1-i}} \; for \; 1 \leqslant i \leqslant l-1 \end{matrix} \right\rangle \\ &= \left\langle x_1, x_{l-1}, x_l \middle| \begin{matrix} R(x_l, x_{l-1}), \\ R(x_l^{(p\beta)^{l-1}}, x_l^{(p\beta)^{l-2}}), R(x_{l-1}^{(p\beta)^{l-2}}, x_{l-1}^{(p\beta)^{l-3}}), \dots, \\ R(x_{l-1}^{(p\beta)^2}, x_{l-1}^{p\beta}), R(x_{l-1}^{p\beta}, x_{l-1}), \\ x_l^{\alpha^{n-1}\gamma} \end{matrix} \right\rangle. \end{aligned}$$

Since $R(x_l^{(p\beta)^i}, x_l^{(p\beta)^{i-1}}) = x_l^{\alpha(p\beta)^i - \beta(p\beta)^{i-1}} = x_l^{(p\beta)^{i-1}(\alpha p\beta - \beta)}$ for $1 \leqslant i \leqslant l-1$, and $\alpha p\beta - \beta = 0 \pmod{\gamma}$ since $p\alpha \equiv 1 \pmod{\gamma}$. Thus, these relations are redundant so can be removed

$$G_\Gamma(R) = \langle x_{l-1}, x_l \,|\, x_l^{\alpha^{n-1}\gamma}, R(x_l, x_{l-1}) \rangle.$$

After we get this presentation, we cannot eliminate x_{l-1} or x_l from the presentation. It is because we are not able to apply Lemma 2.5 further since $(\alpha^{n-1}\gamma, \alpha) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(vii) $\boldsymbol{a_2, a_3, a_6, a_8, a_{10}}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \dots, x_l, \\ y_1, \dots, y_m \end{matrix} \middle| \begin{matrix} x_1^\gamma, R(x_1, x_2), R(x_2, x_3), \dots, R(x_l, x_1) \\ y_1^\eta, R(y_1, y_2), R(y_2, y_3), \dots, R(y_m, y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \dots, x_k = y_m \end{matrix} \right\rangle.$$

We set $\gamma = \alpha^l - \beta^l$, $\eta = \alpha^m - \beta^m$. Since $p_1\alpha \equiv 1 \pmod{\gamma}$, there is an integer $q_1 \in \mathbb{Z}$ such that $p_1\alpha + q_1\gamma = 1$. Moreover, $p_1\alpha \equiv 1 \pmod{\gamma}$ implies that $y_i = y_i^{p_1\alpha} = y_j^{p_1\beta}$ in G. This allows us to adjoin the relation $y_i = y_j^{p_1\beta}$ and to eliminate the generator y_i, and since $p_2\alpha \equiv 1 \pmod{\eta}$, there is an integer $q_2 \in \mathbb{Z}$ such that $p_2\alpha + q_2\eta = 1$. Moreover, $p_2\alpha \equiv 1 \pmod{\eta}$ implies that $x_i = x_i^{p_2\alpha} = x_j^{p_2\beta}$ in G. This allows us to adjoin the relation $x_i = x_j^{p_2\beta}$ and to eliminate the generator x_i as follows:

$$\begin{aligned} y_1 &= y_2^{(p_1\beta)} = y_3^{(p_1\beta)^2} = \dots = y_t^{(p_1\beta)^{t-1}} = \dots = y_m^{(p_1\beta)^{m-1}} \\ x_1 &= x_2^{(p_2\beta)} = x_3^{(p_2\beta)^2} = \dots = x_t^{(p_2\beta)^{t-1}} = \dots = x_l^{(p_2\beta)^{l-1}} \end{aligned}$$

$$G_\Gamma(R)=\left\langle \begin{matrix} x_2,\dots,x_l, \\ y_2,\dots,y_m \end{matrix} \middle| \begin{matrix} x_2^{\beta\gamma}, R(x_2,x_3),\dots,R(x_l,x_2^{p_2\beta}) \\ y_2^{\beta\eta}, R(y_2,y_3),\dots,R(y_m,y_2^{p_1\beta}), \\ x_t=y_{m-k+t}, x_{t+1}=y_{m-k+t+1},\dots,x_k=y_m \end{matrix} \right\rangle$$

$$=\left\langle \begin{matrix} x_3,\dots,x_l, \\ y_3,\dots,y_m \end{matrix} \middle| \begin{matrix} x_3^{\beta^2\gamma}, R(x_3,x_4),\dots,R(x_l,x_3^{(p_2\beta)^2}), \\ y_3^{\beta^2\eta}, R(y_3,y_4),\dots,R(y_m,y_3^{(p_1\beta)^2}), \\ x_t=y_{m-k+t}, x_{t+1}=y_{m-k+t+1},\dots,x_k=y_m \end{matrix} \right\rangle.$$

Simplifying in that way, what remains is

$$G_\Gamma(R)=\left\langle \begin{matrix} x_k, \\ y_m \end{matrix} \middle| \begin{matrix} x_k^\gamma, y_m^\eta \\ x_k^{(p_2\beta)^{k-t}}=x_k^{(p_1\beta)^{k-t}}, x_k^{(p_2\beta)^{k-t-1}},=x_k^{(p_1\beta)^{k-t-1}}\dots,x_k=y_m \end{matrix} \right\rangle$$

$$=\left\langle x_k \middle| \begin{matrix} x_k^\gamma, x_k^\eta \\ x_k^{(p_2\beta)^{k-t}-(p_1\beta)^{k-t}}, x_k^{(p_2\beta)^{k-t-1}-(p_1\beta)^{k-t-1}},\dots,x_k^{p_2\beta-p_1\beta} \end{matrix} \right\rangle$$

$$=\left\langle x_k \middle| \begin{matrix} x_k^\gamma, x_k^\eta \\ x_k^{\beta^{k-t}(p_2^{k-t}-p_1^{k-t})}, x_k^{\beta^{k-t-1}(p_2^{k-t-1}-p_1^{k-t-1})},\dots,x_k^{\beta(p_2-p_1)} \end{matrix} \right\rangle$$

$$=\left\langle x_k \middle| \begin{matrix} x_k^\gamma, x_k^\eta \\ x_k^{(\beta^{k-t},\beta^{k-t-1},\dots,\beta)(p_2^{k-t}-p_1^{k-t},p_2^{k-t-1}-p_1^{k-t-1},\dots,p_2-p_1)} \end{matrix} \right\rangle$$

$$=\left\langle x_k \middle| x_k^\gamma, x_k^\eta, x_k^{\beta(p_2-p_1)} \right\rangle$$

$$=\left\langle x_k \middle| x_k^{(\gamma,\eta,\beta(p_2-p_1))} \right\rangle$$

$$=\left\langle x_k \middle| x_k^{(\gamma,\eta,p_2-p_1)} \right\rangle.$$

Now, $p_1\alpha\equiv 1 \pmod{\gamma}$, and we can say $p_1\alpha\equiv 1 \pmod{(\gamma,\eta)}$,
$p_2\alpha\equiv 1 \pmod{\eta}$, and we can say $p_2\alpha\equiv 1 \pmod{(\gamma,\eta)}$. So, $p_1\alpha-p_2\alpha\equiv 0 \pmod{(\gamma,\eta)}$.
Since $(\alpha,\gamma)=1$ and $(\alpha,\eta)=1$, $\Delta=(\gamma,\eta,(p_1-p_2))=(\gamma,\eta,(p_1-p_2)\alpha)=(\gamma,\eta)$. Then the presentation is

$$G_\Gamma(R)=\langle\, x_k \mid x_k^{(\gamma,\eta)} \,\rangle$$

$$=\langle\, x_k \mid x_k^{\alpha^{(l,m)}-\beta^{(l,m)}} \,\rangle.$$

So $G_\Gamma(R)$ is finite cyclic of order $\alpha^{(l,m)}-\beta^{(l,m)}$.

(viii) $a_1, a_3, a_6, a_8, a_{10}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R)=\left\langle \begin{matrix} x_1,\dots,x_l, \\ y_1,\dots,y_m \end{matrix} \middle| \begin{matrix} R(x_l,x_{l-1}), R(x_{l-1},x_{l-2}),\dots,R(x_{k+1},x_k), \\ R(x_l,x_1), R(x_1,x_2),\dots,R(x_{k-1},x_k), \\ y_1^\eta, R(y_1,y_2), R(y_2,y_3),\dots,R(y_m,y_1), \\ x_t=y_{m-k+t}, x_{t+1}=y_{m-k+t+1},\dots,x_k=y_m \end{matrix} \right\rangle.$$

We set $\eta=\alpha^m-\beta^m$ and since $p_1\alpha\equiv 1 \pmod{\eta}$, there is an integer $q_1\in\mathbb{Z}$ such that $p_1\alpha+q_1\eta=1$. Moreover, $p_1\alpha\equiv 1 \pmod{\eta}$ implies that $y_i=y_i^{p_1\alpha}=y_j^{p_1\beta}$ in G and after applying precisely the same transformations as in the proof of Lemma 2.5 (a), we get

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \dots, x_l, \\ y_m \end{matrix} \middle| \begin{matrix} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_k), \\ R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-1}, x_k), \\ y_m^\eta, \\ x_t = y_m^{(p_1\beta)^{k-t}}, x_{t+1} = y_m^{(p_1\beta)^{k-t-1}}, \dots, x_{k-1} = y_m^{p_1\beta}, \\ x_k = y_m \end{matrix} \right\rangle$$

$$= \left\langle x_1, \dots, x_l \middle| \begin{matrix} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_k), \\ x_k^\eta, R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-1}, x_k), \\ x_t = x_k^{(p_1\beta)^{k-t}}, x_{t+1} = x_k^{(p_1\beta)^{k-t-1}}, \dots, x_{k-1} = x_k^{p_1\beta} \end{matrix} \right\rangle.$$

Since $(\beta, \eta) = 1$, there are integers $p_2, q_2 \in$ such that $p_2\beta + q_2\eta = 1$ so $p_2\beta \equiv 1 \pmod{\eta}$. We can thus apply Lemma 2.5 (b),

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, x_2, \dots, x_{k-1}, \\ x_{k+1}, \dots, x_l \end{matrix} \middle| \begin{matrix} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_{k-1}^{p_2\alpha}) \\ x_{k-1}^{\alpha\eta}, R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-2}, x_{k-1}) \\ x_t = x_k^{(p_1\beta)^{k-t}}, x_{t+1} = x_k^{(p_1\beta)^{k-t-1}}, \dots, \\ x_{k-1} = x_{k-1}^{p_2\alpha\, p_1\beta} \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} x_1, x_2, \dots, x_{k-2}, \\ x_{k+1}, \dots, x_l \end{matrix} \middle| \begin{matrix} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_{k-2}^{(p_2\alpha)^2}), \\ x_{k-2}^{\alpha^2\eta}, R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-3}, x_{k-2}), \\ x_t = x_k^{(p_1\beta)^{k-t}}, x_{t+1} = x_k^{(p_1\beta)^{k-t-1}}, \dots, \\ x_{k-2} = x_{k-2}^{(p_2\alpha)^2(p_1\beta)^2} \end{matrix} \right\rangle.$$

Simplifying in that way, what remains is

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, x_2, \dots, x_t, \\ x_{k+1}, \dots, x_l \end{matrix} \middle| \begin{matrix} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_t^{(p_2\alpha)^{k-t}}), \\ x_t^{\alpha^{k-t}\eta}, R(x_l, x_1), R(x_1, x_2), \dots, R(x_{t-1}, x_t), \\ x_t = x_t^{(p_2\alpha)^{k-t}\,(p_1\beta)^{k-t}} \end{matrix} \right\rangle.$$

Since $(\beta, \alpha^{k-t}\eta) = 1$, we can thus apply Lemma 2.5 (b),

$$G_\Gamma(R) = \left\langle \begin{matrix} x_1, \\ x_{k+1}, x_{k+2}, \dots, x_l \end{matrix} \middle| \begin{matrix} x_1^{\alpha^{k-1}\eta}, R(x_l, x_1), \\ R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, \\ R(x_{k+1}, x_1^{(p_2\alpha)^{k-1}}) \end{matrix} \right\rangle$$

$$= \left\langle x_{k+1}, x_{k+2}, \dots, x_l \middle| \begin{matrix} x_l^{\alpha^k\eta}, R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, \\ R(x_{k+1}, x_l^{(p_2\alpha)^k}), \\ x_i = x_l^{(p_2\alpha)^{l-i}} \text{ for } k+1 \leqslant i \leqslant l-1 \end{matrix} \right\rangle$$

$$= \left\langle x_l \middle| \begin{matrix} x_l^{\alpha^k\eta}, R(x_l, x_l^{p_2\alpha}), R(x_l^{p_2\alpha}, x_l^{(p_2\alpha)^2}), \dots, R(x_l^{(p_2\alpha)^{l-k-2}}, \\ x_l^{(p_2\alpha)^{l-k-1}}), R(x_l^{(p_2\alpha)^{l-k-1}}, x_l^{(p_2\alpha)^k}) \end{matrix} \right\rangle$$

$$= \left\langle x_l \middle| \begin{matrix} x_l^{\alpha^k\eta}, x_l^{\alpha-\beta p_2\alpha}, x_l^{p_2\alpha^2-\beta(p_2\alpha)^2}, \dots, \\ x_l^{\alpha(p_2\alpha)^{l-k-2}-\beta(p_2\alpha)^{l-k-1}}, x_l^{\alpha(p_2\alpha)^{l-k-1}-\beta(p_2\alpha)^k} \end{matrix} \right\rangle$$

$$= \left\langle x_l \middle| \begin{matrix} x_l^{\alpha^k\eta}, x_l^{\alpha-\beta p_2\alpha}, x_l^{p_2\alpha(\alpha-\beta p_2\alpha)}, \dots, \\ x_l^{(p_2\alpha)^{l-k-2}(\alpha-\beta p_2\alpha)}, x_l^{\alpha(p_2\alpha)^{l-k-1}-\beta(p_2\alpha)^k} \end{matrix} \right\rangle.$$

Since $x_l^{\alpha-\beta p_2\alpha} = x_l^{\alpha-\alpha} = 1 \bmod \eta$, we get

$$G_\Gamma(R) = \langle\, x_l \mid x_l^{\alpha^k\eta}, x_l^{\alpha(p_2\alpha)^{l-k-1}-\beta(p_2\alpha)^k} \,\rangle.$$

Supposing $l - k < k$, then

$$\begin{aligned}\alpha(p_2\alpha)^{l-k-1}-\beta(p_2\alpha)^k &= p_2^{l-k-1}\alpha^{l-k}-\beta p_2^k\alpha^k\\ &= p_2^{l-k-1}\alpha^{l-k}-p_2^{k-1}\alpha^k\\ &= p_2^{l-k-1}\alpha^{l-k}(1-p_2^{2k-l}\alpha^{2k-l})\\ &= p_2^{l-k-1}\alpha^{l-k}(p_2^{2k-l}\beta^{2k-l}-p_2^{2k-l}\alpha^{2k-l}) \text{ since } p_2\beta\equiv 1 \pmod{\eta}\\ &= p_2^{l-k-1}\alpha^{l-k}p_2^{2k-l}(\beta^{2k-l}-\alpha^{2k-l})\\ &= p_2^{k-1}\alpha^{l-k}(\beta^{2k-l}-\alpha^{2k-l}).\end{aligned}$$

Thus, we get

$$\begin{aligned}G_\Gamma(R) &= \langle\, x_l \mid x_l^{\alpha^k\eta}, x_l^{p_2^{k-1}\alpha^{l-k}(\beta^{2k-l}-\alpha^{2k-l})}\,\rangle\\ &= \langle\, x_l \mid x_l^{\alpha^{min\{k,l-k\}}(\beta^{(m,2k-l)}-\alpha^{(m,2k-l)})}\,\rangle.\end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{min\{k,l-k\}}(\alpha^{(m,2k-l)}-\beta^{(m,2k-l)})$.
Supposing $l-k>k$, then

$$\begin{aligned}\alpha(p_2\alpha)^{l-k-1}-\beta(p_2\alpha)^k &= p_2^{l-k-1}\alpha^{l-k}-\beta p_2^k\alpha^k\\ &= p_2^{l-k-1}\alpha^{l-k}-p_2^{k-1}\alpha^k\\ &= p_2^{k-1}\alpha^k(p_2^{l-2k}\alpha^{l-2k}-1)\\ &= p_2^{k-1}\alpha^k(p_2^{l-2k}\alpha^{l-2k}-p_2^{l-2k}\beta^{l-2k} \text{ since } p_2\beta\equiv 1 \pmod{\eta}\\ &= p_2^{l-k-1}\alpha^k(\alpha^{l-2k}-\beta^{l-2k}).\end{aligned}$$

Therefore, we get

$$\begin{aligned}G_\Gamma(R) &= \langle\, x_l \mid x_l^{\alpha^k\eta}, x_l^{p_2^{l-k-1}\alpha^k(\alpha^{l-2k}-\beta^{l-2k})}\,\rangle\\ &= \langle\, x_l \mid x_l^{\alpha^k(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})}\,\rangle.\end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^k(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})$.

Supposing $k=l-k$, then $x_l^{p_2^{k-1}\alpha^{l-k}(\beta^{2k-l}-\alpha^{2k-l})}=x_l^{p_2^{k-1}\alpha^k(p_2^{l-2k}\alpha^{l-2k}-1)}=x_l^0=1$.

Therefore, it can be removed from the presentation. Thus, we get $G_\Gamma(R)=\langle\, x_l \mid x_l^{\alpha^k\eta}\,\rangle$. Hence, $G_\Gamma(R)$ is finite cyclic of order $|\alpha^k(\alpha^m-\beta^m)|$.

(ix) $\boldsymbol{a_2, a_3, a_6, a_8, a_{10}, (a_1=1)}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R)=\left\langle \begin{matrix} x_1,\dots,x_l,\\ y_1,\dots,y_m\end{matrix} \;\middle|\; \begin{matrix} R(x_l,x_{l-1}),\\ R(x_k,x_{k+1}),\dots,R(x_{l-2},x_{l-1}),\\ R(x_l,x_1),R(x_1,x_2),\dots,R(x_{k-1},x_k),\\ y_1^\eta,R(y_1,y_2),R(y_2,y_3),\dots,R(y_m,y_1),\\ x_t=y_{m-k+t},x_{t+1}=y_{m-k+t+1},\dots,x_k=y_m\end{matrix}\right\rangle.$$

We set $\eta=\alpha^m-\beta^m$, $p_1\alpha\equiv 1 \pmod{\eta}$ and $p_2\beta\equiv 1 \pmod{\eta}$ as in $(viii)$ then we apply precisely the same transformations as in the proof of $(viii)$ for the relations inside the box to obtain that

$$\begin{aligned}G_\Gamma(R) &= \left\langle x_{k+1},x_{k+2},\dots,x_l \;\middle|\; \begin{matrix} R(x_l,x_{l-1}),\\ x_l^{\alpha^k\eta},R(x_l^{(p_2\alpha)^k},x_{k+1}),R(x_{k+1},x_{k+2}),\dots,\\ R(x_{l-2},x_{l-1})\end{matrix}\right\rangle\\ &= \left\langle x_{k+1},x_{k+2},\dots,x_l \;\middle|\; \begin{matrix} R(x_l,x_{l-1}),\\ x_l^{\alpha^k\eta},R(x_l^{(p_2\alpha)^k},x_{k+1}),R(x_{k+1},x_{k+2}),\dots,\\ R(x_{l-2},x_{l-1}),\\ x_i=x_{l-1}^{(p_2\beta)^{l-1-i}} \text{ for } k+1\leqslant i\leqslant l-1\end{matrix}\right\rangle\\ &= \left\langle x_{l-1},x_l \;\middle|\; \begin{matrix} R(x_l,x_{l-1}),\\ x_l^{\alpha^k\eta},R(x_l^{(p_2\alpha)^k},x_{k+1}),R(x_{k+1},x_{k+2}),\dots,R(x_{l-2},x_{l-1}),\\ x_i=x_{l-1}^{(p_2\beta)^{l-1-i}} \text{ for } k+1\leqslant i\leqslant l-1\end{matrix}\right\rangle\end{aligned}$$

$$= \left\langle x_{l-1}, x_l \;\middle|\; \begin{array}{l} R(x_l, x_{l-1}), \\ x_l^{\alpha^k \eta}, R(x_l^{(p_2\alpha)^k}, x_{l-1}^{(p_2\beta)^{l-k-2}}), R(x_{l-1}^{(p_2\beta)^{l-k-2}}, x_{l-1}^{(p_2\beta)^{l-k-3}}), \\ \dots, R(x_{l-1}^{p_2\beta}, x_{l-1}) \end{array} \right\rangle$$

$$= \left\langle x_{l-1}, x_l \;\middle|\; ^{x_{l-1}, x_l}| R(x_l, x_{l-1}), x_l^{\alpha^k \eta}, R(x_l^{(p_2\alpha)^k}, x_{l-1}^{(p_2\beta)^{l-k-2}}) \right\rangle$$

$$= \left\langle x_{l-1}, x_l \;\middle|\; x_l^{\alpha^k \eta}, x_l^{\alpha} = x_{l-1}^{\beta}, x_l^{\alpha(p_2\alpha)^k} = x_{l-1}^{\beta(p_2\beta)^{l-k-2}} \right\rangle$$

$$= \left\langle x_{l-1}, x_l \;\middle|\; x_l^{\alpha^k \eta}, x_l^{\alpha} = x_{l-1}^{\beta}, x_l^{\alpha(p_2\alpha)^k} = x_l^{\alpha(p_2\beta)^{l-k-2}} \right\rangle$$

$$= \left\langle x_{l-1}, x_l \;\middle|\; x_l^{(\alpha^k \eta, \alpha(p_2\alpha)^k - \alpha(p_2\beta)^{l-k-2})}, x_l^{\alpha} = x_{l-1}^{\beta} \right\rangle.$$

After we get this presentation, we cannot eliminate x_{l-1} or x_l from the presentation by our limited knowledge now (it is because we cannot apply Lemma 2.5 further). Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(x) $a_2, a_4, a_5, a_8, a_{10}, (a_1 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \dots, x_l, \\ y_1, \dots, y_m \end{array} \;\middle|\; \begin{array}{l} R(x_k, x_{k+1}), R(x_{k+1}, x_{k+2}), \dots, R(x_{l-1}, x_l), \\ R(x_k, x_{k-1}), \\ R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \dots, R(x_{k-2}, x_{k-1}), \\ R(y_m, y_1), R(y_1, y_2), \dots, R(y_{t-1}, y_t), \\ R(y_m, y_{m-1}), R(y_{m-1}, y_{m-2}), \dots, R(y_{t+1}, y_t), \\ x_l = y_m, x_1 = y_1, x_2 = y_2, \dots, x_t = y_t \end{array} \right\rangle.$$

There are no directed cycles in that graph. Therefore, we cannot apply Theorem 2.6 and thus it is still an open problem.

(xi) $a_2, a_3, a_6, a_8, a_{10}, (a_9 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle x_1, \dots, x_{l,1}, \dots, y_m \;\middle|\; \begin{array}{l} R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \dots, R(x_{l-1}, x_l), \\ R(x_l, x_1), R(x_1, x_2), \dots, R(x_{t-2}, x_{t-1}), \\ R(x_t, x_{t-1}), \\ R(y_t, y_{t+1}), R(y_{t+1}, y_{t+2}), \dots, \\ R(y_{m-1}, y_m), \\ x_l = y_m, x_1 = y_1, x_2 = y_2, \dots, x_t = y_t \end{array} \right\rangle.$$

There are no directed cycles in that graph. Therefore, we cannot apply Theorem 2.6 and thus it is still an open problem.

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Conflicts of interest

There are no conflicts of interest in this work.

Acknowledgement

The author thanks Gerald Williams for their careful readings and insightful comments on a draft of this article. This paper is derived from the author's doctoral dissertation.

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