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A Fibration Application for Crossed Squares

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Abstract: In this work we showed that the category of crossed squares of algebras has a left adjoint pair, and this category is fibred over the category of cornered crossed modules.

Keywords: Crossed square, Cornered crossed module, fibration

Çaprazlanmış Kareler için Bir Fibrasyon Uygulaması

Özet: Bu çalışmada cebirlerin çaprazlanmış karesinin sol adjoint çifte sahip olduğunu ve bu kategorinin köşeli çaprazlanmış modellerin kategorisi üzerinden fibre olduğunu gösterdik.

Anahtar Kelimeler: Çaprazlanmış Kare , Köşeli Çaprazlanmış Modül , Fibrasyon

1. INTRODUCTION

Crossed squares are defined as algebraic structures for homotopy connected 3-types by Guin-Walery and Loday [1]. Arvasi and Ulualan showed the connection between crossed squares, simplicial groups and 2-crossed modules in [2]. Crossed squares of commutative algebras is due to Ellis [3].

Whitehead in 1949-1950 investigated the algebraic structure for simply connected topological spaces of homotopy 2-types and by defining a classifying space functor from the category of these spaces to that of groups, he introduced the notion of crossed modules of groups [4]. In his work, Whitehead showed the mapping

$$\partial : \pi_2(X, A, *) \to \pi_1(A, *)$$

is a crossed module. Later, Porter, [5], defined the commutative algebra version of crossed modules.

For a $\boldsymbol{k}$-module M, if the k-bilinear map

$$M \times M \to M$$

$$(m_1, m_2) \mapsto m_1 m_2$$

satisfies the following conditions for $m_1, m_2, m_3 \in M$

$$i)\ m_1 m_2 = m_2 m_1$$

$$ii)\ (m_1 m_2) m_3 = m_1 (m_2 m_3)$$

then M is called commutative $\boldsymbol{k}$-algebra (or a commutative algebra over **k**).

A crossed module of commutative algebras [6] is an algebra morphism $\partial : C \to R$ with an action of R on C satisfying $\partial(r \cdot c) = r\partial c$ and $\partial(c) \cdot c' = cc'$ for all $c, c' \in M, r \in R$. The last condition is called the *Peiffer identity*. We will denote such a crossed module by (C, R, ∂).

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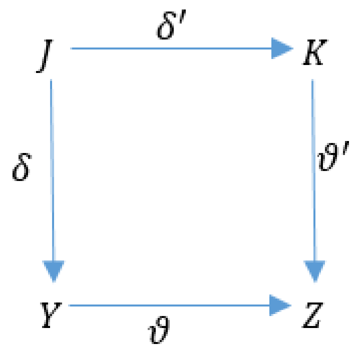
A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair of $\boldsymbol{k}$-algebra morphisms, $\phi : C \to C', \psi : R \to R'$ such that

$$\phi(r \cdot c) = \psi(r)\phi(c).$$

Brown and Sivera [7] showed the forgetful functor from the category of crossed squares of groups to the category of crossed corners of groups is a fibration. In this work we will investigate the commutative algebra version of this functor in analogous way.

2. CROSSED SQUARE

Definition 2.1. A crossed square of commutative algebras [3] is a commutative diagram δ'



together with actions of Z on J,K and Y and a function called h-map

$$h: K \times Y \to J$$

satisfying the following conditions

1. The maps δ, δ' , ϑ' ,ϑ and the composition $\vartheta\delta, = \vartheta'\delta'$ are crossed modules.
2. The maps δ and δ' preserve the action of Z.
3. $k^* h(k, y) = h(k^* k, y) = h(k, k^* y)$
4. $h(k + k', y) = h(k, y) = h(k', y)$
5. $h(k, y + y') = h(k, y) = h(k, y')$
6. $z \cdot h(k, y) = h(z \cdot k, y) = h(k, z \cdot y)$
7. $\delta h(k, y) = k \cdot y$
8. $\delta' h(k, y) = y \cdot k$
9. $h(m, \delta'(j)) = m \cdot j$
10. $h(\delta(j), y) = y \cdot l$

for $k, k' \in K$, $y, y' \in Y$, $z \in Z$,$j \in J$, $k^* \in \boldsymbol{k}$.

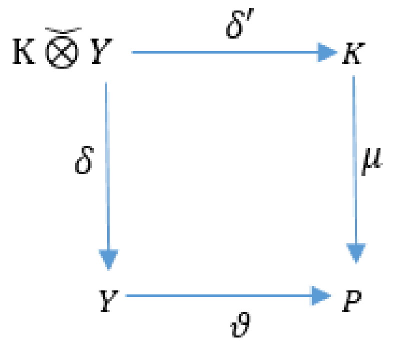
We can give the following result from [3].

Example 2.2. Let $\mu: K \to P$, $\vartheta: Y \to P$ be crossed modules and $K \otimes Y$ be the tensor product with the mapping

$$h: K \times Y \to K \otimes Y$$

$$(k, y) \mapsto k \otimes y$$

then the following diagram



becomes a crossed square. Here the elements of the tensor product are defined as

$\mathrm{K} \widetilde{\otimes} Y = \mathrm{K} \otimes \mathrm{Y} / \{h(p, k) \otimes y = k \otimes h(p, y) : k \in K, y \in Y, p \in P\}$

with mappings

$\mu' : \mathrm{K} \widetilde{\otimes} \mathrm{Y} \to Y$, $\vartheta' : \mathrm{K} \widetilde{\otimes} Y \to K$

$\mathrm{k} \widetilde{\otimes} \mathrm{y} \mapsto h(k, \vartheta(y))$ $\mathrm{k} \widetilde{\otimes} y \mapsto h(\mu(k), y))$

and the h-map

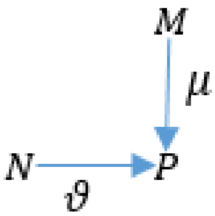
$$h: K \times Y \to K \widetilde{\otimes} Y, \quad . : K \widetilde{\otimes} Y \times K \widetilde{\otimes} Y \to K \widetilde{\otimes} Y$$

$$(k, y) \mapsto k \widetilde{\otimes} y \qquad (k \widetilde{\otimes} y, k' \widetilde{\otimes} y') \mapsto \mathrm{kk}' \widetilde{\otimes} \mathrm{yy}'$$

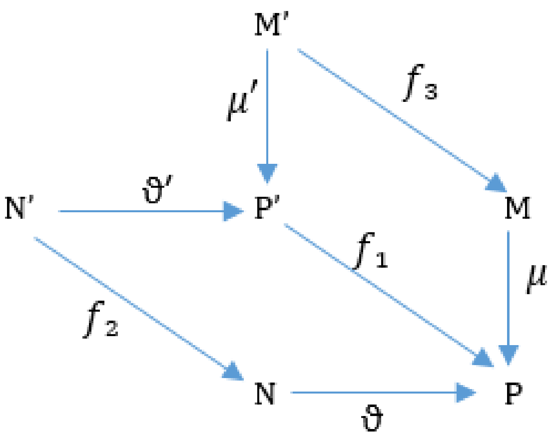
3. CORNERED CROSSED MODULES

Brown and Sivera defined the notion of cornered crossed modules on groups. In this section we will give the analogous definition for commutative algebras.

Let's denote the category of the pairs $\mu: M \to P$, $\vartheta: N \to P$ of crossed module pairs as **XMOD²** . We can illustrate the objects of this category with the following diagram.



We will call **XMOD²** the category of cornered crossed modules. We can express a morphism $f = (f_1, f_2, f_3)$ in **XMOD²** with the following commutative diagram

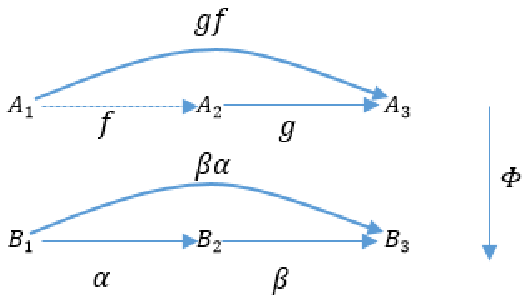


Thus we get $f_1\mu' = \mu f_3$ and $\vartheta f_2{}' = f_1\vartheta'$.

4. FIBRATION of CATEGORIES

We can give the following basic definition from [7].

Definition 3.1. Let $\Phi: \boldsymbol{C} \to \boldsymbol{D}$ be a functor. A morphism $g: A_2 \to A_3$ in $\boldsymbol{X}$ over $\beta := \Phi(\beta)$ is called cartesian if and only if for all $\alpha: B_1 \to B_2$ in $\boldsymbol{B}$ and $\theta: A_1 \to A_3$ with $\Phi(\theta) = \beta\alpha$ there is a unique morphism $f: A_1 \to A_2$ with $\Phi(f) = \alpha$ and $\theta = gf$. We can express this as follows

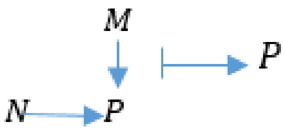


A morphism $\gamma: A_1 \to A_2$ is called vertical if and only if $\Phi(\gamma)$ is an identity morphism in $\boldsymbol{D}$. In particular, for $I \in D$ we write $\boldsymbol{C_I}$, called the fibre over **I** , for the subcategory of C consisting of these morphims γ with $\Phi(\gamma) = id_I$.

Definition 3.2. The functor $\Phi: \boldsymbol{C} \to \boldsymbol{D}$ is fibration or category fibred over D if and only if for all $\beta: B_2 \to B_3$ in D and $C \in \boldsymbol{C_I}$ there is a cartesian morphism $g: A_2 \to A_3$ over β such a g is called a cartesian lifting of C along β.

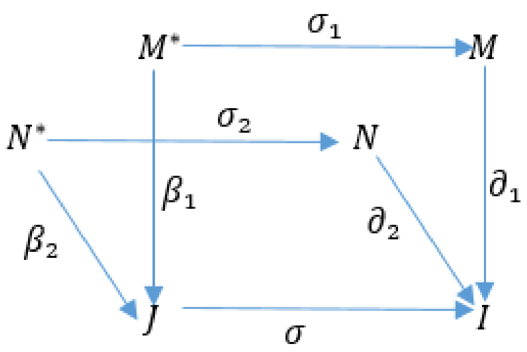
Theorem 3.3. The forgetfull functor

$$\Phi: \boldsymbol{XMod^2} \to \boldsymbol{Alg}$$



is a fibration.

Proof. We should show that the category of cornered crossed smodules is fibred over algebras. For all $u: J \to I$ morphisms in the category of algebras and $X \in XMod_I^2$ we need to show the morphism $\varphi: Y \to X$ is cartesian over u. Now we will construct a cornered crossed module over J. This is illustrated with the following diagram



Define

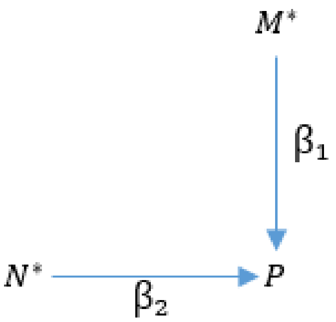
$$\sigma^*(M,N) = (M^*,N^*) \subset (J\, \widetilde{\otimes}\, M)\times(J\, \widetilde{\otimes}\, N)$$

$$\sigma(j) = \partial_1(m) = \partial_2(n)$$

$$\beta_1(j \otimes m) \mapsto j$$

$$\beta_2(n \otimes j) \mapsto j$$

due to β_1: $M^* \to J$ and β_2:$N^* \to J$ are crossed modules. It is clear that the following diagram



is a cornered crossed module. Thus the morphism $(\sigma_1\sigma_2, \sigma)$ becomes cartesian morphism. Therefore the forgetfull functor Φ is a fibration of categories.

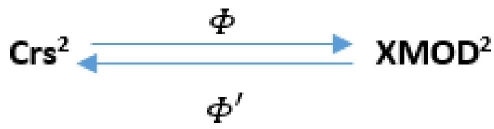
Now we show that the category **Crs**^{2} of crossed modules have a left adjoint pair that is **XMOD**^{2} the category of cornered crossed squares.

Theorem 3.4. The forgetful functor

$$\Phi: \boldsymbol{Crs^2} \rightarrow \boldsymbol{XMod^2}$$

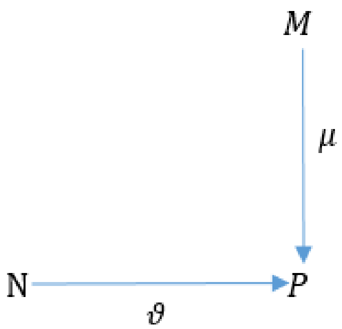
is a fibration and have a left adjoint.

Proof.

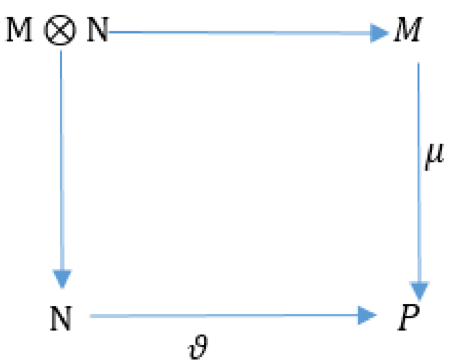


Let Φ be the forgetful functor and Φ' be the the functor taking a cornered crossed module to the crossed square via tensor product. Then Φ has a left adjoint.

Let the following diagram



be an object in **XMOD**^{2}. As we know from example2.2. that the following diagram



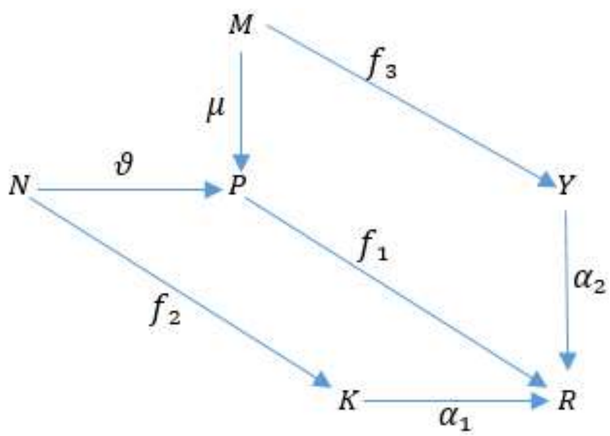
is a crossed square. Thus for every object C in **XMOD²** , Φ′(C) is an object in **Crs²** . Next we will show that the morphism

$$\eta_C : C \rightarrow \Phi(\Phi'(C))$$

is a universal morphism. We need to show for any morphism $f : D \rightarrow D'$ in **Crs²** there is a morphism

$\Phi(f) : C \rightarrow \Phi(D')$ and is a bijection from $MOR_{Crs^2}\ (D, D')$ to $MOR_{XMOD^2}\ (C, \Phi(D'))$.

Let $(f_1, f_2, f_3) = f : \Phi(C) \rightarrow D$ be a morphism of cornered crossed squares given with the following commutative diagram



Since C is an object in **Crs²**, $h_C : N \times M \rightarrow L$ is the h-map and for $\Phi(D')$, $h_D : K \times_R Y \rightarrow K \otimes Y$ is the h-map. Define

$$\eta_{C,D} : Mor(\Phi(C), D) \rightarrow Mor(C, \Phi'(D))$$

$$f \mapsto \eta_{C,D}\ (f) \quad = \eta_{C,D}(f_1, f_2, f_3)$$
$$= (f_1, f_2, f_3, h_D(f_2 \times f_3))$$

and

$$\Psi_{C,D} : Mor(C, \Phi'(D)) \rightarrow Mor(\Phi(C), D)$$

$$g \mapsto \Psi(g_1, g_2, g_3, g_4) = (g_1, g_2, g_3)$$

then we get

$$(\Psi_{C,D} \circ \eta_{C,D}\ )(f)$$
$$= \Psi_{C,D}(\eta_{C,D}\ (f_1, f_2, f_3))$$
$$= \Psi_{C,D}(f_1, f_2, f_3, h_D(f_2 \times f_3)) = (f_1, f_1, f_3)$$

$$\Psi_{C,D} \circ \eta_{C,D} = 1_{XMOD^2}$$

Similarly

$$(\eta_{C,D} \circ \Psi_{C,D})(g)$$
$$= \eta_{C,D}\ \left(\Psi_{C,D}(g^1, g^2, g^3, g^4)\right)$$
$$= \eta_{C,D}\ (g_1, g_2, g_3)$$

$$= (g_1, g_2, g_3, h_D(g_2 \times g_3))$$

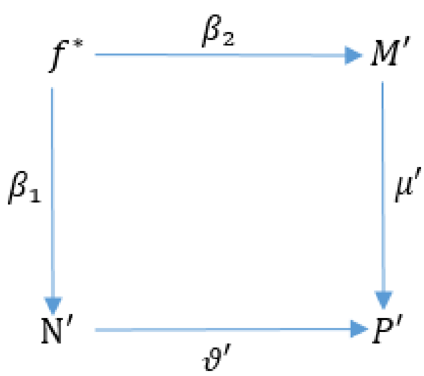
since the tensor product is universal

$$h_D(g_2 \times g_3) = g_4$$

is unique . Thus

$$\eta_{C,D} \circ \Psi_{C,D} = 1_{Crs^2}$$

This shows that the $\boldsymbol{Crs^2}$ has a left adjoint. The following diagram



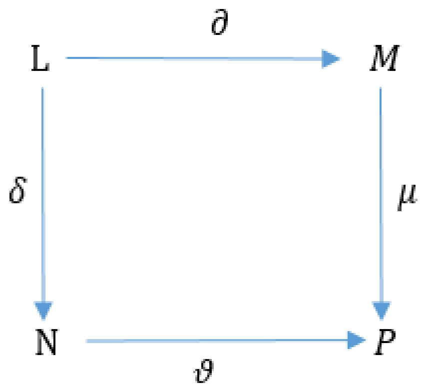
with

$$f^* = \{(n', m', l) \in N \times M \times L : \vartheta'(n)$$
$$= \mu'(m), f_3(m')$$
$$= \partial(l), f_2{}'(n') = \delta(l)\}$$

$$\beta_1(n', m', l) = m'$$

$$\beta_2(n', m', l) = n'$$

is a crossed square. Let the following diagram be any crossed square



Define $\alpha: f^* \to L$,$\alpha(n', m', l) = l$, then (f^*, β_1, β_2) is the pullback (μ', ϑ'). Since the categories with pullback and left adjoint has a fibration we deduce that the category $\boldsymbol{Crs}^2$ is fibred over the category $\boldsymbol{XMOD}^{\mathbf{2}}$.

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