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The Unit-transmuted Lindley Distribution with Applications

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ABSTRACT

This paper introduces the unit-transmuted Lindley (unit-TL) distribution. The statistical properties of the unit-TL distribution defined between (0,1) are discussed in detail. Several estimation techniques are used to estimate the parameters of the unit-TL distribution. An analysis through simulation is carried out to evaluate the efficacy of the suggested model. Furthermore, a unique regression model is developed for bounded response variables based on the unit-TL distribution. To illustrate the importance of the suggested model in precisely describing restricted datasets, two distinct datasets are examined.

Keywords: Proportion data, Transmuted Lindley, Least squares, Simulation.

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Introduction

A lot of real-world scenarios require data (such percentages and proportions) that are restricted to a particular range. This is especially true in economic environments where nondurable consumption is given a percentage of income and industry market shares [1]. Such datasets require the use of flexible distributions to correctly represent them. Despite being widely utilized in many scientific studies, the Beta distribution has certain drawbacks, most notably the inability to describe its functions in an explicit form. Compared to the Beta distribution, the Topp-Leone distribution is simpler and has become more well-known. It was first presented by [2]. Similar to this, the Kumaraswamy distribution was used by many scientist, which was first presented by [3] and made widespread by [4].

Researchers used some mathematical transformations to propose distributions on (0,1) interval. These are $Y = X/(X+1)$ and $Y = \exp(-X)$. Using these transformations, researchers introduced several distributions such as unit-BS by [5], log-XG by[6], log-exponential power by [7], log-Bilal [8], log-WE by [9], a new kind of unit-Lindley by[10], unit-Chen by [11], continuous Bernoulli by [12], unit generalized half-normal by [13] and unit Gompertz by [14].

This study aims to introduce an innovative distribution with closed and accessible statistical features, defined on the unit interval. Compared to popular distributions like Beta, Kumaraswamy, and Topp-Leone, this one has a number of advantages. A closed form of the probability density function (pdf) and moments are among the statistical characteristics of the unit-TL distribution that may be determined. It allows for the introduction of a regression model of the proposed distribution and shows a better fit than other widely recognized distributions constructed on the unit interval.

The paper is divided as follows: A thorough summary of the mathematical characteristics connected to the suggested distribution is given in Section 2. The methods for estimating parameters, such as weighted least squares, least squares, and maximum likelihood, are covered in Section 3. The simulation study that evaluates the effectiveness of the parameter estimation techniques in finite samples are shown in Section 4. In Section 5, an innovative regression model that uses the generalized linear model approach is introduced as a complement to the Beta regression model. To demonstrate the adaptability of the unit-TL distribution in comparison to popular distributions in (0,1) interval, two real datasets are investigated in Section 6. The paper gets to its conclusion in Section 7.

The Unit-Transmuted Lindley Distribution

The pdf of the Lindley distribution is

$$f(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)\,exp(-\theta x)\,, x > 0. \tag{1}$$

where $\theta > 0$ is scale parameter. It is possible to represent the pdf provided in (1) as a combination of gamma and exponential distributions. The cumulative distribution function (cdf) is

$$F(x) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1} exp(-\theta x)\,, x \geq 0. \tag{2}$$

Several authors have examined the Lindley distribution and its statistical characteristics were deduced by [15]. There are several Lindley distribution generalizations in the statistical literature. [16] presented a novel Lindley distribution, known as the transmuted Lindley (TL) distribution, with the following pdf using the transmutation idea.

$$f(x;\theta,\lambda)=\frac{\theta^2}{\theta+1}(1+x)\,exp(-\theta x)\left(1-\lambda+2\lambda\frac{1+\theta+\theta x}{\theta+1}exp(-\theta x)\right),x>0. \tag{3}$$

Here, the parameter $|\lambda|\leq 1$ is the transmutation parameter. The cdf of (3) is

$$F(x;\theta,\lambda)=\left(1-\frac{1+\theta+\theta x}{\theta+1}exp(-\theta x)\right)\left(1+\lambda\frac{1+\theta+\theta x}{\theta+1}exp(-\theta x)\right),x\geq 0. \tag{4}$$

Now, using $Y=X/(X+1)$ transformation in (3), we have

$$f(y;\theta,\lambda)=\frac{\theta^2}{\theta+1}(1-y)^{-3}\,exp\left(-\frac{\theta y}{1-y}\right)\left(1-\lambda+2\lambda\frac{1+\theta+\frac{\theta y}{1-y}}{\theta+1}exp\left(-\frac{\theta y}{1-y}\right)\right),0<y<1. \tag{5}$$

The Equation (5) is symbolized by $Y\sim$ unit-TL(θ,λ). Its cdf is

$$F(y;\theta,\lambda)=\left(1-\frac{1+\theta+\frac{\theta y}{1-y}}{\theta+1}exp\left(-\frac{\theta y}{1-y}\right)\right)\left(1+\lambda\frac{1+\theta+\frac{\theta y}{1-y}}{\theta+1}exp\left(-\frac{\theta y}{1-y}\right)\right). \tag{6}$$

Figure 1 shows the pdf plots of the unit-TL distribution for various values of the parameters. Both a left- and a right-skewed unit-TL distribution are possible.

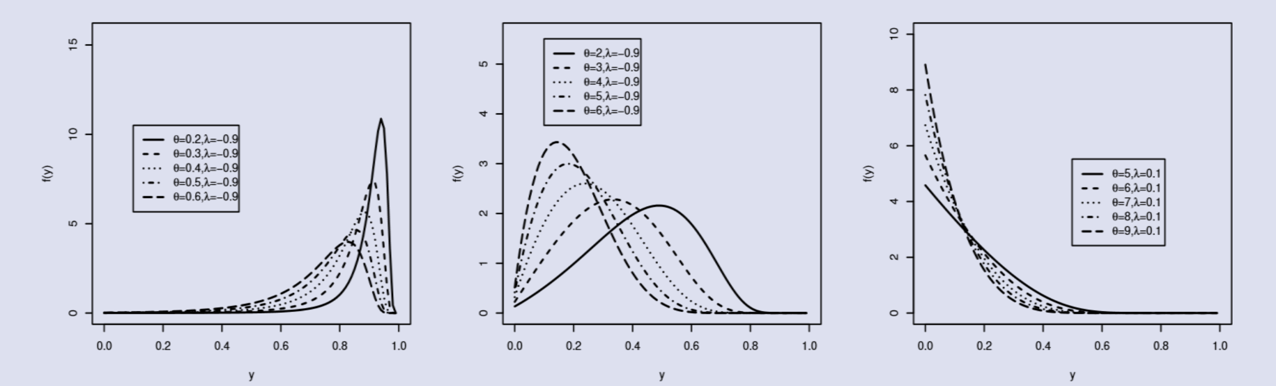

Figure 1. The pdf plots of the model.

The survival function is obtained as $S(y)=1-F(y)$, and given by

$$S(y;\theta,\lambda)=1-\left(1-\frac{1+\theta+\frac{\theta y}{1-y}}{\theta+1}exp\left(-\frac{\theta y}{1-y}\right)\right)\left(1+\lambda\frac{1+\theta+\frac{\theta y}{1-y}}{\theta+1}exp\left(-\frac{\theta y}{1-y}\right)\right),0\leq y\leq 1. \tag{7}$$

Using the pdf and cdf, the hazard rate function (hrf) is

$$\begin{aligned}H(y;\theta,\lambda)&=\frac{\theta^2}{\theta+1}(1-y)^{-3}\,exp\left(-\frac{\theta y}{1-y}\right)\left(1-\lambda+2\lambda\frac{1+\theta+\frac{\theta y}{1-y}}{\theta+1}exp\left(-\frac{\theta y}{1-y}\right)\right)\\&\times\left[1-\left(1-\frac{1+\theta+\frac{\theta y}{1-y}}{\theta+1}exp\left(-\frac{\theta y}{1-y}\right)\right)\left(1+\lambda\frac{1+\theta+\frac{\theta y}{1-y}}{\theta+1}exp\left(-\frac{\theta y}{1-y}\right)\right)\right]^{-1}.\end{aligned} \tag{8}$$

The hrf forms of the unit-TL model is illustrated (see, Figure 2). The unit-TL has only an increasing form of the hrf.

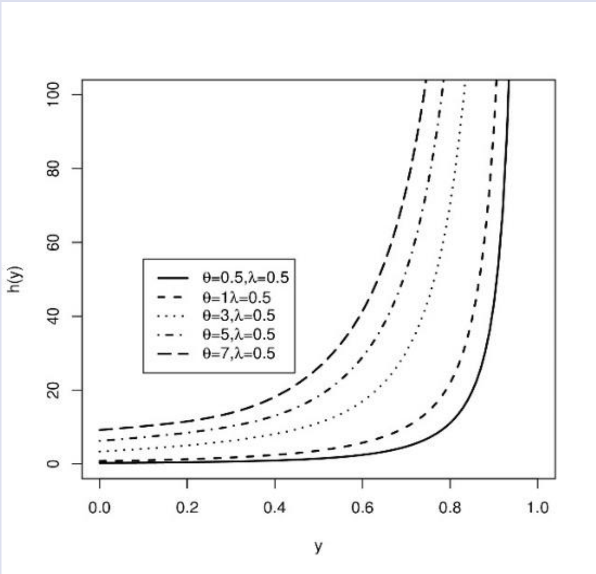

Figure 2. The hazard plots of unit-TL.

Generating random variables from the unit-TL distribution can be done using the properties of the Lindley and transmuted distributions. The cdf of the transmuted distribution is

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x) \tag{9}$$

where $|u| \leq 1$ and $G(x)$ is the cdf of the Lindley distribution for the transmuted-Lindley distribution. The quantile function of the transmuted distribution can be obtained for the general class of these distributions, as follows

$$Q_G = \left(\frac{\lambda - \sqrt{2\lambda - 4\lambda u + \lambda^2 + 1} + 1}{2\lambda}\right) \tag{10}$$

where $Q_G(\cdot)$ is the quantile function of the Lindley distribution for transmuted-Lindley distribution. The qlindley function defined in the LindleyR package of the R software can be used for that purpose. The below algorithm can be easily implemented in R software to generate random variables from the unit-TL distribution.

1. Define λ and θ parameters.
2. Generate random variables from standard uniform distribution, $u \sim U(0,1)$.
3. Calculate $x_u = \frac{\lambda - \sqrt{2\lambda - 4\lambda u + \lambda^2 + 1} + 1}{2\lambda}$
4. Using the quantile function of the Lindley distribution, calculate $x = \mathrm{qlindley}(x_u, \theta)$
5. Apply the following transformation $y = \frac{x}{(x+1)}$
6. Repeat steps 2-5 n times.

The integration in (11) may be solved to obtain the moments of the unit-TL distribution.

$$E(Y^k) = \frac{\theta^2}{\theta+1}\int_0^1 y^k\,(1-y)^{-3}\,exp\left(-\frac{\theta y}{1-y}\right)\left(1-\lambda+2\lambda\frac{1+\theta+\frac{\theta y}{1-y}}{\theta+1}exp\left(-\frac{\theta y}{1-y}\right)\right)dy. \tag{11}$$

(11) cannot have an analytical solution. For a given value of *k*, however, the integration in (11) can be solved. We have the mean of the unit-TL distribution for *k* = 1.

$$E(Y) = \frac{\theta\left(1-\frac{\lambda}{2}\right)+1}{(\theta+1)^2} \tag{12}$$

Estimation

In this section, three parameter estimation methods are described for the unit-TL distribution.

Maximum likelihood estimation

Assume that the unit-TL distribution is the distribution of the random samples $y_1, y_2, y_3, \ldots, y_n$. The $\ell(\theta,\lambda)$ function is provided by

$$\ell(\theta,\lambda) = n\,log\left(\frac{\theta^2}{\theta+1}\right) - 3\sum_{i=1}^{n} log(1-y_i) - \theta\sum_{i=1}^{n}\frac{y_i}{1-y_i} + \sum_{i=1}^{n} log\left(\left(1-\lambda+2\lambda\frac{1+\theta+\frac{\theta y_i}{1-y_i}}{\theta+1}exp\left(-\frac{\theta y_i}{1-y_i}\right)\right)\right) \tag{13}$$

We obtain the normal equations by calculating partial derivatives of (13) with regard to θ and λ.

$$\frac{\partial \ell}{\partial \theta} = n(\theta+1)\theta^{-2}\left(\frac{2\theta}{\theta+1} - \frac{\theta^2}{(\theta+1)^2}\right) - z_i + \frac{\begin{pmatrix} -2\lambda y_i \exp(-\theta z_i)(\theta+\theta z_i+1)\{(\theta+1)(1-y_i)\}^{-1} \\ -2\lambda \exp(-\theta z_i)(\theta+\theta z_i+1)(\theta+1)^{-2} \\ +2\lambda \exp(-\theta z_i)(z_i+1)(\theta+1)^{-1} \end{pmatrix}}{1-\lambda+2\lambda y_i(\theta+1)^{-1}\exp(-\theta z_i)(\theta+\theta z_i+1)},$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{2\exp(-\theta z_i)(\theta+\theta z_i+1)(\theta+1)^{-1} - 1}{1-\lambda+2\lambda\exp(-\theta z_i)(\theta+\theta z_i+1)(\theta+1)^{-1}}.$$

Where $z_i = y_i/(1-y_i)$ and $(\hat{\theta}, \hat{\lambda})$ are the maximum likelihood estimates (MLEs) of the following equations when they are solved simultaneously. These equations incorporate complex functions, which makes the solution impossible to get the MLEs in an explicit form. As such, the solution requires the application of numerical methods. Using the statistical software R, one may determine the MLEs of the parameters.

Least Squares (LS) and Weighted LS (WLS) Estimations

Let $Y_{(i)}$ for $i = 1,2,3,\ldots,n$ be the random variable denotes the ordered samples from the unit-TL distribution. To obtain the LSEs estimations, the function given in (14) is minimized.

$$\sum_{i=1}^{n}\left[\left(1 - \frac{1+\theta+\frac{\theta y_{(i)}}{1-y_{(i)}}}{\theta+1} exp\left(-\frac{\theta y_{(i)}}{1-y_{(i)}}\right)\right)\left(1 + \lambda\frac{1+\theta+\frac{\theta y_{(i)}}{1-y_{(i)}}}{\theta+1} exp\left(-\frac{\theta y_{(i)}}{1-y_{(i)}}\right)\right) - \frac{i}{n+1}\right]^2. \quad (14)$$

Also, the equation (15) is minimized to get WLSEs of the model parameters.

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left(1 - \frac{1+\theta+\theta z_{(i)}}{\theta+1} exp(-\theta z_{(i)})\right)\left(1 + \lambda\frac{1+\theta+\theta z_{(i)}}{\theta+1} exp(-\theta z_{(i)})\right) - \frac{i}{n+1}\right]^2. \quad (15)$$

Simulation

Using the estimation methods mentioned in previous section, bias, mean squared error (MSE) and mean relative error (MRE) are calculated. The anticipation is to observe that when the sample size increases, the bias and MSE should be near the zero value and MRE should be near the one value. The parameters of the unit-TL distribution are defined as θ = 2, λ = 0.5. The size of the sample is raised from 50 to 500 by 5 units.

Figures 3-5 present a graphical summary of the simulation findings. Large sample sizes are associated with biases and MSEs that are, as predicted, close to zero in estimating methods.

The MREs are also close to the one. Nevertheless, compared to other estimate techniques, the MLE method's biases and MSE values approach the required values more quickly. Furthermore, when the sample size is small, the bias and MSE of the MLE approach are lower than other methods. Therefore, we advise estimating the unit-TL distribution's parameters using the MLE approach.

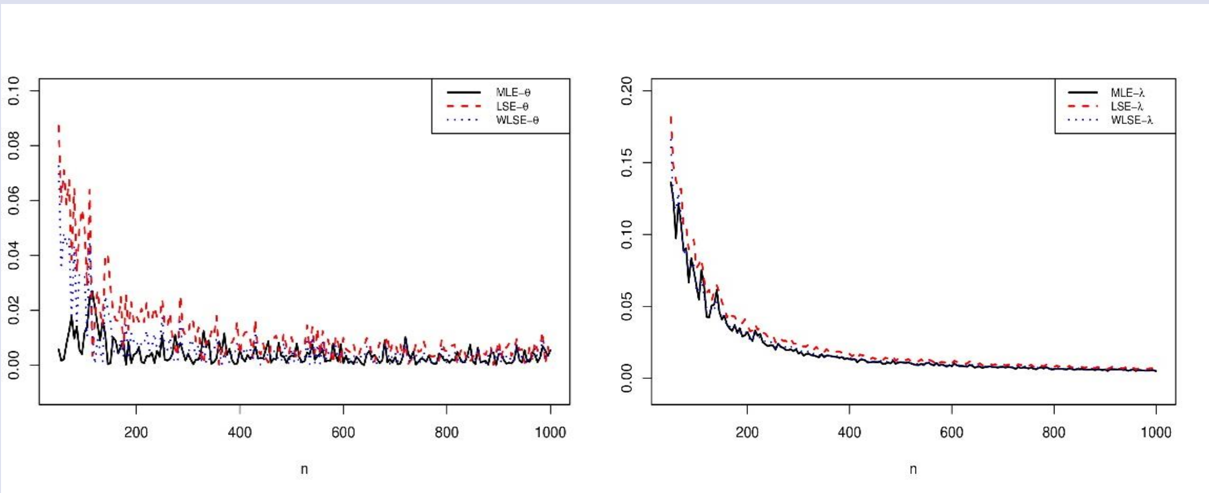

Figure 3. Estimated biases.

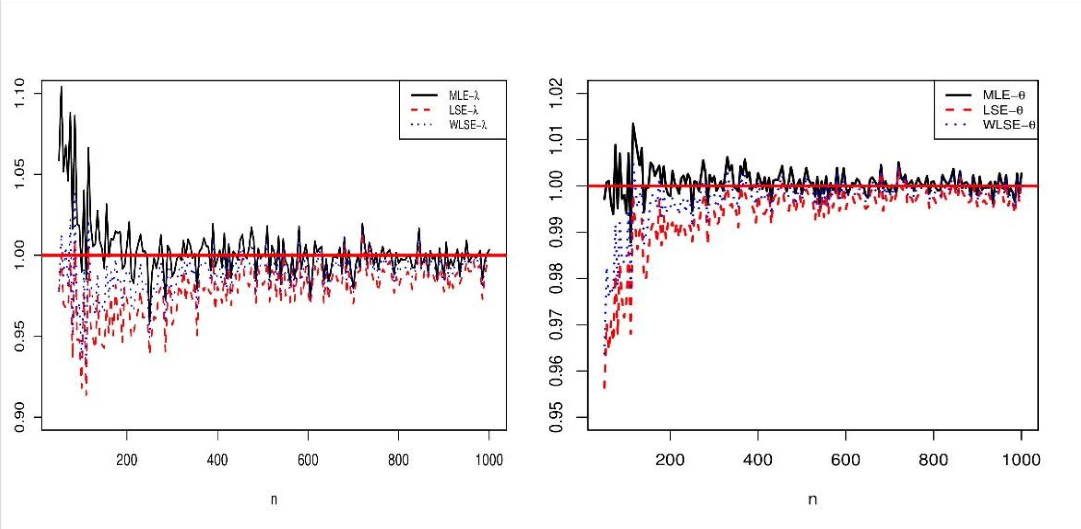

Figure 4. Estimated MREs.

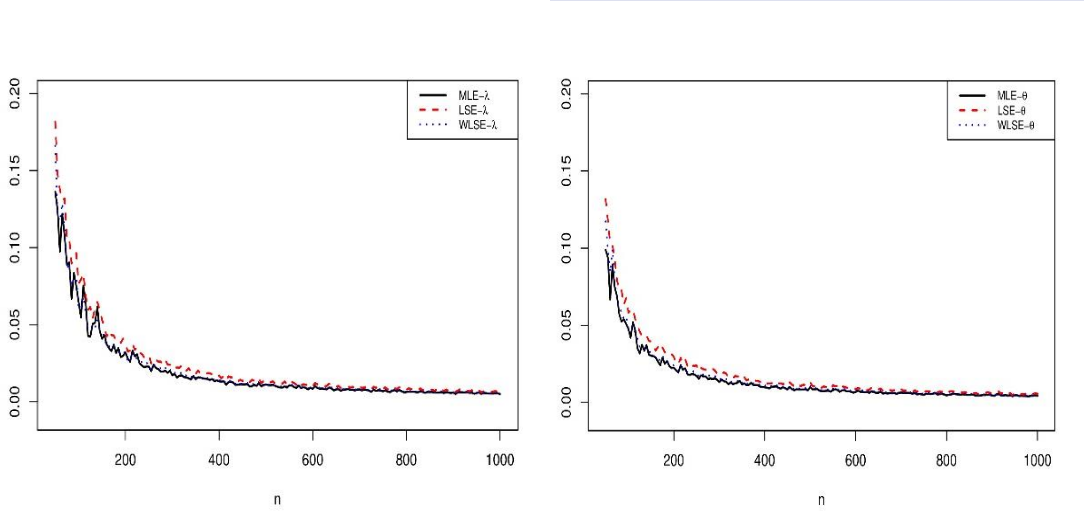

Figure 5. Estimated MSEs.

The Unit-Transmuted Lindley Regression Model

We present an alternative regression model that offers a new approach to modeling of bounded dependent variable with covariates. Let $\theta = (2\mu)^{-1}\left[\left(1-\frac{\lambda}{2}-2\mu\right)+\sqrt{4\mu+\left(1-\frac{\lambda}{2}\right)^2-4\mu\left(1-\frac{\lambda}{2}\right)}\right]$, the pdf is

$$f(y;\mu,\lambda) = \frac{\gamma(\mu,\lambda)^2}{\gamma(\mu,\lambda)+1}(1-y)^{-3}\,exp\left(-\frac{\gamma(\mu,\lambda)y}{1-y}\right)\times\left(1-\lambda+2\lambda\frac{1+\gamma(\mu,\lambda)+\frac{\gamma(\mu,\lambda)y}{1-y}}{\gamma(\mu,\lambda)+1}exp\left(-\frac{\gamma(\mu,\lambda)y}{1-y}\right)\right), \quad (16)$$

Where

$$\gamma(\mu,\lambda) = (2\mu)^{-1}\left[\left(1-\frac{\lambda}{2}-2\mu\right)+\sqrt{4\mu+\left(1-\frac{\lambda}{2}\right)^2-4\mu\left(1-\frac{\lambda}{2}\right)}\right]. \quad (17)$$

In the re-parametrization, we have $E(Y)=\mu$. As in the beta regression method, the independent variables are connected to the dependent variable via link function. Since the dependent variable is defined on (0,1) interval, we use the logit-link function

$$\mu_i = \frac{exp(x_i^T\beta)}{1+exp(x_i^T\beta)}, i=1,\dots,n. \quad (18)$$

Substituting (18) in (16), the log-likelihood function is

$$\ell(\beta,\lambda)=\sum_{i=1}^{n} ln\left(\frac{\gamma(\mu_i,\lambda)^2}{\gamma(\mu_i,\lambda)+1}\right)-3\sum_{i=1}^{n} ln(1-y_i)-\sum_{i=1}^{n}\gamma(\mu_i,\lambda)\frac{y_i}{1-y_i} + \sum_{i=1}^{n} ln\left(1-\lambda+2\lambda\frac{1+\gamma(\mu_i,\lambda)+\frac{\gamma(\mu_i,\lambda)y_i}{1-y_i}}{\gamma(\mu_i,\lambda)+1}exp\left(-\frac{\gamma(\mu_i,\lambda)y_i}{1-y_i}\right)\right) \tag{19}$$

where μ_i is as in (18). The maximization of the log-likelihood function described in Equation (19) is achieved through the optim function in the R software. The asymptotic standard errors are then calculated by help of the inverse of the observed information matrix.

The residuals, as defined by [17] are employed to assess deviations from the assumption of error. These residuals are defined as:

$$\hat{e}_i=-ln[1-F(y_i)], i=1,2,\dots,n, \tag{20}$$

where $F(y_i)$ is the estimated cdf.

Applications

Water Capacity Data

This section employs a real data application to compare the unit-TL model with Beta, Kumaraswamy, and Topp-Leone distributions. Table 1 presents the MLEs, A, W, AIC and BIC. Here, A and W represent the Anderson-Darling and Cramer-von Mises, respectively. Also, KS represents the Kolmogorov-Smirnov test statistic value.

The dataset consists of monthly water capacity data from the Shasta reservoir in California, USA, spanning from February 1991 to 2010. The reservoir has a maximum capacity of 4552000 AF, and the data was normalized to the interval [0,1] using a normalization equation. [18] previously analyzed this dataset.

The hazard shape information can assist in selecting an appropriate model. [19] developed a useful tool for this purpose, called as TTT plot. A straight diagonal shape in the TTT plot indicates a constant hazard, while a convex shape suggests decreasing hazards and a concave shape indicates increasing hazards. A bathtub shape in the hazard occurs when it transitions from convex to concave. Examination of Figure 6 reveals that the hrf shape of the data is increasing. Therefore, utilizing a unit-TL distribution appears to be an effective choice for modeling this dataset.

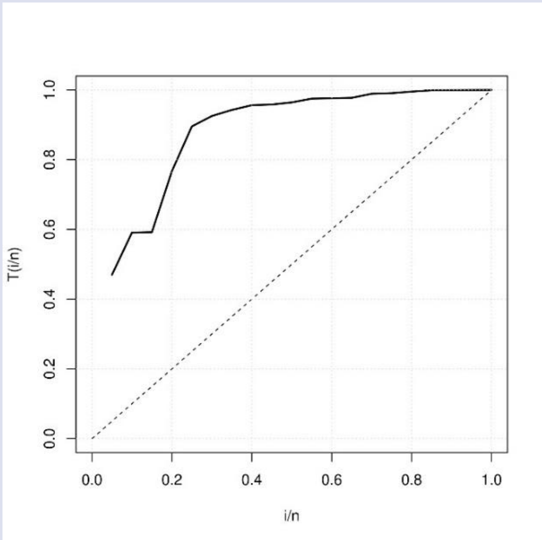

Figure 6. TTT plot of the dataset.

Table 1 displays the estimated parameters, standard errors, and other statistics for the monthly water capacity dataset. The values in Table 1 clearly show that the goodness-of-fit statistics for the unit-TL distribution are at their lowest values. For the monthly water capacity dataset, the proposed distribution may thus be thought to be the best fit model.

Table 1. Estimated parameters of the fitted models.

Distributions	Parameters		$-\ell$	AIC	BIC	A	W	K-S	p
Beta	7.3154	2.9098	-12.5619	-21.1239	-19.1324	1.6192	0.2796	0.2359	0.1834
	2.3180	0.8754							
Kumaraswamy	6.3476	4.4893	-13.4747	-22.9494	-20.9580	1.4245	0.2407	0.2209	0.2447
	1.5575	2.0409							
Topp-Leone	8.6664		-11.5876	-21.1753	-20.1795	1.7864	0.3134	0.2549	0.1241
	1.9379								
unit-TL	0.6764	0.9990	-16.0715	-28.1431	-26.1516	0.9005	0.1394	0.1795	0.4847
	0.1120	0.3937							

Figure 7 displays the fitted functions of the model on the dataset. As can be seen from the right panel of Figure 7, out of all the models, the unit-TL distribution provides the best match to the monthly water capacity dataset.

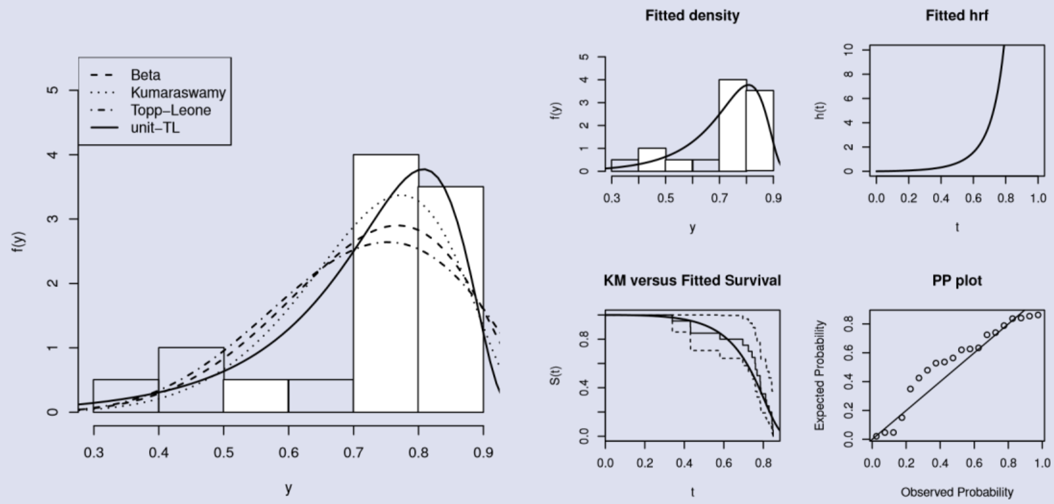

Figure 7: Comparison of the fitted densities (left-panel) and pdf, sf, hrf and PP plot of the unit-TL (right-panel).

Application of regression

In this part, the unit-TL regression model is checked against the beta regression model using the OECD Better Life Index (BLI) dataset. We use the R software's betareg package to obtain the parameters of the beta regression model.

The purpose of this application is to determine how the variables of water quality (x_1), air pollution (x_2) and murder rate (x_3) relate to self-reported health (y). One may get the dataset that was utilized in this study at the website of the OECD.

The logit link function is used in both regression models. As a result, the following represents the regression structure for μ_i.

$$\text{logit}(\mu_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} \tag{20}$$

Table 2: Results of the fitted regression models.

Parameters	Beta			unit-TL		
	Estimate	SE	p	Estimate	SE	p
β_0	0.9068	1.1491	0.4300	0.9844	1.3438	0.4638
β_1	0.5049	1.1506	0.6608	0.3943	1.3481	0.7699
β_2	-0.0424	0.0188	0.0237	-0.0450	0.0221	0.0416
β_3	-0.6673	1.9246	0.7288	-0.9082	2.1119	0.6672
ϕ	14.4130	3.2140	<0.001	-	-	-
λ	-	-	-	-0.9990	0.7302	0.1709
$-\ell$		-28.0600			-30.1429	
AIC		-46.1200			-50.2858	
BIC		-37.9321			-42.0979	

Table 2 provides a summary of the models. Interestingly, in both regression models, the parameter air pollution is found significant that has a detrimental effect on self-reported health.

Utilizing the computed AIC, and BIC values is essential in selecting the most suitable model. With the unit-TL distribution exhibiting the lowest values for these statistics, it is evident that the proposed model outperforms the alternative model in terms of fitting performance for the dataset. Furthermore, Figure 8 illustrates the Cox-Snell residuals for the unit-TL regression model, demonstrating the proximity of the plotted points to the diagonal line.

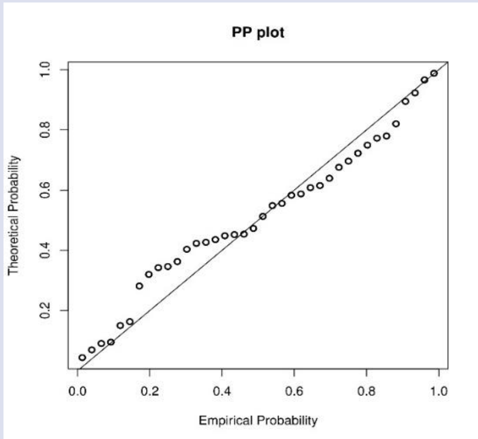

Figure 8. PP plot of the residuals.

Conclusion

We introduce a unique distribution with restricted support and two parameters. A number of statistical properties are obtained. Through a simulation study, the estimate of unknown parameters of the unit-TL distribution is explored using weighted least squares, maximum likelihood, and least squares approaches. To compare the performance and adaptability of the unit-TL distribution with competitive distributions, two datasets are analyzed. Additionally, the unit-TL regression constructed for the modeling of the bounded dependent variable is presented.

Conflict of Interest

There is no conflict of interest.

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