Cumhuriyet Sci. J., 46(1) (2025) 109-114
DOI: https://doi.org/10.17776/csj.1606770

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Some Results on the Oscillation of a Class of Generalized Fractional Integro-Differential Equations

Funda Raziye Mert[1,a] Selami Bayeğ [2,b,*]

[1] *Department of Software Engineering, Faculty of Computer and Informatics, Adana Alparslan Türkeş Science and Technology University, 01250, Adana, Türkiye.*

[2] *Department of Industrial Engineering, Faculty of Engineering, University of Turkish Aeronautical Association, 06790, Ankara, Türkiye.*

**Corresponding author*

Research Article

History
Received: 24/12/2024
Accepted: 01/03/2025

ABSTRACT

In this study, we analyze the oscillatory behavior of solutions to a specific class of fractional integro-differential equations. First, we derive sufficient conditions that ensure nonoscillatory solutions exhibit a well-defined asymptotic behavior. Building on this result, we establish a series of oscillation theorems that provide deeper insight into the qualitative nature of solutions. To validate our theoretical findings, we present a concrete example that demonstrates the applicability of our main results. These contributions aim to advance the theoretical framework of fractional equations, offering new perspectives on their dynamic behavior and potential applications in mathematical modeling.

Keywords: Fractional integro-differential equation, Riemann-Liouville Derivative, Riemann-Liouville Integral, Oscillation

[a] *rmert@atu.edu.tr* *https://orcid.org/0000-0001-6613-2733*
[b] *sbayeg@thk.edu.tr* *https://orcid.org/0000-0001-7014-1739*

Introduction

Fractional calculus is an extension of classical differentiation and integration to arbitrary order. Its history is as old as the classical calculus. Fractional derivatives and integrals have been shown to be better than classical tools for describing many real-life phenomena in science and engineering. Hence, many and varied sectors of engineering and science, including fluid mechanics, electromagnetics, electrochemistry, biological population models, viscoelasticity, signals processing, and signals optics use fractional calculus.

Fractional differential equations have recently attracted considerable attention. The study of fractional differential equations is interdisciplinary and is encountered in diverse fields such as plasma physics, biomathematics, fluid dynamics, mathematical biology, control systems, elasticity, biotechnology, quantum mechanics, optics, and complex systems. Since the derivatives there are of fractional order, they can approximate real data more flexibly, see [1].

The variety of fractional operators is what distinguishes fractional calculus from other mathematical disciplines. This allows the scientists working on modeling real life phenomena to choose the best operator for their model. To model real life phenomena more accurately, researchers have needed several other fractional operators in addition to the Riemann-Liouville and Caputo ones. An interested reader can look at [2-7] for some newly introduced fractional operators. These operators are defined as particular cases of fractional operators depending on a kernel function.

Oscillation is a substantial aspect of the qualitative behavior of solutions of differential and difference equations. Its theory is important to study the oscillatory phenomena in technology and natural and social sciences. A major problem in oscillation theory is proving the existence or non-existence of an oscillatory solution to a given equation or system. Additionally, the behavior of other solutions relative to a particular oscillatory (non-oscillatory) solution is also studied. Many articles on theoretical aspects of oscillation theory are published every year. Surprisingly, this significant area of research has substantial applications and is not entirely theoretical. Oscillation theory has important applications in physics, biology, ecology, physiology, etc. Studying oscillations provides a better understanding of the dynamics of solutions of equations that model applied problems encountered in engineering, technology, and science. Despite its importance, when we look at the literature, we see that there are few studies related to the oscillation of fractional differential and difference equations. As far as we know, Grace et al. [8] first studied the oscillation for a fractional differential equation. After that, corresponding results for ordinary differential and difference equations have been extended to fractional differential and difference equations, see [9-19]. There is also very little research on the oscillation theory of fractional integro-differential equations, see [20-23]. This work investigates the oscillatory behavior of solutions to a fractional Volterra integro-differential equation using the method introduced in [24]. We believe that our study will inspire further research on fractional integro-differential equations. We consider the fractional integro-differential equation

$$\begin{cases} D_a^{\mu,\mathfrak{S}} x(t) = \mathfrak{z}(t) - \int_a^t \mathcal{M}(t,v)\mathcal{Y}(v,x(v))\,dv, t \geq a \geq 0, 0 < \mu < 1, \\ \lim_{t\to a^+} I_a^{1-\mu,\mathfrak{S}} x(t) = b_1 , \end{cases} \tag{1}$$

where, $\mathfrak{z}$, $\mathcal{M}$, and $\mathcal{Y}$ are continuous, $b_1 \in \mathbb{R}$, and $D_a^{\mu,\mathfrak{S}}$ and $I_a^{1-\mu,\mathfrak{S}}$ are the left fractional general derivative and integral operators in the Riemann-Liouville setting, respectively.

We concentrate only on those solutions of Eq. (1) not identically zero eventually. Henceforth, we mean such solutions. Such solutions are called oscillatory if they are not eventually of one sign and nonoscillatory otherwise. The equation itself is oscillatory in case all its solutions oscillate.

PRELIMINARIES

We now present some basic definitions and important lemmas. Let $\mu \in \mathbb{C}$ with $Re(\mu) > 0$, $I = (a,b)$ be an interval, and $\mathfrak{S} \in C^1(I)$ be an increasing function with $\mathfrak{S}'(t) \neq 0\ \forall t \in I$. The left fractional integral and derivative in the Riemann-Liouville sense of a function θ with respect to another function $\mathfrak{S}$ are defined in [25, 26] as

$$I_a^{\mu,\mathfrak{S}}\theta(t) = \frac{1}{\Gamma(\mu)}\int_a^t (\mathfrak{S}(t) - \mathfrak{S}(v))^{\mu-1}\mathfrak{S}'(v)\theta(v)dv \tag{2}$$

and

$$\begin{aligned} D_a^{\mu,\mathfrak{S}}\theta(t) &= {}_{\mathfrak{S}}D_t^n I_a^{n-\mu,\mathfrak{S}}\theta(t) \\ &= \frac{1}{\Gamma(n-\mu)}\, {}_{\mathfrak{S}}D_t^n \int_a^t (\mathfrak{S}(t) - \mathfrak{S}(v))^{n-\mu-1}\mathfrak{S}'(v)\theta(v)dv, \end{aligned} \tag{3}$$

respectively, where $n = [\mu]$ and

$${}_{\mathfrak{S}}D_t^n = \left(\frac{1}{\mathfrak{S}'(t)}\frac{d}{dt}\right)^n.$$

In the special cases $\mathfrak{S}(t) = t$ and $\mathfrak{S}(t) = \ln t$ in Eq. (2) and Eq. (3), we have the Riemann-Liouville and Hadamard fractional operators, respectively.

We assume throughout that $\mathfrak{S}(t) > 0\ \forall t \in I$.

Lemma 1 [26] Let $\mu \in \mathbb{C}$ with $Re(\mu) > 0$, $n = -[-Re(\mu)]$, $\theta \in L(a,b)$, and $(I_a^{\mu,\mathfrak{S}}\theta)(t) \in AC_{\mathfrak{S}}^n[a,b]$. Then,

$$I_a^{\mu,\mathfrak{S}} D_a^{\mu,\mathfrak{S}}\theta(t) = \theta(t) - \sum_{j=1}^n \frac{(I_a^{j-\mu,\mathfrak{S}}\theta)(a^+)}{\Gamma(\mu-j+1)}(\mathfrak{S}(t) - \mathfrak{S}(a))^{\mu-j}. \tag{4}$$

Lemma 2 [27] Let $\mu, \nu \in \mathbb{C}$ with $Re(\mu) > 0$ and $Re(\nu) > 0$. Then

$$I_a^{\mu,\mathfrak{S}}(\mathfrak{S}(t) - \mathfrak{S}(a))^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)}(\mathfrak{S}(t) - \mathfrak{S}(a))^{\nu+\mu-1}. \tag{5}$$

Lemma 3 [28] For non-negative real numbers $\mathcal{K}$ and $\mathcal{R}$,

$$\mathcal{K}^\alpha - (1-\alpha)\mathcal{R}^\alpha - \alpha\mathcal{K}\mathcal{R}^{\alpha-1} \leq 0,\ \ 0 < \alpha < 1. \tag{6}$$

The equality holds if and only if $\mathcal{K} = \mathcal{R}$

Main Results

We assume that the following hypotheses are met:

(H1) $\mathfrak{z}: [a,\infty) \to \mathbb{R}$ is continuous;

(H2) $\mathcal{M}: [a,\infty) \times [a,\infty) \to \mathbb{R}$ is continuous and there exist continuous functions $\zeta_1, \zeta_2: [a,\infty) \to (0,\infty)$ such that

$$0 \leq \mathcal{M}(t,v) \leq \zeta_1(t)\zeta_2(v)\ \text{ for }\ t \geq v \geq a;$$

(H3) $\mathcal{Y}: [a,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous function $\eta: [a,\infty) \to (0,\infty)$ and a real number α, $0 < \alpha \leq 1$ such that

$$0 < x\mathcal{Y}(t,x) \leq \eta(t)|x|^{\alpha+1}\ \text{ for }\ t \geq a,\ x \neq 0;$$

For a given continuous function $\xi: [a,\infty) \to (0,\infty)$, we define

$$q_\pm(t) := \mathfrak{z}(t) \pm (1-\alpha)\alpha^{\alpha/(1-\alpha)}\zeta_1(t)\int_a^t \xi^{\alpha/(\alpha-1)}(v)\zeta_2^{1/(1-\alpha)}(v)\eta^{1/(1-\alpha)}(v)\,dv,\quad 0 < \alpha < 1.$$

To begin, we provide sufficient conditions for each nonoscillatory solution of Eq. (1) to fulfill

$$x(t) = O(\mathfrak{S}^\mu(t))\ \text{ as }\ t \to \infty.$$

Theorem 1 Let $0 < \alpha < 1$ and the hypotheses (H1)–(H3) hold. Assume that

$$\zeta_1(t) \le M_1 \text{ for } t \ge a \tag{7}$$

for some real number $M_1 > 0$ and

$$\int_a^\infty \mathfrak{S}^\mu(v)\xi(v)\,dv < \infty. \tag{8}$$

If

$$\limsup_{t\to\infty} \frac{1}{\mathfrak{S}^\mu(t)} I_a^{\mu,\mathfrak{S}}[q_+(t)] < \infty, \text{and } \liminf_{t\to\infty} \frac{1}{\mathfrak{S}^\mu(t)} I_a^{\mu,\mathfrak{S}}[q_-(t)] > -\infty, \tag{9}$$

then every nonoscillatory solution $x(t)$ of Eq. (1) satisfies

$$\limsup_{t\to\infty} \frac{|x(t)|}{\mathfrak{S}^\mu(t)} < \infty. \tag{10}$$

Proof 1 Assume that $x(t)$ is a nonoscillatory solution of Eq. (1), say $x(t) > 0$ for $t \ge T_1$, for some $T_1 \ge a$. From Eq. (1), we have

$$D_a^{\mu,\mathfrak{S}} x(t) = \mathfrak{z}(t) - \int_a^t \mathcal{M}(t,v)\mathcal{Y}(v,x(v))\,dv$$
$$= \mathfrak{z}(t) - \int_a^{T_1} \mathcal{M}(t,v)\mathcal{Y}(v,x(v))\,dv - \int_{T_1}^t \mathcal{M}(t,v)\mathcal{Y}(v,x(v))\,dv.$$

Let $L := \min\{\mathcal{Y}(t,x(t)) : t \in [a,T_1]\} \le 0$ and $\kappa := -L\int_a^{T_1} \zeta_2(v)\,dv \ge 0$.

Since $\mathcal{M}(t,v)\mathcal{Y}(v,x(v)) \ge L\mathcal{M}(t,v) \ge L\zeta_1(t)\zeta_2(v), \quad a \le v \le T_1,$

we have

$$\int_a^{T_1} \mathcal{M}(t,v)\mathcal{Y}(v,x(v))\,dv \ge L\zeta_1(t)\int_a^{T_1} \zeta_2(v)\,dv$$

and

$$-\int_a^{T_1} \mathcal{M}(t,v)\mathcal{Y}(v,x(v))\,dv \le -L\zeta_1(t)\int_a^{T_1} \zeta_2(v)\,dv = \kappa\zeta_1(t).$$

In view of $\mathcal{M}(t,v)\mathcal{Y}(v,x(v)) > 0$, we have

$$-\int_{T_1}^t \mathcal{M}(t,v)\mathcal{Y}(v,x(v))\,dv < 0$$

and

$$-\int_{T_1}^t \mathcal{M}(t,v)\mathcal{Y}(v,x(v))\,dv < 0 \le \zeta_1(t)\int_{T_1}^t \zeta_2(v)\eta(v)x^\alpha(v)\,dv.$$

Thus, we get

$$D_a^{\mu,\mathfrak{S}} x(t) \le \mathfrak{z}(t) + \kappa\zeta_1(t) + \zeta_1(t)\int_{T_1}^t \zeta_2(v)\eta(v)x^\alpha(v)\,dv$$
$$= \mathfrak{z}(t) + \kappa\zeta_1(t) + \zeta_1(t)\int_{T_1}^t \left(\zeta_2(v)\eta(v)x^\alpha(v) - \xi(v)x(v)\right)dv + \zeta_1(t)\int_{T_1}^t \xi(v)x(v)\,dv. \tag{11}$$

In (6), by setting $\mathcal{K} := (\zeta_2\eta)^{1/\alpha}x$ and $\mathcal{R} := \left(\frac{1}{\alpha}\xi(\zeta_2\eta)^{-1/\alpha}\right)^{\frac{1}{\alpha-1}}$, we get

$$\zeta_2(v)\eta(v)x^\alpha(v) - \xi(v)x(v) \le (1-\alpha)\alpha^{\alpha/(1-\alpha)}\xi^{\alpha/(\alpha-1)}(v)\zeta_2^{1/(1-\alpha)}(v)\eta^{1/(1-\alpha)}(v).$$

Hence, inequality (11) gives

$$D_a^{\mu,\mathfrak{S}} x(t) \le q_+(t) + \kappa\zeta_1(t) + \zeta_1(t)\int_a^t \xi(v)x(v)\,dv, \quad t \ge a$$

and in view of (7),

$$D_a^{\mu,\mathfrak{S}} x(t) \le q_+(t) + \kappa M_1 + M_1\int_a^t \xi(v)x(v)\,dv, \quad t \ge a. \tag{12}$$

Now, applying $I_a^{\mu,\mathfrak{S}}$ to (12) and using (4) and (5) with $\nu = 1$, we get

$$x(t) \le \frac{b_1}{\Gamma(\mu)}(\mathfrak{S}(t) - \mathfrak{S}(a))^{\mu-1} + \frac{\kappa M_1}{\Gamma(\mu+1)}(\mathfrak{S}(t) - \mathfrak{S}(a))^\mu + I_a^{\mu,\mathfrak{S}}[q_+(t)]$$
$$+ M_1 I_a^{\mu,\mathfrak{S}}\left[\int_a^t \xi(v)x(v)\,dv\right]. \tag{13}$$

By interchanging the order of integration, we have

$$I_a^{\mu,\mathfrak{S}}\left[\int_a^t \xi(v)x(v)\,dv\right] = \frac{1}{\Gamma(\mu)}\int_a^t \mathfrak{S}'(v)(\mathfrak{S}(t)-\mathfrak{S}(v))^{\mu-1}\int_a^v \xi(u)x(u)\,dudv$$
$$= \int_a^t \xi(u)x(u)\int_u^t \frac{\mathfrak{S}'(v)(\mathfrak{S}(t)-\mathfrak{S}(v))^{\mu-1}}{\Gamma(\mu)}\,dvdu$$
$$= \int_a^t \xi(u)x(u) I_u^{\mu,\mathfrak{S}}(1)\,du$$
$$= \frac{1}{\Gamma(\mu+1)}\int_a^t (\mathfrak{S}(t)-\mathfrak{S}(u))^{\mu}\xi(u)x(u)\,du. \tag{14}$$

Using (14) in (13), it follows

$$x(t) \le \frac{b_1}{\Gamma(\mu)}(\mathfrak{S}(t)-\mathfrak{S}(a))^{\mu-1} + \frac{\kappa M_1}{\Gamma(\mu+1)}(\mathfrak{S}(t)-\mathfrak{S}(a))^{\mu} + I_a^{\mu,\mathfrak{S}}[q_+(t)] + \frac{M_1}{\Gamma(\mu+1)}\mathfrak{S}^{\mu}(t)\int_a^t \xi(v)x(v)\,dv$$

and hence,

$$\frac{x(t)}{\mathfrak{S}^{\mu}(t)} \le c_1 + \frac{M_1}{\Gamma(\mu+1)}\int_a^t \mathfrak{S}^{\mu}(v)\xi(v)\frac{x(v)}{\mathfrak{S}^{\mu}(v)}\,dv, \quad t \ge t_1 > a,$$

where taking into account (9), $c_1 > 0$ is an upper bound for

$$\frac{b_1}{\Gamma(\mu)}\frac{(\mathfrak{S}(t)-\mathfrak{S}(a))^{\mu-1}}{\mathfrak{S}^{\mu}(t)} + \frac{\kappa M_1}{\Gamma(\mu+1)}\frac{(\mathfrak{S}(t)-\mathfrak{S}(a))^{\mu}}{\mathfrak{S}^{\mu}(t)} + \frac{I_a^{\mu,\mathfrak{S}}[q_+(t)]}{\mathfrak{S}^{\mu}(t)}.$$

Proceeding as in the proof of the well-known Gronwall's inequality, we obtain

$$\frac{x(t)}{\mathfrak{S}^{\mu}(t)} \le \left(c_1 + \frac{M_1}{\Gamma(\mu+1)}\int_a^{t_1}\xi(v)x(v)\,dv\right)e^{\frac{M_1}{\Gamma(\mu+1)}\int_{t_1}^t \mathfrak{S}^{\mu}(v)\xi(v)\,dv}, \quad t \ge t_1,$$

and using (8),

$$\limsup_{t\to\infty}\frac{x(t)}{\mathfrak{S}^{\mu}(t)} < \infty.$$

If $x(t) < 0$, then we set $y := -x$. It follows that y satisfies Eq. (1) with $\mathfrak{z}(t)$ replaced by $-\mathfrak{z}(t)$ and $\mathcal{Y}(t,x)$ by $-\mathcal{Y}(t,-y)$, respectively. Continuing in the same manner, we obtain

$$\limsup_{t\to\infty}\frac{-x(t)}{\mathfrak{S}^{\mu}(t)} < \infty.$$

The proof is accomplished.

Now, by using the asymptotic result in Theorem 1, we will construct theorems for the oscillation of Eq. (1) in Theorems 2 and 4. We will provide sufficient conditions in Theorem 3 that, for the case where $\alpha = 1$, nonoscillatory solutions of Eq. (1) fulfill (10).

Theorem 2 Let $0 < \alpha < 1$ and the hypotheses (H1)–(H3) hold. Assume that (7) and (9) are satisfied and that

$$\limsup_{t\to\infty} I_a^{\mu,\mathfrak{S}}[\zeta_1(t)] < \infty \tag{15}$$

and

$$\limsup_{t\to\infty} \mathfrak{S}^{\mu}(t)\int_a^t \mathfrak{S}^{\mu}(v)\xi(v)\,dv < \infty. \tag{16}$$

If

$$\liminf_{t\to\infty} I_a^{\mu,\mathfrak{S}}[q_+(t)] = -\infty, \quad \limsup_{t\to\infty} I_a^{\mu,\mathfrak{S}}[q_-(t)] = \infty, \tag{17}$$

then Eq. (1) is oscillatory.

Proof 2 Let $x(t)$ be a nonoscillatory solution of Eq. (1), say $x(t) > 0$ for $t \ge T_1$ for some $T_1 \ge a$. The proof when $x(t) < 0$ is similar.

Continuing as in the proof of Theorem 1, we obtain

$$x(t) \le \frac{b_1}{\Gamma(\mu)}(\mathfrak{S}(t)-\mathfrak{S}(a))^{\mu-1} + I_a^{\mu,\mathfrak{S}}[q_+(t)] + \kappa I_a^{\mu,\mathfrak{S}}[\zeta_1(t)] + \frac{M_1}{\Gamma(\mu+1)}\mathfrak{S}^{\mu}(t)\int_a^t \mathfrak{S}^{\mu}(v)\xi(v)\frac{x(v)}{\mathfrak{S}^{\mu}(v)}\,dv. \tag{18}$$

Moreover, (16) implies (8), and hence the result of Theorem 1 holds. Together with (15), this indicates that the last two integrals of (18) are bounded. Taking limit inferior on both sides and using (17) yields a contradiction with the fact that $x(t)$ is eventually positive. The proof is completed.

Corollary 1 Let $0 < \alpha < 1$ and the hypotheses (H1)–(H3) hold. Assume that (7), (15), and (16) are satisfied and that

$$\limsup_{t\to\infty}\frac{1}{\mathfrak{S}^{\mu}(t)} I_a^{\mu,\mathfrak{S}}[\mathfrak{z}(t)] < \infty, \quad \liminf_{t\to\infty}\frac{1}{\mathfrak{S}^{\mu}(t)} I_a^{\mu,\mathfrak{S}}[\mathfrak{z}(t)] > -\infty,$$

$$\limsup_{t\to\infty}\int_a^t \frac{\mathfrak{S}'(v)(\mathfrak{S}(t)-\mathfrak{S}(v))^{\mu-1}}{\Gamma(\mu)}\zeta_1(v)\int_a^v \xi^{\alpha/(\alpha-1)}(u)\zeta_2^{1/(1-\alpha)}(u)\eta^{1/(1-\alpha)}(u)\,du\,dv < \infty.$$

If

$$\liminf_{t\to\infty} I_a^{\mu,\mathfrak{S}}[\mathfrak{z}(t)] = -\infty \text{ and } \limsup_{t\to\infty} I_a^{\mu,\mathfrak{S}}[\mathfrak{z}(t)] = \infty, \tag{19}$$
then Eq. (1) is oscillatory.

Analogously, when $\alpha = 1$, we can easily prove the following theorems.

Theorem 3 Let $\alpha = 1$ and the hypotheses (H1)-(H3) hold. Besides (7), assume that
$\int_a^\infty \mathfrak{S}^\mu(v)\eta(v)\zeta_2(v)\,dv < \infty$.
If
$$\limsup_{t\to\infty} \frac{I_a^{\mu,\mathfrak{S}}[\mathfrak{z}(t)]}{\mathfrak{S}^\mu(t)} < \infty \text{ and } \liminf_{t\to\infty} \frac{I_a^{\mu,\mathfrak{S}}[\mathfrak{z}(t)]}{\mathfrak{S}^\mu(t)} > -\infty, \tag{20}$$
then, every nonoscillatory solution of Eq. (1) satisfies (10).

Theorem 4 Let $\alpha = 1$ and the hypotheses (H1)-(H3) hold. Assume that (7), (15), (19), and (20) are satisfied. If
$\limsup_{t\to\infty} \mathfrak{S}^\mu(t) \int_a^t \mathfrak{S}^\mu(v)\eta(v)\zeta_2(v)\,dv < \infty$,
then, Eq. (1) is oscillatory.

Example 1 Let $\alpha = 1$. Consider the integro-differential equation
$$\begin{cases} D_1^{1/2,t} x(t) = \frac{1}{t} - \int_1^t \frac{x(v)}{tv^2}\,dv, t \geq 1, \\ \lim_{t\to 1^+} I_1^{1/2,t} x(t) = 1\,. \end{cases} \tag{21}$$
We have $\mu = 1/2, a = 1, \mathfrak{S}(t) = t, \mathfrak{z}(t) = \frac{1}{t}, \mathcal{M}(t,v) = \frac{1}{t}$, and $\mathcal{Y}(t,x) = \frac{x}{t^2}$. Let's take $\zeta_1(t) = \frac{1}{t}, \zeta_2(v) = 1$, and $\eta(t) = \frac{1}{t^2}$. It is clear that $\zeta_1(t)$ is bounded from above. We have
$$\int_1^\infty \frac{1}{v^{3/2}}\,dv = 2 < \infty$$
and $I_1^{1/2,t}[\frac{1}{t}] = \frac{1}{\Gamma(1/2)} \int_1^t (t-v)^{-1/2} \frac{1}{v} dv = \frac{2\ln\left(1+\sqrt{\frac{t-1}{t}}\right)+\ln t}{\sqrt{\pi t}}$,
which implies that
$$\limsup_{t\to\infty} \frac{I_1^{1/2,t}[\frac{1}{t}]}{t^{1/2}} = 0 < \infty \text{ and } \liminf_{t\to\infty} \frac{I_1^{1/2,t}[\frac{1}{t}]}{t^{1/2}} = 0 > -\infty.$$
Hence, all the conditions of Theorem 3 are satisfied, and every nonoscillatory solution of Eq. (21) has the asymptotic property (10).

Conclusions

In this study, we analyzed the oscillatory behavior of solutions to a class of fractional integro-differential equations. Firstly, in Theorem 1, we gave sufficient conditions under which every nonoscillatory solution of Eq. (1) satisfies the asymptotic property (10). Afterward, by using the asymptotic result obtained in Theorem 1, we established theorems for the oscillation of Eq. (1) in Theorems 2 and 4. In Theorem 3, we presented sufficient conditions under which every nonoscillatory solution of Eq. (1) satisfies (10) for the case $\alpha = 1$. To reinforce the theoretical results, we presented a concrete example that illustrates the applicability and effectiveness of our main findings. These contributions deepen the understanding of fractional integro-differential equations and provide a foundation for further exploration in this area.

Conflict of interests

There are no conflicts of interest in this work.

References

[1] Singh J., Kumar D., Baleanu D., New aspects of fractional Biswas-Milovic model with Mittag-Leffler law, *Math. Model. Nat. Phenom.*, 14(3) (2019).

[2] Kilbas A.A., Hadamard-type fractional calculus, *J. Korean Math. Soc.*, 38(6) (2001) 1191-1204.

[3] Katugampola U.N., New approach to generalized fractional integral, *Appl. Math. Comput.,* 218(3) (2011) 860-865.

[4] Katugampola U.N., A new approach to generalized fractional derivatives, *Bull. Math. Anal. Appl.*, 6(4) (2014) 1-15.

[5] Jarad F., Abdeljawad T., Baleanu D., On the generalized fractional derivatives and their Caputo modification, *J. Nonlinear Sci. Appl.*, 10(5) (2017) 2607-2619.

[6] Jarad F., Abdeljawad T., Baleanu D., Caputo-type modification of the Hadamard fractional derivatives, *Advances in Difference Equations*, 2012 (2012) 1-8.

[7] Jarad F., Uğurlu E., Abdeljawad T., Baleanu D., On a new class of fractional operators, *Adv. Differ. Equ.*, 247 (2017).

[8] Grace S.R., Agarwal R.P., Wong P.J.Y., Zafer A., On the oscillation of fractional differential equations, *Fract. Calc. Appl. Anal.*, 15 (2012) 222-231.

[9] Zhu P., Xiang Q., Oscillation criteria for a class of fractional delay differential equations, *Adv. Differ. Equ.*, 2018:403 (2018).

[10] Bolat Y., On the oscillation of fractional order delay differential equations with constant coefficients, *Commun. Nonlinear Sci. Numer. Simul.*, 19 (2014) 3988-3993.

[11] Chen D.X., Oscillation criteria of fractional differential equations, *Adv. Differ. Equ.*, 2012:33 (2012).

[12] Abdalla B., Abdeljawad T., Oscillation criteria for kernel function dependent fractional dynamic equations, *Discrete & Continuous Dynamical Systems - S*, 14(10) (2021).

[13] Zhou Y., Ahmad B., Chen F., Alsaedi A., Oscillation for fractional partial differential equations, *Bull. Malays. Math. Soc.*, 42 (2019) 449-465.

[14] Aphithana A., Ntouyas S.K., Tariboon J., Forced oscillation of fractional differential equations via conformable derivatives with damping term, *Bound. Value Probl.*, 2019 (2019) 1-16.

[15] Chen D., Qu P., Lan Y., Forced oscillation of certain fractional differential equations, *Adv. Differ. Equ.*, 2013 (2013) 125-134.

[16] Alzabut J., Abdeljawad T., Sufficient conditions for the oscillation of nonlinear fractional difference equations, *J. Fract. Calc. Appl.*, 5(1) (2014) 177-187.

[17] Abdalla B., On the oscillation of q-fractional difference equations, *Adv. Differ. Equ.*, 2017:254 (2017) 1-11.

[18] Abdalla B., Oscillation of differential equations in the frame of nonlocal fractional derivatives generated by conformable derivatives, *Adv. Differ. Equ.*, 2018:107 (2018) 1-15.

[19] Abdalla B., Abdeljawad T., On the oscillation of Hadamard fractional differential equations, *Adv. Differ. Equ.*, 2018:409 (2018).

[20] Aslıyüce S., Güvenilir A.F., Zafer A., Oscillation criteria for a certain class of fractional order integro-differential equations, *Hacet. J. Math. Stat.*, 46(2) (2017) 199-207.

[21] Mert R., Bayeğ S., Abdeljawad T., Abdalla B., On the oscillation of kernel function dependent fractional integro-differential equations, *Rocky Mt. J. Math.*, 52(4) (2022) 1451-1460.

[22] Restrepo J.E., Suragan D., Oscillatory solutions of fractional integro-differential equations, *Math. Methods Appl. Sci.*, 43(15) (2020) 9080-9089.

[23] Restrepo J.E., Suragan D., Oscillatory solutions of fractional integro-differential equations II, *Math. Methods Appl. Sci.*, 44(8) (2021) 7262-7274.

[24] Grace S.R., Zafer A., Oscillatory behavior of integro-dynamic and integral equations on time scales, *Appl. Math. Lett.*, 28 (2014) 47-52.

[25] Kilbas A.A., Srivastava M.H., Trujillo J.J., Theory and Application of Fractional Differential Equations, North Holland Mathematics Studies 204, (2006).

[26] Jarad F., Abdeljawad T., Generalized fractional derivatives and Laplace transform, *Discrete and Continuous Dynamical Systems-S*, 13 (2020) 709-722.

[27] Samko S.G., Kilbas A.A., Marichev O.I., Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, (1993).

[28] Hardy G.H., Littlewood J.E., Polya G., Inequalities, Cambridge University Press, (1988).