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Point Estimation for the Inverse Rayleigh Distribution under Type-II Left and Right Censoring

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ABSTRACT

The Inverse Rayleigh distribution is frequently utilized in reliability and survival analysis. This study focuses on deriving modified maximum likelihood estimators for the scale parameter of the Inverse Rayleigh distribution under Type-II left and right censoring. The efficacy of the proposed estimators is assessed through comparison with Anderson-Darling, Kolmogorov-Smirnov, and Cramér-von Mises type estimators via Monte Carlo simulations across various censoring schemes and parameter configurations. Additionally, a numerical example is presented to illustrate the proposed methodology. The simulation study demonstrates that the proposed estimators outperform the others. Additionally, given their explicit nature, the proposed estimators can serve as initial values for obtaining the maximum likelihood estimator.

Keywords: Anderson-Darling statistic, Cramér-von mises, Left and right censoring, Kolmogorov-Smirnov statistics, Modified maximum likelihood estimation.

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Introduction

The Rayleigh distribution has long been a cornerstone in statistical modeling due to its simplicity and broad applicability in diverse fields such as physics, engineering, and environmental studies as a special case of the Weibull distribution. Thanks to its versatility, the Rayleigh distribution has found applications ranging from reliability analysis to quality control. Recent studies, such as [1]-[3], highlight its continued relevance in addressing real-world problems.

The inverse Rayleigh (IR) distribution, derived from the Rayleigh distribution, extends this utility by providing a flexible model for reliability and survival analysis. Specifically, the IR distribution emerges when the reciprocal of a Rayleigh-distributed random variable is considered. If X follows a Rayleigh distribution, then $Y = \frac{1}{X}$ follows an IR distribution. Its probability density function (pdf) and cumulative distribution function (cdf) are given, respectively, by

$$f(x;\theta) = \frac{2\theta^2}{x^3} exp\left\{-\frac{\theta^2}{x^2}\right\}, x > 0, \theta > 0 \quad (1)$$

$$F(x;\theta) = exp\left\{-\frac{\theta^2}{x^2}\right\}, \quad (2)$$

where θ is a scale parameter.

The IR distribution has been studied in various contexts. Some recent studies are given as follows: [4] focused on a group acceptance sampling plan for truncated life tests. [5] discussed the estimation problem, both from a Bayesian and non-Bayesian perspective based on lower record values. [6] studied the characteristics of shrinkage test-estimators considering an asymmetric loss function. [7] explored the Bayesian estimation of parameter and the reliability function of the IR distribution. [8] examined the estimation of parameters for the IR distribution based on Type-I hybrid censored samples. [9] developed a moving average control chart for monitoring failures under a time-truncated test when item lifetimes follow Rayleigh and IR distributions, evaluating its performance using average run lengths (ARL). [10] introduced E-Bayesian and Hierarchical Bayesian estimation methods for estimating the scale parameter and reversed hazard rate of the IR distribution. [11] derived Bayes estimators for the parameter of an IR distribution under symmetric and asymmetric loss functions. Recently, [12] discussed the estimation of process capability index when the underlying distribution follows IR distribution.

Type-II censored samples are frequently encountered in various applications, such as life testing. In this type of censoring, only the smallest or largest observations are not observed. The censoring mechanism involves testing n items until the first non-event or failure time is observed, leading to the termination of the experiment. Statistical inferences based on Type-II censored samples have been addressed in many studies, for example, they can be found in [13-18] and the referenced cited therein. For detailed insights into Type-II censored samples, we refer to the works of [19] and [20].

The motivation for this study arises from the significant gap in the literature regarding the IR distribution under Type II left and right censoring. Existing methods for obtaining maximum likelihood (ML) estimates often rely on search algorithms such as Nelder-Mead or BFGS, which are sensitive to initial values and computationally demanding. To address this issue, we aim to develop a more efficient and robust method that eliminates dependence on initial values, simplifying the estimation process while maintaining accuracy. In this regard, this study discusses point estimation for the IR distribution parameter under Type-II left and right censoring. ML, modified maximum likelihood (MML), least squares (LS), Anderson-Darling type (AD), and Cramér-von Mises type (CvM) estimation methods are proposed in Section 2. The performance of the proposed estimators is compared through a simulation study in Section 3. In Section 4, a numerical example is also proposed for illustrative purposes. Section 5 closes the paper with concluding remarks.

Point Estimation Under Type-II Left and Right Censoring

In this section, we present some explicit estimators by modifing ML estimator under Type-II left and right censoring. We compare the proposed estimators with the ML, AD, CvM, and LS estimation methods based on simulated samples. The AD, CvM, and LS methods are adapted for Type-II censored sample. It is noted that [21-25] used these estimators based on complete sample.

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ represent the ordered statistics from $IR(\theta)$ distribution. Then, the likelihood function based on Type-II censored data with a censoring of r_1 observations on the left and r_2 observations on the right can be expressed in general form as follows:

$$L(\theta|\mathbf{x}) \propto \left(\prod_{i=r_1+1}^{n-r_2} f\left(x_{(i)};\theta\right)\right)\left(F\left(x_{(a)}\right)\right)^{r_1}\left(1-F\left(x_{(b)}\right)\right)^{r_2}.$$

Hence, the log-likelihood function is given by

$$\ell(\theta|\boldsymbol{x}) \propto 2(n-r_1-r_2)\log(\theta) - \theta^2 \sum_{i=r_1+1}^{n-r_2} \frac{1}{x_{(i)}^2} - \frac{r_1\theta^2}{x_{(a)}^2} + r_2 \log\left(1-\exp\left\{-\frac{\theta^2}{x_{(b)}^2}\right\}\right), \tag{3}$$

where $a = r_1 + 1$ and $b = n - r_2$. Then associated gradient found to be

$$\frac{d\ell(\theta|\boldsymbol{x})}{d\theta} = \frac{2(n-r_1-r_2)}{\theta} - 2\theta \sum_{i=r_1+1}^{n-r_2} \frac{1}{x_{(i)}^2} - \frac{2r_1\theta}{x_{(a)}^2} + \frac{2r_2\theta \exp\left\{-\frac{\theta^2}{x_{(b)}^2}\right\}}{x_{(b)}^2\left(1-\exp\left\{-\frac{\theta^2}{x_{(b)}^2}\right\}\right)}. \tag{4}$$

The observed information matrix can be obtained by differentiating (4) with respect to the parameter θ and negating the resulting expressions. Hence the observed information matrix is obtained as

$$-\frac{d^2\ell(\theta|\mathbf{x})}{d\theta^2} = -\frac{2(n-r_1-r_2)}{\theta^2} - 2\sum_{i=a}^{b}\frac{1}{x_{(i)}^2} - \frac{2r_1}{x_{(a)}^2} - \frac{2r_2\exp\left\{-\frac{\theta^2}{x_{(b)}^2}\right\}\left(-x_{(b)}^2 + x_{(b)}^2\exp\left\{-\frac{\theta^2}{x_{(b)}^2}\right\} + 2\theta^2\right)}{x_{(b)}^4\left(-1+\exp\left\{-\frac{\theta^2}{x_{(b)}^2}\right\}\right)^2}.$$

Then, the ML estimator of parameter θ can be defined as

$$\hat{\theta}_1 = \arg\max\ \ell(\theta|\mathbf{x}).$$

$\hat{\theta}_1$ can be also obtained by the solution of likelihood equation $d\ell(\theta|\mathbf{x})/d\theta = 0$, however it easily be seen that there is no explicit solution for θ. Therefore, numerical methods such as Newton-Raphson or Brent can be used to achieve the solution. Fixed-point iteration, a widely recognized numerical method, can be easily implemented as an alternative. Here, we give a fixed-point iteration for the solution of the likelihood equation. Utilizing the likelihood equation, the fixed-point iterations can be obtained as

$$\theta^{(h+1)} = \frac{G\left(\theta^{(h)}\right)x_{(a)}}{4\left(\sum_{i=a}^{b}\frac{x_{(a)}^2}{x_{(i)}^2}+r_1\right)} + \frac{4x_{(a)}\left\{x_{(a)}^2\left((n-r_2)\sum_{i=a}^{b}\frac{1}{x_{(i)}^2}+\frac{G\left(\theta^{(h)}\right)}{16}\right)-r_1^2-\left(\sum_{i=a}^{b}\frac{x_{(a)}^2}{x_{(i)}^2}-n+r_2\right)r_1\right\}^{1/2}}{4\left(\sum_{i=a}^{b}\frac{x_{(a)}^2}{x_{(i)}^2}+r_1\right)}, \tag{5}$$

where

$$G\left(\theta^{(h)}\right) = \frac{2r_2\theta^{(h)}\exp\left\{-\left(\frac{\theta^{(h)}}{x_{(b)}}\right)^2\right\}}{x_{(b)}^2\left(1-\exp\left\{-\left(\frac{\theta^{(h)}}{x_{(b)}}\right)^2\right\}\right)}.$$

It is noted that the search methods may suffer from the initial values. It is desired to explicit estimators which do not need an initial value. They can also be used as initial values for searching algorithms. Let us start to construct the MML estimators. Consider the transformation $Z_{(b)} = X_{(b)}/\theta$. It is clear that $Z_{(b)}$ is the bth order statistic from the IR(1) distribution which is independent of parameter θ. Then, the likelihood equation can be re-written as

$$\frac{2(n-r_1-r_2)}{\theta} - 2\theta \sum_{i=r_1+1}^{n-r_2} \frac{1}{x_{(i)}^2} - \frac{2r_1\theta}{x_{(a)}^2} + \frac{r_2}{\theta} z_b g(z_b) = 0, \tag{6}$$

$$g(z) = \frac{f(z;1)}{1-F(z;1)} = \frac{2\,exp\left(-\frac{1}{z^2}\right)}{z^3\left\{1-exp\left(-\frac{1}{z^2}\right)\right\}}. \tag{7}$$

Eq. (6) does not also admit an explicit solution for θ due to the complex nature of $g(z)$. [13] utilized hyperbolic approximation as a method to approach $z_b g(z_b)$, and we intend to employ the same technique initially in addressing our own problem. Hence, we use an equation hyperbola in the first quadrant

$$zg(z) = K_1,$$

where the value of K_1 can be obtained by using any two points h_1 and h_2 on the curve that is very close to each other, and denoted by K_1. Then, as $h \to \infty$, h_1 and h_2 tend to a common value of h. Therefore,

$$-\frac{g(h_2)-g(h_1)}{h_2-h_1} = \frac{K_1}{h_1 h_2}, \tag{8}$$

and h is given by

$$h = \frac{1}{\sqrt{log\left(\frac{n+1}{b}\right)}}$$

which is solution of the equation $F(h;1) = \frac{n-r_2}{n}$. Then (8) is reduced to

$$K_1 = -h^2\left[\frac{d}{dz}g(z)\right]_{z=h} = \frac{2\,exp\left(-\frac{1}{h^2}\right)\left\{3h^2 - 3h^2\,exp\left(-\frac{1}{h^2}\right)+2\right\}}{h^4\left\{\left(1-exp\left(-\frac{1}{h^2}\right)\right)\right\}^2}. \tag{9}$$

Substituting (9) into (6), we have the modified likelihood equation

$$\frac{d\ell(\theta|x)}{d\theta} = \frac{2(n-r_1-r_2)}{\theta} - 2\theta \sum_{i=r_1+1}^{n-r_2} \frac{1}{x_{(i)}^2} - \frac{2r_1\theta}{x_{(a)}^2} + \frac{r_2}{\theta}K_1 = 0, \tag{10}$$

and the solution to this equation with respect to θ gives the MML, which is given by

$$\hat{\theta}_2 = x_{(a)}\left(\frac{2(n-r_1-r_2)+r_2K_1}{2\left(x_{(a)}^2\sum_{i=r_1+1}^{n-r_2}\frac{1}{x_{(i)}^2}+r_1\right)}\right)^{\frac{1}{2}}. \tag{11}$$

Now, we propose an alternative method for approximate to $z_b g(z_b)$. Traditionally, $E(Z_b)$ can be treated as Z_b, which can be easily approximated by

$$\hat{z}_b \approx \frac{1}{\sqrt{log\left(\frac{n+1}{b}\right)}}, \tag{12}$$

where, (12) arises from the facts

$$Z_b \overset{d}{=} F^{-1}(U_{(b)};1)$$

and

$$E(Z_b) \approx F^{-1}\left(\frac{b}{n+1};1\right),$$

where $U_{(b)}$ is the bth order statistic from standard uniform random variable. Then, the second MML is obtained as

$$\hat{\theta}_3 = x_{(a)}\left(\frac{2(n-r_1-r_2)+r_2K_2}{2\left(x_{(a)}^2\sum_{i=r_1+1}^{n-r_2}\frac{1}{x_{(i)}^2}+r_1\right)}\right)^{\frac{1}{2}}, \tag{13}$$

where

$$K_2 = \hat{z}_b g(\hat{z}_b).$$

The works of [26] and [27] suggest revising the modified Maximum Likelihood Estimation (MMLE). According to the revising methodology, we replace z_b with $z_b = \frac{x_{(b)}}{\theta}$ and calculate the updated estimate $\hat{\theta}$ using Eq.(13). This process is done until $\hat{\theta}$ converges adequately. It is note worthy that more than a few updates (e.g., 3 or 5) may be necessary to stabilize the estimator.

Remark 1. In our simulation study, we observed that the revised MML is almost identical to the ML. From this, it can be concluded that there is no need to employ search methods dependent on the initial values, such as Nelder-Mead or BFGS, to obtain the ML. That is ML estimate can be obtained by updating the MMLE.

Let us define the following modified objective functions to obtain AD, CvM, and LS type estimates for the parameter θ based on Type-II left and right censoring:

$$Q_{AD}(\theta) = -n + \frac{1}{n}\sum_{j=r^*+1}^{n-r^*} (2j-1)\left(\frac{\theta^2}{x_{(j)}^2}\right) + \left[log\left(1-exp\left\{-\frac{\theta^2}{x_{(n-j+1)}^2}\right\}\right)\right],$$

$$Q_{CvM}(\theta) = \frac{1}{12n} + \sum_{j=r_1+1}^{n-r_2} \left(exp\left\{ -\frac{\theta^2}{x_{(j)}^2} \right\} - \frac{j-0.5}{n} \right)^2,$$

$$Q_{LS}(\theta) = \sum_{j=r_1+1}^{n-r_2} \left(exp\left\{ -\frac{\theta^2}{x_{(j)}^2} \right\} - \frac{j}{n+1} \right)^2,$$

where

$$r^* = \begin{cases} r_2 & , \quad r_1 \le r_2 \\ r_1 & , \quad r_1 > r_2. \end{cases}$$

Then, the AD, CvM, and LS type estimates of θ are given, respectively, by

$$\hat{\theta}_4 = \arg\min_{\theta} Q_{AD}(\theta), \quad (14)$$

$$\hat{\theta}_5 = \arg\min_{\theta} Q_{CvM}(\theta), \quad (15)$$

$$\hat{\theta}_6 = \arg\min_{\theta} Q_{LS}(\theta). \quad (16)$$

In this study, all minimization problems are solved via numerical method Nelder-Mead, which is available in the R function optim.

Simulation Study

In this section, we perform Monte Carlo simulation to assess the performance of the estimators given in Section 2. The performances of the proposed estimators are compared in terms of mean squared error (MSE) and bias criteria.

Table 1. MSE and bias (in parantheses) for the estimates of $\theta = 1$

n	r_1	r_2	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$	$\hat{\theta}_5$	$\hat{\theta}_6$
30	1	1	0.0095	0.0094	0.0095	0.0104	0.0118	0.0117
			(0.0132)	(0.0121)	(0.0132)	(0.0051)	(0.0090)	(0.0062)
	3	3	0.0102	0.0099	0.0102	0.0112	0.0122	0.0121
			(0.0121)	(0.0047)	(0.0122)	(0.0029)	(0.0056)	(0.0034)
	6	6	0.0115	0.0108	0.0116	0.0127	0.0132	0.0131
			(0.0163)	(-0.0145)	(0.0166)	(0.0073)	(0.0086)	(0.0073)
	1	6	0.0093	0.0088	0.0093	0.0124	0.0117	0.0115
			(0.0141)	(-0.0112)	(0.0143)	(0.0083)	(0.0137)	(0.0092)
	6	1	0.0117	0.0117	0.0117	0.0127	0.0132	0.0132
			(0.0155)	(0.0142)	(0.0156)	(0.0067)	(0.0045)	(0.0050)
60	3	3	0.0047	0.0047	0.0047	0.0053	0.0059	0.0059
			(0.0079)	(0.0061)	(0.0079)	(0.0034)	(0.0052)	(0.0039)
	6	6	0.0048	0.0046	0.0048	0.0054	0.0059	0.0058
			(0.0067)	(0.0001)	(0.0067)	(0.0024)	(0.0038)	(0.0027)
	12	12	0.0056	0.0056	0.0056	0.0062	0.0064	0.0064
			(0.0088)	(-0.0201)	(0.0089)	(0.0045)	(0.0050)	(0.0044)
	3	12	0.0047	0.0047	0.0047	0.0063	0.0059	0.0058
			(0.0067)	(-0.0174)	(0.0068)	(0.0035)	(0.0060)	(0.0038)
	12	3	0.0054	0.0054	0.0054	0.0060	0.0063	0.0063
			(0.0073)	(0.0052)	(0.0073)	(0.0024)	(0.0013)	(0.0015)
100	5	5	0.0027	0.0027	0.0027	0.0031	0.0034	0.0034
			(0.0049)	(0.0033)	(0.0049)	(0.0022)	(0.0032)	(0.0024)
	10	10	0.0029	0.0028	0.0029	0.0033	0.0035	0.0035
			(0.0049)	(-0.0013)	(0.0049)	(0.0022)	(0.0030)	(0.00249
	20	20	0.0033	0.0036	0.0033	0.0037	0.0038	0.0038
			(0.0056)	(-0.0225)	(0.0057)	(0.0026)	(0.0029)	(0.0025)
	5	20	0.0027	0.0030	0.0027	0.0036	0.0033	0.0033
			(0.0038)	(-0.0196)	(0.0039)	(0.0019)	(0.0034)	(0.0021)
	20	5	0.0032	0.0031	0.0032	0.0036	0.0037	0.0037
			(0.0048)	(0.0029)	(0.0048)	(0.0021)	(0.0013)	(0.0014)
200	10	10	0.0013	0.0013	0.0013	0.0016	0.0017	0.0017
			(0.0019)	(0.0004)	(0.0019)	(0.0008)	(0.0014)	(0.0010)
	20	20	0.0014	0.0014	0.0014	0.0016	0.0017	0.0017
			(0.0018)	(-0.0041)	(0.0019)	(0.0005)	(0.0009)	(0.0005)
	40	40	0.0016	0.0022	0.0016	0.0018	0.0019	0.0019
			(0.0020)	(-0.0255)	(0.0021)	(0.0005)	(0.0006)	(0.0004)
	10	40	0.0013	0.0017	0.0013	0.0018	0.0017	0.0017
			(0.0017)	(-0.0213)	(0.0017)	(0.0002)	(0.0011)	(0.0004)
	40	10	0.0015	0.0015	0.0015	0.0018	0.0019	0.0019
			(0.0025)	(0.0008)	(0.0025)	(0.0011)	(0.0008)	(0.0008)
500	25	25	0.0005	0.0005	0.0005	0.0006	0.0007	0.0007
			(0.0007)	(-0.0007)	(0.0007)	(0.0003)	(0.0006)	(0.0004)
	50	50	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007
			(0.0007)	(-0.0051)	(0.0008)	(0.0000)	(0.0002)	(0.0000)
	100	100	0.0006	0.0013	0.0006	0.0007	0.0007	0.0007
			(0.0007)	(-0.0264)	(0.0008)	(0.0003)	(0.0003)	(0.0003)
	25	100	0.0005	0.0010	0.0005	0.0007	0.0007	0.0007
			(0.0009)	(-0.0218)	(0.0009)	(0.0006)	(0.0009)	(0.0006)
	100	25	0.0006	0.0006	0.0006	0.0007	0.0008	0.0008
			(0.0010)	(-0.0006)	(0.0010)	(0.0004)	(0.0002)	(0.0002)

Since the θ is a scale parameter, $\theta = 1$ is considered in the simulations without loss of generality. Various censoring schemes and sample sizes of $n = 30, 60, 100, 200,$ and 500 are included in the simulations. Censoring schemes are selected by considering various combinations of right and left censoring, with the number of observations censored from the right and left sides comprising 5%, 10%, and 20% of the dataset. In the case of $n = 30$, r_1 and r_2 are fixed as 1 for a 5% censorship situation. The average of the bias and MSE of the estimators with 10000 replications are given in Table 1.

According to the results from Table 1, the $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$ exhibit similar performance, and they slightly outperform current methods in terms of both MSE and bias considered in this study. When $r_1 = r_2$, an increase in the number of censored observations results in higher MSE values, aligning with expectations. As anticipated, both n and the count of censored observations contribute to an increase in MSE values. Moreover, when $r_1 > r_2$, the MSE tends to be larger. It is possible to reach the ultimate conclusion that censoring should be at right in Type-II scheme. The most significant result extracted from this simulation study is that the proposed estimators perform equally well as the ML estimator and even better than the others. Therefore, there is no objection to using our proposed explicit estimators instead of the ML estimator, which requires numerical methods.

Real Data Analysis

In this section, the strengths of 1.5 cm glass fibres ([28]) dataset is used for illustration purposes. The Table 2 provides the ML estimate of the parameter θ for the complete dataset, along with various goodness of fit values such as the Akaike Information Criterion (AIC), Corrected Akaike's Information Criterion (CAIC), Bayesian Information Criterion (BIC), Anderson-Darling (A), Kolmogorov-Smirnov (KS), and their corresponding p-values (in parentheses). As the p-values for the goodness-of-fit tests shown in Table 2 exceed 0.05, we cannot reject the hypothesis that the data originates from the IR distribution at a significance level of 0.05.

Table 2. Some results fot the glass fibres data for the complete data

$\hat{\theta}$	$\ell(\hat{\theta})$	AIC	CAIC	BIC	A	KS
0.1574	- 31.5405	- 61.0810	- 60.9210	- 59.7852	1.4936 (0.1781)	0.1758 (0.3738)

Let us censoring the complete data with scheme $r_1 = 1$ and $r_2 = 2$. Then, the Type-II left and right censored data is produced by: 0.11, 0.12, 0.12, 0.12, 0.12, 0.13, 0.13, 0.14, 0.15, 0.15, 0.15, 0.16, 0.16, 0.16, 0.17, 0.20, 0.20, 0.20, 0.21, 0.23, 0.26, 0.32, 0.33, 0.33. Using this Type-II left and right censored data, point estimates of θ are given in Table 3.

Table 3. Point estimates for the censored glass fibres data

$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{\theta}_4$	$\hat{\theta}_5$	$\hat{\theta}_6$
0.1562	0.1558	0.1565	0.1431	0.1435	0.1430

Fixed-point iteration given in (5) is also considered for the censored data. The convergence of the fixed-point iterations to the ML is illustrated in Figure 1, where the dashed line indicates the ML estimate.

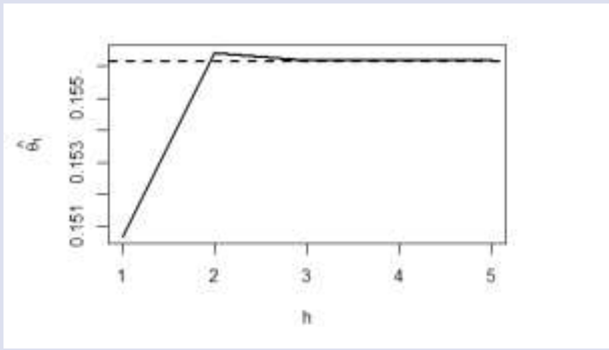

Figure 1. The convergence of the fixed-point iterations to the ML

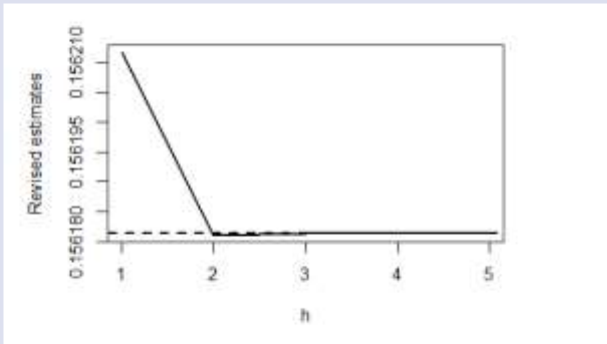

Figure 2. Convergence of the revised estimates of $\hat{\theta}_3$ to the $\hat{\theta}_1$

Figure 2 also demonstrates revising steps for $\hat{\theta}_3$. Based on Figure 2, it can be seen that only a small number of updates are required to reach the ML estimate.

Conclusion

This study is focused on estimating the parameter of the inverse Rayleigh (IR) distribution under Type-II left and right censoring scheme. In this regard, several point estimation methods are introduced, including ML, AD, CvM, and LS. Additionally, two novel MML approaches are proposed, designed to address the challenges of parameter estimation under Type-II left and right censoring. The results of the simulation study demonstrate that the proposed MML estimators consistently outperform the other methods across various scenarios. These estimators are particularly noteworthy for their explicit solutions, which eliminate the need for iterative search methods and simplify the estimation process. This makes the MML estimators promising alternatives for the point estimation of the IR distribution parameter in Type-II left and right censoring contexts. Furthermore, the explicit nature of the proposed MML estimators enhances their computational efficiency and establishes them as excellent initial values for obtaining the ML estimator. By providing reliable starting points, they help overcome challenges typically associated with iterative methods, such as dependency on initial values and computational complexity. This combination of simplicity and effectiveness underscores the value of the proposed MML estimators as practical tools for parameter estimation in censored data scenarios.

Conflicts of interest

All authors declare that they have no conflict of interest.

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