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Exotic 4 -Manifolds from Genus-4 Lefschetz Fibrations

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ABSTRACT

In this paper, we study minimal simply connected 4-manifolds with $b_2^+ = 3$ which admit genus-4 Lefschetz fibrations over the 2-sphere. We first explicitly construct a genus-4 Lefschetz fibration over the 2-sphere using the monodromy of generalized Matsumoto fibration of genus 3 and the monodromy of the smallest genus-2 fibration given by Baykur and Korkmaz. We then construct two genus-4 Lefschetz fibrations over the 2-sphere that are exotic minimal symplectic 4-manifolds belonging to the homeomorphism classes of $3\mathbb{C}P^2\#15\overline{\mathbb{C}P^2}$ and $3\mathbb{C}P^2\#14\overline{\mathbb{C}P^2}$ by performing the fiber sum operation and then lantern substitution.

Keywords: Lefschetz fibrations, Symplectic 4-manifolds, Exotic manifolds, Mapping class groups.

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Introduction

Smooth 4-manifolds that are homeomorphic to standard manifolds but non-diffeomorphic to them are called exotic. The existence of such simply-connected manifolds is less clear when the Euler characteristic of these manifolds is small, which has gained great interest. Producing exotic minimal symplectic structures on 4-manifolds is also an interesting topic that has used many constructions techniques such as knot surgery, Luttinger surgery, rational blow-down, generalized fiber sum by Akhmedov, Fintushel and Stern, Gompf, Jongil Park, and several others [1,2,6,13, 17-23, 26,28,31-33,35].

Many ideas can also be applied to get exotic structures on simply connected manifolds satisfying $b_2^+ = 3$ and having relatively small b_2^-, a subject with a rich history. It is known that the celebrated $K3$ surface $E(2)$ is such a 4-manifold with $b_2^+ = 3$ and $b_2^- = 19$. Gompf [23] discovered symplectic 4-manifolds with $b_2^+ = 3$ and $14 \leq b_2^- \leq 18$. Later, it was shown that many of those Gompf's examples are exotic copies of $3\mathbb{C}P^2\#\mathrm{n}\overline{\mathbb{C}P^2}$ using Donaldson's polynomials invariants (cf. [34] and [38]). About a quarter of a century ago, Park [28,29,30] constructed exotic copies of $3\mathbb{C}P^2\#\mathrm{n}\overline{\mathbb{C}P^2}$ with $10 \leq n \leq 13$. Next, Stipsicz and Szabó [36] obtained similar results for $n = 8$ and 9. Such symplectic 4-manifolds with smaller b_2^- have been constructed by Akhmedov, Park and the others. Their examples are exotic copies of $3\mathbb{C}P^2\#\mathrm{n}\overline{\mathbb{C}P^2}$ with $4 \leq n \leq 7$.

In the past decades, many authors have used Lefschetz fibration structures to derive exotic copies of some 4-manifolds (see [3,4,5,7,10,11]). A Lefschetz fibration is, roughly speaking, a fibering of a smooth 4-manifold by a compact orientable surface with finite number of singularities. Since the celebrated works of Donaldson and Gompf in late 1990s, Lefschetz fibrations have played an essential role in the study of 4-manifold topology. Donaldson showed that Lefschetz fibration structures are found on all symplectic 4-manifolds, after blowing-ups if necessary, which results in a combinatorial way of studying the topology of 4-manifolds if they have a Lefschetz fibration structure. Conversely, by the remarkable work of Gompf, any 4-manifold that admits a Lefschetz fibration structure carries a symplectic structure when its fiber genus is at least two.

Recently, some authors [3,4,5,10] constructed exotic 4-manifolds admitting Lefschetz fibration structures by providing their monodromies which enables a combinatorial technique. In [10] the authors used genus-2 Lefschetz fibrations to construct exotic copies of $3\mathbb{C}P^2\#\mathrm{n}\overline{\mathbb{C}P^2}$ with $12 \leq n \leq 19$. The author [4] gave examples of exotic manifolds $3\mathbb{C}P^2\#\mathrm{n}\overline{\mathbb{C}P^2}$ with $13 \leq n \leq 19$ using genus-3 Lefschetz fibration structures.

Very recently, the author [5] derived some exotic 4-manifolds with $b_2^+ = 1$ via small genus-4 Lefschetz fibrations. We would like to state that the goal of this paper is to derive exotic 4-manifolds with $b_2^+ = 3$ through genus-4 Lefschetz fibration structures with few singular fibers on various smooth symplectic 4-manifolds.

In this present paper, we first obtain an identity $U = 1$ in the mapping class group of the closed connected oriented surface of genus-4 derived from generalized Matsumoto's relation for genus 3 [12,25] and Baykur-Korkmaz's relation [10]. Then we get a genus-4 Lefschetz fibration associated with the monodromy U. We construct a minimal genus-4 Lefschetz fibration by taking the twisted fiber sum of two copies of this fibration to obtain an exotic copy of $3\mathbb{C}P^2\#15\overline{\mathbb{C}P^2}$. Finally, we apply the lantern substitution to get a further exotic 4-manifold which is an exotic copy of $3\mathbb{C}P^2\#14\overline{\mathbb{C}P^2}$ (for the main result, see Theorem 6).

Preliminaries and Background Results

Our purpose in this section is to give several preliminary definitions and review some useful facts.

Mapping Class Groups

Let Σ_g^n be a compact connected oriented smooth genus-g surface having $n \geq 0$ boundary components. If $n = 0$, the number n will be omitted from the notation, and write Σ_g. The mapping class group $\mathrm{Mod}(\Sigma_g^n)$ of Σ_g^n is defined as the group of all isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_g^n \to \Sigma_g^n$ which fix the boundary pointwise, modulo isotopies of the same type. Throughout the paper the curves on Σ_g^n and the diffeomorphisms of $\Sigma_g^n \to \Sigma_g^n$ should be understood up to isotopy. We always use the convention of functional notation, so that if α and β are two mapping classes, then $\alpha\beta$ means that β acts on Σ_g first.

Now, let us recall some basic properties of Dehn twists which we will repeatedly use without referring to them explicitly. For two simple closed curves a and b on Σ_g^n and $f \in \mathrm{Mod}(\Sigma_g^n)$,

- If a and $b$are disjoint, then $t_a t_b = t_b t_a$ *(Commutativity).*
- If $f(a) = b$ then $f t_a f^{-1} = t_b$ *(Conjugation).*
- For four boundary parallel curves x_1, x_2, x_3 and x_4 on the surface Σ_0^4 and three interior simple closed curves y_1, y_2 and y_3 shown in Figure 1, we get $t_{x_1} t_{x_2} t_{x_3} t_{x_4} = t_{y_1} t_{y_2} t_{y_3}$ *(Lantern relation).*

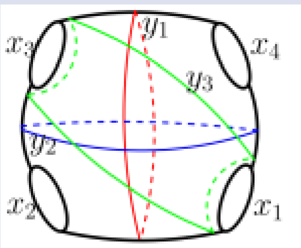

Figure 1. Lantern relation

Lefschetz fibrations

A Lefschetz fibration of a closed, connected, oriented smooth 4-manifold X is a smooth surjection $f: X \to \mathbb{S}^2$ with finite number of critical points $\{p_1, p_2, \ldots, p_n\}$ so that about each p_i it is of the form $f(z_1, z_2) = z_1 z_2$ respecting some complex coordinates that are compatible with the fixed global orientations of X and $\mathbb{S}^2$. Here $\mathbb{S}^2$denotes the 2-sphere. (Note that one can consider any closed orientable surface instead of the base space $\mathbb{S}^2$; however, in this study, we only consider the sphere $\mathbb{S}^2$.) The genus of a Lefschetz fibration is defined to be the genus of a regular fiber. The preimage of a critical value is said to be a singular fiber. One can obtain it by collapsing a simple closed curve (called a vanishing cycle) on a regular fiber to a point. Every singular fiber of a Lefchetz fibration may contain only one critical point (after a small perturbation if necessary), which we will assume. A vanishing cycle is said to be irreducible if it is nonseparating, otherwise it is said to be reducible. Throughout the paper, to avoid trivial examples, we consider Lefschetz fibrations which are relatively minimal, namely they have no fiber which contains a (-1)-sphere, and also assume they have at least one singular fiber.

We describe a genus-g Lefschetz fibration by a positive factorization, called monodromy, which is a word of $\mathrm{Mod}(\Sigma_g)$. The monodromy of a genus- g Lefschetz fibration $f: X \to \mathbb{S}^2$ with vanishing cycles $c_1, c_2, \ldots, c_k$ (chosen according to a Hurwitz system) is given by a positive factorization in $\mathrm{Mod}(\Sigma_g)$,

$$t_{c_1} t_{c_2} \ldots t_{c_k} = 1$$

up to Hurwitz moves (replacing subwords $t_{c_i} t_{c_{i+1}}$ with $t_{c_{i+1}} t_{t_{c_{i+1}}^{-1}(c_i)}$, or vice versa) and global conjugations (replacing every t_{c_i} with $t_{\phi(c_i)}$, for some $\phi \in \mathrm{Mod}(\Sigma_g)$). A Lefschetz fibration $f: X \to \mathbb{S}^2$ admits a section if there exists a map $\sigma: \mathbb{S}^2 \to X$ such that $f \circ \sigma = id_{\mathbb{S}^2}$. A Lefschetz fibration having the monodromy $t_{c_1} t_{c_2} \ldots t_{c_k} = 1$ in $\mathrm{Mod}(\Sigma_g)$ admits m disjoint sections $S_1, S_2, \ldots, S_m$, where the self-intersection of S_j is equal to $-n_j$, if its monodromy has a lifting to $\mathrm{Mod}(\Sigma_g^m)$ of the form

$$t_{\tilde{c}_1} t_{\tilde{c}_2} \cdots t_{\tilde{c}_k} = t_{\delta_1}^{n_1} t_{\delta_2}^{n_2} \cdots t_{\delta_m}^{n_m},$$

where each n_i is an interger, $\delta_1, \ldots, \delta_m$ are distinct boundary parallel curves and each $t_{\tilde{c}_i}$ is a Dehn twist mapped to t_{c_i} under the capping homomorphism $\mathrm{Mod}(\Sigma_g^m) \to \mathrm{Mod}(\Sigma_g)$.

For $i = 1,2$, let $f_i: X_i \to \mathbb{S}^2$ be a genus-g Lefschetz fibration with monodromy factorization $W_i = 1$ and a regular fiber F_i. Let $\phi: F_2 \to F_1$ be an orientation-preserving diffeomorphism and $r: \mathbb{S}^1 \to \mathbb{S}^1$ an orientation-reversing diffeomorphism. The twisted fiber sum of the Lefcshetz fibrations f_1 and f_2 is obtained by deleting a fibered neighborhood of F_i from X_i and gluing them along their boundaries via $r \times \phi$, which has the monodromy factorization $W_1 W_2^{\phi}$ (here W_2^{ϕ} refers to the conjugated word, that is $W_2^{\phi} = t_{\phi(a_1)} t_{\phi(a_2)} \cdots t_{\phi(a_n)}$ if $W_2 = t_{a_1} t_{a_2} \cdots t_{a_n}$).

The Euler characteristic of a genus-g Lefschetz fibration X with k singular fibers is

$$e(X) = 4 - 4g + k.$$

One can use Endo and Nagami's useful techniques in which they introduced the notion called the signatures of relators (see [16] for more details) for computing the signature of a Lefschetz fibration over $\mathbb{S}^2$. Let us mention the definition of signature of relators and the results that we will use later.

Let $\mathcal{F}$ denote the free group generated by all isotopy classes of simple closed curves on Σ_g. There is a natural homomorphism $\varrho\colon \mathcal{F} \to \mathrm{Mod}(\Sigma_g)$ sending a simple closed curve a to the Dehn twist t_a. The homomorphism ϱ is surjective since the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by Dehn twists. An element of $\mathrm{Ker}\varrho$ is called a relator. A relator ρ is expressed as a word $\rho = {c_1}^{\epsilon_1}{c_2}^{\epsilon_2}\cdots{c_n}^{\epsilon_n}$, where each c_i is a simple closed curve on Σ_g and each $\epsilon_i = \pm 1$ for $i = 1,2,\dots,n$. If $\epsilon_i = 1$ for all $i = 1,2,\dots,n$, then the relator is said to be positive. For instance, the following word is a relator which comes from the lantern relation

$$L = y_1 y_2 y_3 x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1},$$

where the curves as in Figure 1.

We have an explicit homomorphism $c_g\colon \mathrm{Ker}\varrho \to \mathbb{Z}$ inducing the evaluation map $H_2(\mathrm{Mod}(\Sigma_\mathrm{g})) \to \mathbb{Z}$ for the 2-cocycle τ_g, where $\tau_g\colon \mathrm{Mod}(\Sigma_\mathrm{g}) \times \mathrm{Mod}(\Sigma_\mathrm{g}) \to \mathbb{Z}$ is the Meyer's signature cocycle (see Proposition 2.3 of [16]). For a relator ρ, let $s(\rho)$ be the sum of the exponents of Dehn twists about separating simple closed curves which are contained in the word ρ. The signature of ρ is defined as

$$I_g(\rho) \coloneqq -c_g(\rho) - s(\rho).$$

This definition can be extended to the elements of the free group $\mathcal{F}$.

We list signatures of some relators which we will use later (for proofs, see [16]).

- $I_g(\rho_1\rho_2) = I_g(\rho_1) + I_g(\rho_2)$, where ρ_1, ρ_2 are relators.
- $I_g(L) = +1$, where L is the lantern relator.
- $I_g\left(\left(B_0 B_1 \cdots B_g C\right)^2\right) = -4$ if g is even.
- $I_g\left(\left(B_0 B_1 \cdots B_g a^2 b^2\right)^2\right) = -8$ if g is odd.

(The last two words above are relators obtained from generalized Matsumoto's relation that will be explained in the next subsection.)

The following theorem holds:

Theorem 1 [16] For a Lefschetz fibration of genus-g $f\colon X \to \mathbb{S}^2$ with the monodromy factorization $t_{c_1}t_{c_2}\dots t_{c_k} = 1$, so that $c_1 c_2 \cdots c_k \in \mathrm{Ker}\varrho$ is a positive relator. Then the signature $\sigma(X)$ of X is given by

$$\sigma(X) = I_g(c_1 c_2 \cdots c_k).$$

Using this technique, the signature of such a Lefschetz fibration is equal to the sum of signatures of some basic relators into which its monodromy decomposes.

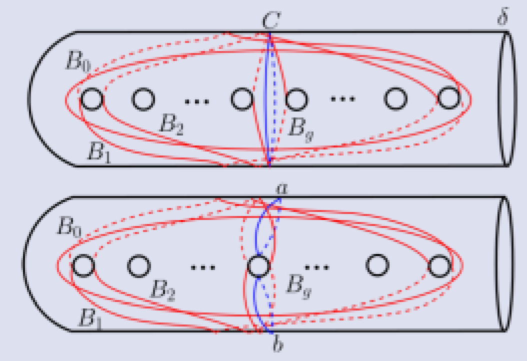

Figure 2. The Dehn twist curves B_i, C, a and b on Σ_g^1.

Let W_g be the following word:

$$W_g = \begin{cases} (t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}t_C)^2 & \text{if } g = 2k, \\ (t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}t_a^2 t_b^2)^2 & \text{if } g = 2k+1, \end{cases} \qquad (1)$$

where the curves $B_0, B_1, \dots, B_g, a, b$ and C are depicted in Figure 2. The word W_g equals to the identity in $\mathrm{Mod}(\Sigma_g)$ when δ is capped off by a disk. The identity $W_g = t_\delta$ in $\mathrm{Mod}(\Sigma_g^1)$ was shown in [25] (here δ denotes the curve parallel to the boundary of the surfaceΣ_g^1). Let $M_g \to \mathbb{S}^2$ be the Lefschetz fibration associated with the monodromy $W_g = 1$. The total space M_g is diffeomorphic to $\Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{C}P^2}$ if $g = 2k$ and it is diffeomorphic to $\Sigma_k \times \mathbb{S}^2 \# 8\overline{\mathbb{C}P^2}$ if $g = 2k+1$.

The smallest genus two Lefschetz fibration

Baykur and Korkmaz [10] constructed a genus-2 positive factorization consisting of seven positive Dehn twists, which yields the smallest genus-2 Lefschetz fibration whose total space is diffeomorphic to $\mathbb{T}^2 \times \mathbb{S}^2 \# 3\overline{\mathbb{C}P^2}$. They gave a lifting of this relation to $\mathrm{Mod}(\Sigma_2^1)$. A further lift to $\mathrm{Mod}(\Sigma_2^2)$ was given by Stipsicz and Yun [37]. Recently, Baykur [9] gave a yet further lift to $\mathrm{Mod}(\Sigma_2^3)$, which can be rewritten as follows.

$$t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{B_2} t_C = t_{\delta_1} t_{\delta_2} t_{\delta_3} \qquad (2)$$

where all curves appearing in the relation above are depicted in Figure 3 (here δ_i's are boundary components of Σ_2^3). For a comprehensive proof, see [8].

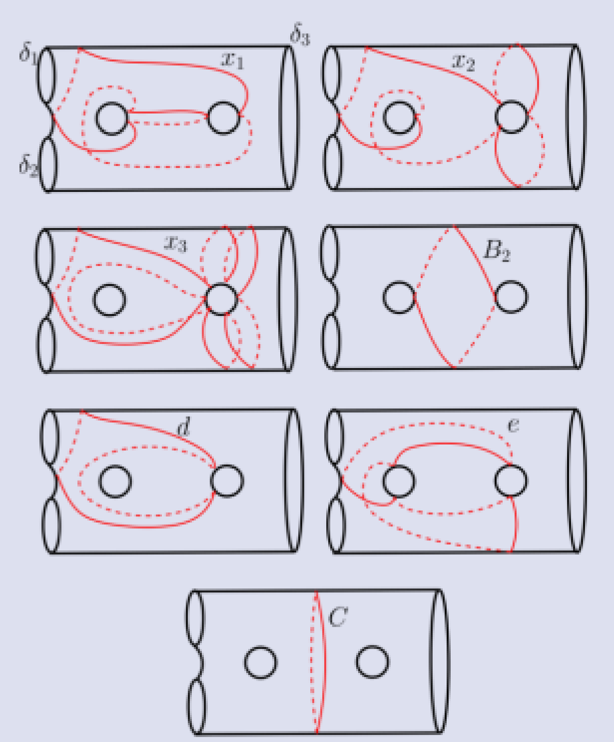

Figure 3. The curves x_1, x_2, x_3, B_2, d, e and C on Σ_2^3.

Symplectic sum

Let Y_1 and Y_2 denote symplectic 4-manifolds containing embedded symplectic surfaces $V_{Y_1} \subset Y_1$ and $V_{Y_2} \subset Y_2$ of genus $g \geq 0$ whose homology classes satisfy $[V_{Y_1}]^2 + [V_{Y_2}]^2 = 0$. The symplectic sum $Y_1 \#_{V_{Y_1}=V_{Y_2}} Y_2$ along V_{Y_1} and V_{Y_2} is defined as $(Y_1 \setminus \mathcal{N}V_{Y_1}) \cup_\phi (Y_2 \setminus \mathcal{N}V_{Y_2})$, where $\mathcal{N}V_{Y_1}$ (resp. $\mathcal{N}V_{Y_2}$) is the open disk normal bundle of V_{Y_1} in Y_1 (resp. V_{Y_2} in Y_2) and the map $\phi: \partial\mathcal{N}V_{Y_1} \to \partial\mathcal{N}V_{Y_2}$ between the boundaries of $\mathcal{N}V_{Y_1}$ and $\mathcal{N}V_{Y_2}$ which is lifted from an orientation-preserving diffeomorphism $V_{Y_1} \to V_{Y_2}$. The following theorem can be used to decide whether a symplectic sum is minimal or not.

Theorem 2 [14,39]. Let M be the symplectic sum of Y_1 and Y_2 along V_{Y_1} and V_{Y_2} , in the notation above.
Then the following holds.

i. If there exists an embedded symplectic sphere of self-intersection (-1) contained in either $Y_1 \setminus V_{Y_1}$ or $Y_2 \setminus V_{Y_2}$, then M is not minimal.
ii. If one summand, say Y_2 , is $\mathbb{CP}^2$ and $V_{Y_2} = V_{\mathbb{CP}^2}$ is an embedded sphere of self-intersection $(+4)$ that is in the class $[V_{\mathbb{CP}^2}] = 2[H] \in H_2(\mathbb{CP}^2, \mathbb{Z})$, where H is a degree-1 curve in $\mathbb{CP}^2$ and the other summand Y_1 has at least two disjoint exceptional spheres E_i of self-intersection (-1) each meeting V_{Y_1} positively and transversely in a single point with $[E_i].[V_{Y_1}] = 1$, then M is not minimal.
iii. If one summand, say Y_2 , is an $\mathbb{S}^2$-bundle over a genus-g surface and V_{Y_2} is a section of this fiber bundle, then M is minimal if and only if Y_1 is minimal.
iv. M is minimal in all other cases.

Constructing of Genus-4 Lefschetz Fibrations

Our purpose in this section is to construct Lefschetz fibrations of genus- 4 over $\mathbb{S}^2$. To do this, we first derive a factorization of t_δ in $\mathrm{Mod}(\Sigma_4^1)$ consisting of 20 positive Dehn twists that will be our building block using the so-called *breeding* technique used in [4,5,7,9,10,24]. Afterwards, we construct two positive factorizations which will be the monodromies of genus- 4 Lefschetz fibrations over $\mathbb{S}^2$ by first taking the fiber sum and then using the lantern substitution.

Let us consider the surface Σ_4^1 depicted in Figure 4. We first embed the generalized Matsumoto relation for genus-3 given in (1) into $\mathrm{Mod}(\Sigma_4^1)$ in such a way that the boundary parallel curve δ shown in Figure 2 is sent to the curve $\overline{C}$ shown in Figure 4. (Here, the notation β_i is used instead of B_i to distinguish them from the curves coming from other factorizations.) Hence, we get the following relation:

$$t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2 = t_{\overline{C}},$$

which can be expressesd as

$$t_{\overline{C}}^{-1}t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2 = 1$$

We then embed the relation (2) into $Mod(\Sigma_4^1)$ so that the boundary parallel curves δ_1, δ_2 and δ_3 are mapped to the curves a, b and δ in Figure 4, respectively. Therefore, the following relation holds in $Mod(\Sigma_4^1)$:

$$t_e t_{x_1}t_{x_2}t_{x_3}t_d t_{\overline{B_2}} t_{\overline{C}} = t_a t_b t_\delta.$$

(Here the notations $\overline{B_2}$ and $\overline{C}$ are used instead of the curves B_2 and C to distinguish them from the ones appearing in the equation (1).) One can rewrite this relation as

$$t_a^{-1}t_b^{-1}t_e t_{x_1}t_{x_2}t_{x_3}t_d t_{\overline{B_2}} t_{\overline{C}} = t_\delta \tag{4}$$

using the fact that the curves a and b are disjoint from all curves appearing in the factorization (4). Then we breed the relations (3) and (4) by combining them as

$$\left(t_a^{-1}t_b^{-1}t_e t_{x_1}t_{x_2}t_{x_3}t_d t_{\overline{B_2}} t_{\overline{C}}\right)\left(t_{\overline{C}}^{-1}t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2t_{\beta_0}t_{\beta_1} t_{\beta_2}t_{\beta_3}t_a^2t_b^2\right) = t_\delta,$$

which implies the following relation:

$$\left(t_e t_{x_1}t_{x_2}t_{x_3}t_d t_{\overline{B_2}} t_{\overline{C}}\right)\left(t_{\overline{C}}^{-1}t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2t_{\beta_0}t_{\beta_1}t_{\beta_2} t_{\beta_3}t_a^2t_b^2\right) = t_\delta t_a t_b.$$

Hence, one can obtain the following identity in $\mathrm{Mod}(\Sigma_4^1)$:

$$t_e t_{x_1}t_{x_2}t_{x_3}t_d t_{\overline{B_2}} t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a t_b = t_\delta \tag{5}$$

using some cancellations. Set

$$U = t_e t_{x_1}t_{x_2}t_{x_3}t_d t_{\overline{B_2}} t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a^2t_b^2t_{\beta_0}t_{\beta_1}t_{\beta_2}t_{\beta_3}t_a t_b$$

so that $U = 1$ in $\mathrm{Mod}(\Sigma_4)$ by capping off the boundary component of Σ_4^1. Let X_U denote the total space of the genus-4 Lefschetz fibration associated with the monodromy $U = 1$. Note that it has 20 singular fibers, and it admits a (-1)-section.

By the Euler characteristic formula, the Euler characteristic $e(X_U)$ of X_U is computed as follows:

$$e(X_U) = 4 - 4g + (\#\text{singular fibers}) \\ = 4 - 4(4) + 20 = 8.$$

We use Endo and Nagami's method in order to compute the signature of X_U, which states that it can be computed as a sum of signatures of relators associated with some relations in the mapping class group. Since the signature does not change after an embedding of any relation into a higher genus surface, we only compute the signatures of genus-3 relation (3) and genus-2 relation (4). The signature of the relator associated with the relation (3) is equal to the signature of the genus-3 Lefschetz pencil with one base point on $\mathbb{T}^2 \times \mathbb{S}^2 \# 7\overline{\mathbb{C}P^2}$, which is -7. The signature of the relator coming from the relation (4) is the same as that of the genus-2 Lefschetz pencil with two base points on $\mathbb{T}^2 \times \mathbb{S}^2 \# \overline{\mathbb{C}P^2}$, which is -1. One can conclude that $\sigma(X_U) = -8$.

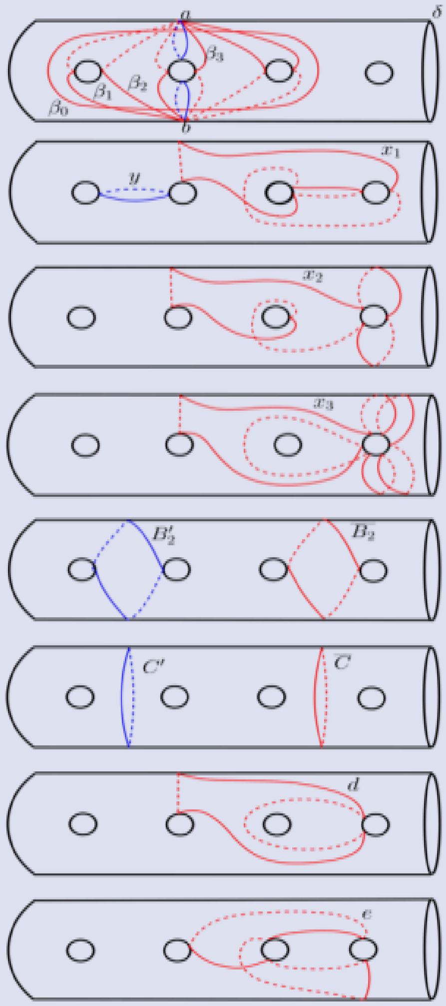

Figure 4. The curves $\beta_i, x_1, x_2, x_3, B_2', \overline{B_2}, C', \overline{C}, d, e, a, b, y$ on Σ_4^1..

For constructing further genus-4 Lefschetz fibrations, we rewrite generalized Matsumoto's relation for genus-4 given in (1) as

$$W_4 = t_{B_0}^2 V = t_\delta,$$

in $\mathrm{Mod}(\Sigma_4^1)$, where

$$V = t_{t_{B_0}{}^{-1}(B_1)} t_{t_{B_0}{}^{-1}(B_2)} t_{t_{B_0}{}^{-1}(B_3)} t_{t_{B_0}{}^{-1}(B_4)} t_{t_{B_0}{}^{-1}(C)} \\ t_{B_1} t_{B_2} t_{B_3} t_{B_4} t_C.$$

Let $\alpha = t_{B_0} t_{a_1}$, where the curve a_1 is in Figure 5. It can be verified that $\alpha(B_0) = a_1$. The conjugation of W_4 with α gives rise to the following identity:

$$W_4^\alpha = t_{\alpha(B_0)}^2 V^\alpha = t_{a_1}^2 V^\alpha = t_\delta.$$

Let

$y_1 = \alpha t_{B_0}^{-1}(B_1)$	$y_6 = \alpha(B_1)$
$y_2 = \alpha t_{B_0}^{-1}(B_2)$	$y_7 = \alpha(B_2)$
$y_3 = \alpha t_{B_0}^{-1}(B_3)$	$y_8 = \alpha(B_3)$
$y_4 = \alpha t_{B_0}^{-1}(B_4)$	$y_9 = \alpha(B_4)$
$y_5 = \alpha t_{B_0}^{-1}(C)$	$y_{10} = \alpha(C)$

such that

$$W_4^\alpha = t_{a_1}^2 t_{y_1} t_{y_2} \cdots t_{y_{10}} = t_\delta.$$

Using the positive factorization $U = t_\delta$ in (5), we get the identity

$$UW_4^\alpha = \left(t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{\overline{B_2}} t_{\beta_0} t_{\beta_1} t_{\beta_2} t_{\beta_3} t_a^2 t_b^2 t_{\beta_0} t_{\beta_1} t_{\beta_2} \\ t_{\beta_3} t_a t_b\right)\left(t_{a_1}^2 t_{y_1} t_{y_2} \cdots t_{y_{10}}\right) = t_\delta^2 \qquad (6)$$

in $Mod(\Sigma_4^1)$. One can observe that the factorization $t_a t_b t_{a_1} t_{a_1}$ appears in UW_4^α and the curves $\{a, b, a_1, a_1\}$ bound a four-holed sphere. This allows us to use lantern relation. Hence, using the relation $t_a t_b t_{a_1} t_{a_1} = t_{C'} t_y t_{B_2'}$, we get the following identity:

$$t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{\overline{B_2}} t_{\beta_0} t_{\beta_1} t_{\beta_2} t_{\beta_3} t_a^2 t_b^2 t_{\beta_0} t_{\beta_1} t_{\beta_2} t_{\beta_3} (t_{C'} t_y \\ t_{B_2'}) t_{y_1} t_{y_2} \cdots t_{y_{10}} = t_\delta^2, \qquad (7)$$

where the Dehn twist curves C', y, and B_2' are illustrated as in Figure 4. Let U_1 and U_2 be the positive factorizations of t_δ^2 in (6)and (7) and X_1 and X_2 be the smooth 4-manifolds which admit the genus-4 Lefschetz fibrations associated to the monodromies U_1 and U_2 , respectively. Note that they admit a section (of self-intersection

$$(-e(X_i) = 4 - 4g + (\#\text{singular fibers})2)). \\ = 4 - 4(4) + 33 - i = 21 - i$$

For $i = 1{,}2$, the Euler characteristic of X_i , $e(X_i)$, is

The signature of X_1 , $\sigma(X_1)$, is computed as

$$\sigma(X_1) = \sigma(X_U) + \sigma(M_4) \\ = (-8) + (-4) = -12.$$

Since the monodromy factorization U_2 for the Lefschetz fibration on X_2 is obtained from the monodromy factorization U_1 for the one on X_1 by a lantern relation, the signature $\sigma(X_2)$ of X_2 satisfies

$$\sigma(X_2) = \sigma(X_1) + I_g(L) \\ = (-12) + 1 = -11.$$

Exotic Minimal 4-Manifolds with $b_2^+ = 3$

This section presents exotic $3\mathbb{C}P^2\#15\overline{\mathbb{C}P^2}$ and $3\mathbb{C}P^2\#14\overline{\mathbb{C}P^2}$ which both admit minimal genus-4 Lefschetz fibrations over $\mathbb{S}^2$.

Lemma 4. The 4-manifold X_1 is simply-connected.

Proof. Consider the monodromy of the Lefschetz fibration $X_1 \to \mathbb{S}^2$ given in (6). To compute the group $\pi_1(X_1)$, we cap off the boundary component δ, which leads to the following in $\mathrm{Mod}(\Sigma_4)$:

$$t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{\overline{B_2}} t_{\beta_0} t_{\beta_1} t_{\beta_2} t_{\beta_3} t_a^2 t_b^2 t_{\beta_0} t_{\beta_1} t_{\beta_2} t_{\beta_3} t_a t_b \\ t_{a_1}^2 t_{y_1} t_{y_2} \dots t_{y_{10}} = 1.$$

Since X_1 admits a section, by the theory of Lefschetz fibrations, we get the isomorphism:

$$\pi_1(X_1) \cong {}^{\pi_1(\Sigma_4)}/_{N},$$

where N is the normal closure of the subset of $\pi_1(\Sigma_4)$ containing all the vanishing cycles for X_1. So, $\pi_1(X_1)$ admits the following presentation with the standard generators a_i, b_i (for $i = 1,2,3,4$) and defining relations:

$$b_4^{-1} b_3^{-1} b_2^{-1} b_1^{-1} (a_1 b_1 a_1^{-1})(a_2 b_2 a_2^{-1})(a_3 b_3 a_3^{-1})(a_4 b_4 \ a_4^{-1}) = 1,$$

$$e = x_1 = x_2 = x_3 = d = \overline{B_2} = \beta_0 = \beta_1 = \beta_2 = \beta_3 \\ = a = b = a_1 = y_m = 1, \quad m = 1,2,\dots,10,$$

where the curves a_i and b_i's are illustrated in Figure 5. Hence, $\pi_1(X_1)$ has the relations as follows (among many others):

$$a_1 = 1 \qquad (8)$$

$$a = a_2 = 1 \qquad (9)$$

$$x_1 = a_2 b_3 a_3^{-1} a_4 b_3 a_3^{-1} a_4 b_4 = 1, \qquad (10)$$

$$\overline{B_2} = a_3 b_4 a_4 b_4^{-1} = 1, \qquad (11)$$

$$\beta_0 = b_1 b_2 b_3 = 1, \qquad (12)$$

$$\beta_2 = a_1 b_2 b_3 [b_4, a_4] a_3 b_3^{-1} = 1, \qquad (13)$$

$$\beta_3 = a_2 b_2 b_3 [b_4, a_4] a_3 b_3^{-1} a_3^{-1} a_2 = 1, \qquad (14)$$

$$y_7 = b_1 a_1^{-1} b_1 b_2 b_3 b_4^2 a_4^{-1} = 1, \qquad (15)$$

$$y_9 = b_1 b_2 b_3 b_4 b_3 a_3^{-1} a_4 b_4 a_4^{-1} b_1 b_2 a_2^{-1} = 1 \qquad (16)$$

where the vanishing cycles y_7 and y_9 are depicted in Figure 5. Here, for the commutator of α and β, we use the notation $[\alpha, \beta]$ (i.e. $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$).

Using the relations (8), (9), (13) and (14), we obtain $a_3 = 1$. And so, the relation (11) gives $a_4 = 1$. Also, by the relation (13), we have $b_2 = 1$. To show that $b_1 = b_3 = b_4 = 1$, first consider the relations (12) and (15), which yield $b_1 = b_3^{-1} = b_4^{-2}$. From these identities, the relation (16) turns into $b_4^2 = 1$, which gives $b_1 = b_3 = 1$. This finishes the proof since the relation (10) gives the requested identity $b_4 = 1$.

Lemma 5. The 4-manifold X_2 is simply-connected.

Prof. By capping off the boundary component δ in (5), in $\mathrm{Mod}(\Sigma_4)$, the following holds:

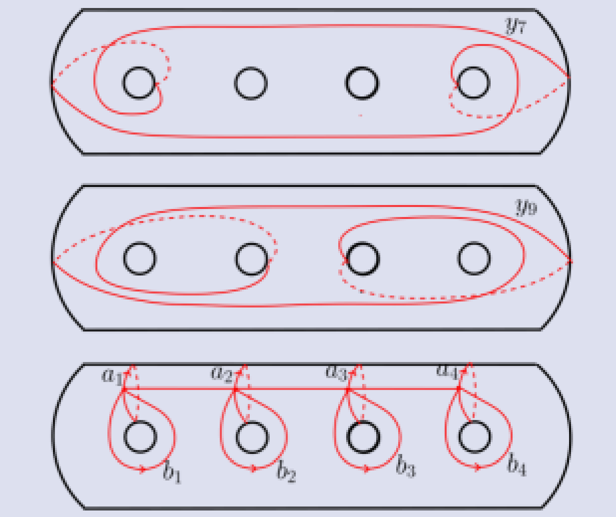

Figure 5. The Dehn twist curves y_7, y_9 and the generators of $\pi_1(\Sigma_4)$

$$t_e t_{x_1} t_{x_2} t_{x_3} t_d t_{\overline{B_2}} t_{\beta_0} t_{\beta_1} t_{\beta_2} t_{\beta_3} t_a^2 t_b^2 t_{\beta_0} t_{\beta_1} t_{\beta_2} t_{\beta_3} t_y t_{C'} t_{B_2'} \\ t_{y_1} t_{y_2} \dots t_{y_{10}} = 1$$

Similar to the computation of the fundamental group $\pi_1(X_1)$, the group $\pi_1(X_2)$ has the presentation with standard generators $a_1, a_2, a_3, a_4, b_1, b_2, b_3andb_4$ with defining relations

$$b_4^{-1} b_3^{-1} b_2^{-1} b_1^{-1} (a_1 b_1 a_1^{-1})(a_2 b_2 a_2^{-1})(a_3 b_3 a_3^{-1})(a_4 b_4 \\ a_4^{-1}) = 1,$$

$$e = x_1 = x_2 = x_3 = d = \overline{B_2} = \beta_0 = \beta_1 = \beta_2 = \beta_3 \\ = a = b = y = C' = B_2' = y_m = 1, \quad m = 1,2,\dots,10.$$

Thus, the relations $(8), (10) - (16)$ and with the following additional relation (among many others) hold in $\pi_1(X_2)$:

$$y = a_1 a_2^{-1} = 1, \qquad (17)$$

The relations (8) and (17) imply that $a_1 = a_2 = 1$. In the rest of the proof, one can follow the same steps as in the proof of Lemma 5.

Theorem 6. The 4-manifolds X_1 and X_2 are exotic $3\mathbb{C}P^2\#15\overline{\mathbb{C}P^2}$ and $3\mathbb{C}P^2\#14\overline{\mathbb{C}P^2}$, respectively.

Proof. Recall that for each $i = 1,2$, the Euler characteristic and the signature of X_i are as follows: $e(X_i) = 21 - i$ and $\sigma(X_i) = -13 + i$. By lemmata 4 and 5, $\pi_1(X_i) = 1$. Thus, the topological invariants $e(X_i)$ and $\sigma(X_i)$ of X_i are as follows.

$$e(X_i) = 2 - 2b_1(X_i) + b_2(X_i) \\ = 2 + b_2^+(X_i) + b_2^-(X_i) = 21 - i,$$

and

$\sigma(X_i) = b_2^+(X_i) - b_2^-(X_i) = -13 + i,$

which imply that $\left(b_2^+(X_i), b_2^-(X_i)\right) = (3,16 - i)$.

By Freedman's well-known classification theorem, one can conclude that X_i is in the homeomorphism classes of $3\mathbb{C}P^2\#(16-i)\overline{\mathbb{C}P^2}$. We can use [39] to conclude that X_1 is minimal. A lantern substitution can be considered as a rational blowdown surgery along a sphere of self-intersection -4 [15,18]. Also, this operation can be regarded as the symplectic sum. Thus, $X_2 = X_1\#_{V_{X_1}=V_{\mathbb{C}P^2}}\mathbb{C}P^2$ such that V_{X_1} is a symplectic sphere of self-intersection (-4) in X_1 and $V_{\mathbb{C}P^2}$ is a symplectic sphere of self-intersection $(+4)$ in $\mathbb{C}P^2$ that belongs to the class of $[V_{\mathbb{C}P^2}] = 2[H] \in H_2(\mathbb{C}P^2;\mathbb{Z})$. Since X_1 is minimal, we can exclude all the possible nonminimal cases in Theorem 2 and conclude that X_2 is minimal. However, there exists a smoothly embedded (-1)-sphere in $3\mathbb{C}P^2\#(16-i)\overline{\mathbb{C}P^2}$, that is, for each i, it is nonminimal. Thus, X_i is certainly not diffeomorphic to $3\mathbb{C}P^2\#(16-i)\overline{\mathbb{C}P^2}$. We also remark that $3\mathbb{C}P^2\#(16-i)\overline{\mathbb{C}P^2}$ are known to be nonsymplectic, which implies the exoticness of symplectic X_i for each $i = 1,2$.

Conflict of interest

There are no conflicts of interest in this work.

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