Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.38-4, Supplement (2017) 130-137

On Some New Ideal Convergent Sequence Spaces of M_λ-Method of Summability

Nazlım Deniz ARAL[1*], Şükran KONCA[1]

[1]*Department of Mathematics, Bitlis Eren University, 1300, Bitlis, Turkey*

Received: 22.06.2017; Accepted: 23.10.2017 *http://dx.doi.org/10.17776/csj.363614*

Abstract: In the present work, we introduce some ideal convergent sequence spaces by using M_λ- summability method which is defined by P. N. Natarajan [On the (M,λ_n)-method of summability, Analysis] as a typically generalization of Nörlund method. Further, we examine some of their topological properties.

Keywords: I-convergence, Orlicz function, M_λ-method of summability, sequence space

M_λ- Toplanabilme Metoduyla Tanımlanan Bazı Yeni İdeal Yakınsak Dizi Uzayları

Özet: Bu çalışmada Nörlund metodunun bir genelleşmesi olarak P.N. Natarajan [On the (M,λ_n)-method of summability, Analysis] tarafından tanımlanan M_λ-toplanabilme metodu kullanılarak bazı ideal yakınsak dizi uzayları tanımlanmıştır. Ayrıca, bu uzayların bazı topolojik özellikleri incelenmiştir.

Anahtar Kelimeler: I-yakınsaklık, Orlicz fonksiyonu, M_λ-toplanabilme metodu, dizi uzayı

1. INTRODUCTION

The Orlicz spaces were introduced by Birnbaum and W. Orlicz [1] in 1931. Krasnosel'skii and Rutickii [2] detailed study on Orlicz spaces. Lindberg [3] studied various properties of Orlicz sequence spaces and their subspaces. For further results, see, [4, 5].

The notion of I-convergence was introduced by Kostyrko, Salat and Wilczynski [6] corresponds to a generalization of the statistical convergence. Related papers can be seen in for example, [7-11].

Natarajan [12] has introduced a very new method of summability which is called (M,λ_n) method in 2013 and studied some of its properties concerning its regularity, consistency and translativity. He also has proved an inclusion theorem and an equivalence theorem. Recently, Aral and Küçükaslan [13] have defined M_λ-statistical convergence and given some inclusion results for different λ's, in addition to some relations between statistical convergence and M_λ-statistical convergence given. There are still some open problems on this new method of summability, for example one of them, there have not been any ideal convergent sequence spaces defined yet. In this paper, we have defined some spaces of ideal convergent sequences defined by M_λ-method of summability and Orlicz functions. We also examine some of topological properties of these

* Corresponding author. *Email address: ndaral@beu.edu.tr*

sequence spaces. By this way, we aim to fill this gap.

2. DEFINITIONS AND PRELIMINARIES

Before beginning of the presentation of the main results, we recall the following definitions. Throughout the paper, for brevity, by the notation $\lim_k x_k$ we mean $\lim_{k\to\infty} x_k$ and by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ we mean the set of all natural numbers, the set of all real numbers and the set of all complex numbers respectively. For the convenience, we also use the notation M_λ instead of (M,λ_n) representation given in the work of Natarajan [12].

Definition 1 [6] Let $X \neq \emptyset$ and $P(X) = 2^X$ be the family of all subsets of X. Then, a family of sets $I \subset 2^X$ is said to be an ideal on X if and only if I satisfies these conditions:

1. $\emptyset \in I$,

2. $A, B \in I$ imply $A \cup B \in I$,

3. $A \in I$, $B \subset A$ imply $B \in I$.

An ideal I is called non-trivial if $I \neq \emptyset$ and $X \notin I$, that is $I \neq 2^X$. A non-trivial ideal $I \subset 2^X$ is called admissible if $\{x\} \in I$ for each $x \in X$.

Definition 2 [14] Let X be a linear metric space. A function $p: X \longrightarrow \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,

2. $p(-x) = p(x)$, for all $x \in X$,

3. $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$,

4. If (λ_n) is a sequence of scalars with $\lambda_n \longrightarrow \lambda$ as $n \longrightarrow \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \longrightarrow 0$ as $n \longrightarrow \infty$, then $p(\lambda_n x_n - \lambda x) \longrightarrow 0$ as $n \longrightarrow \infty$.

Definition 3 [14] An Orlicz function is a function $M: [0,\infty) \longrightarrow [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

If the convexity of an Orlicz function is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function. If M is an Orlicz function, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$. An Orlicz function M is said to satisfy Δ_2-condition for all values of x, if there exists a constant $K > 0$, such that $M(2x) \leq KM(x)$ for all $x \geq 0$. The Δ_2-condition is equivalent to $M(Lx) \leq KLM(x)$, for all values of $x > 0$ and for $L > 1$ [2].

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$, for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. It is well known that a sequence space E is normal implies that E is monotone [14].

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of strictly positive real numbers with $0 < p_k \leq sup_{k\in\mathbb{N}} p_k = G$, and let $D = \max\{1, 2^{G-1}\}$. Then we have

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k} \qquad (1)$$

for all $a_k, b_k \in \mathbb{C}$ [15].

Let $\lambda = (\lambda_n)$ be a sequence such that $\sum_{n=1}^{\infty} |\lambda_n| < \infty$.

Definition 4 [13] A real valued sequence $x = (x_n)$ is said to be M_λ-convergent to $l \in \mathbb{R}$, called the M_λ-limit of x and denoted by $x_n \longrightarrow l\ (M_\lambda)$, if $\sum_{k=0}^{n} \lambda_{n-k} x_k \longrightarrow l$, $n \longrightarrow \infty$.

3. Main Results

Now, we present our main results. Throughout the paper I will be considered as a non-trivial admissible ideal.

Let M be an Orlicz function and let $q = (q_n)$ be a bounded sequence of positive real numbers. We define the following sequence spaces,

$$c^I(M,q)^{M_\lambda} := \left\{x = (x_n) \in w : I - \lim_n M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k} x_k - L\right|}{\rho}\right)^{q_n} = 0 \; for\ some\ L\ and\ \rho > 0\right\},$$

$$c_0^I(M,q)^{M_\lambda} := \left\{x = (x_n) \in w : I - \lim_n M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k} x_k\right|}{\rho}\right)^{q_n} = 0 \; for\ some\ \ \rho > 0\right\},$$

$$l_\infty(M,q)^{M_\lambda} := \left\{x = (x_n) \in w : sup_{n\in\mathbb{N}} M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k} x_k\right|}{\rho}\right)^{q_n} < \infty \; for\ some\ \ \rho > 0\right\}. \quad (2)$$

We can write

$$m^I(M,q)^{M_\lambda} = c^I(M,q)^{M_\lambda} \cap l_\infty(M,q)^{M_\lambda} \text{ and } m_0^I(M,q)^{M_\lambda} = c_0^I(M,q)^{M_\lambda} \cap l_\infty(M,q)^{M_\lambda}.$$

For some special cases we obtain the followings:

1. If $q = (q_n) = 1$ for all $n \in \mathbb{N}$, then the sequence spaces given by (2) reduce to the following sequence spaces.

$$c^I(M)^{M_\lambda} := \left\{x = (x_n) \in w : I - \lim_n M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k} x_k - L\right|}{\rho}\right) = 0 \; for\ some\ L\ and\ \rho > 0\right\},$$

$$c_0^I(M)^{M_\lambda} := \left\{x = (x_n) \in w : I - \lim_n M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k} x_k\right|}{\rho}\right) = 0 \; for\ some\ \ \rho > 0\right\},$$

$$l_\infty(M)^{M_\lambda} := \left\{x = (x_n) \in w : sup_{n\in\mathbb{N}} M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k} x_k\right|}{\rho}\right) < \infty \; for\ some\ \ \rho > 0\right\}.$$

2. If we take $M(x) = x$ in (2), then we obtain the followings:

$$(c^I)^{M_\lambda} := \left\{x = (x_n) \in w : I - \lim_n\left(\left|\sum_{k=0}^{n} \lambda_{n-k} x_k - L\right|\right) = 0 \; for\ some\ L\right\},$$

$$(c_0^I)^{M_\lambda} := \left\{x = (x_n) \in w : I - \lim_n\left(\left|\sum_{k=0}^{n} \lambda_{n-k} x_k\right|\right) = 0\right\},$$

$$(l_\infty)^{M_\lambda} := \{x = (x_n) \in w : sup_{n\in\mathbb{N}}\left(\left|\sum_{k=0}^{n} \lambda_{n-k} x_k\right|\right) < \infty\}.$$

3. When we take $M(x) = x$ and $q = (q_n) = 1$ for all $n \in \mathbb{N}$ in the equation (2) then we have

$$c^I(q)^{M_\lambda} := \left\{x = (x_n) \in w : I - \lim_n\left(\left|\sum_{k=0}^{n} \lambda_{n-k} x_k - L\right|^{q_n}\right) = 0 \; for\ some\ L\right\},$$

$$c_0^I(q)^{M_\lambda} := \left\{x = (x_n) \in w : I - \lim_n\left(\left|\sum_{k=0}^{n} \lambda_{n-k} x_k\right|^{q_n}\right) = 0\right\},$$

$$l_\infty(q)^{M_\lambda} := \{x = (x_n) \in w : sup_{n\in\mathbb{N}}\left(\left|\sum_{k=0}^{n} \lambda_{n-k} x_k\right|^{q_n}\right) < \infty\}.$$

4. Let $p = (p_n)$ be a sequence of positive natural numbers and $P_n = \sum_{k=0}^{n} p_k \longrightarrow \infty$. Take into consider $\lambda = (\lambda_n)$ as

$$\lambda_n = \left(\frac{p_0}{P_n}, \frac{p_1}{P_n}, \dots, \frac{p_n}{P_n}, 0, 0, \dots\right)$$

then M_λ-convergence coincide with N_p-convergence and the sequence spaces given by (2) reduces to the following sequence spaces for $q = (q_n) = 1$ $(\forall n \in \mathbb{N})$ and $M(x) = x$;

$$(c^I)^{N_p} := \left\{x = (x_n) \in w : I - \lim_n \left(\left|\frac{1}{P_n}\sum_{k=0}^{n} x_k - L\right|\right) = 0 \; for\ some\ L\right\} ,$$

$$(c_0^I)^{N_p} := \left\{x = (x_n) \in w : I - \lim_n \left(\left|\frac{1}{P_n}\sum_{k=0}^{n} x_k\right|\right) = 0\right\} ,$$

$$(l_\infty)^{N_p} := \left\{x = (x_n) \in w : sup_{n\in\mathbb{N}} \left(\left|\frac{1}{P_n}\sum_{k=0}^{n} x_k\right|\right) < \infty\right\} .$$

Theorem 1 Let M be an Orlicz function and let $q = (q_n)$ be a bounded sequence of positive real numbers. Then, the spaces $c^I(M,q)^{M_\lambda}$, $c_0^I(M,q)^{M_\lambda}$, $m^I(M,q)^{M_\lambda}$ and $m_0^I(M,q)^{M_\lambda}$ are linear.

Proof. Let $x, y \in c^I(M,q)^{M_\lambda}$. Then, there exist positive numbers ρ_1 and ρ_2 such that

$$I - \lim_n M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k}x_k - L_1\right|}{\rho_1}\right)^{q_n} = 0, \text{ for some } L_1 \in \mathbb{C},$$

$$I - \lim_n M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k}y_k - L_2\right|}{\rho_2}\right)^{q_n} = 0, \text{ for some } L_2 \in \mathbb{C}.$$

For a given $\varepsilon > 0$, we have

$$K_1 = \left\{k \in \mathbb{N} : M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k}x_k - L_1\right|}{\rho_1}\right)^{q_n}\right\}, \quad K_2 = \left\{k \in \mathbb{N} : M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k}y_k - L_2\right|}{\rho_2}\right)^{q_n}\right\}. \qquad (3)$$

Let $\rho_3 = max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$ and let $\alpha, \beta \in \mathbb{C}$ be scalars. Since M is non-decreasing convex function, so by using inequality (1), we have

$$\lim_n M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k}(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)\right|}{\rho_3}\right)^{q_n}$$

$$\leq \lim_n M\left(\frac{|\alpha|\left|\sum_{k=0}^{n} \lambda_{n-k}x_k - L_1\right|}{\rho_3} + \frac{|\beta|\left|\sum_{k=0}^{n} \lambda_{n-k}y_k - L_2\right|}{\rho_3}\right)^{q_n}$$

$$\leq \lim_n M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k}x_k - L_1\right|}{\rho_1}\right)^{q_n} + \lim_n M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k}y_k - L_2\right|}{\rho_2}\right)^{q_n} .$$

We have from (3),

$$\left\{k \in \mathbb{N} : \lim_n M\left(\frac{\left|\sum_{k=0}^{n} \lambda_{n-k}(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)\right|}{\rho_3}\right)^{q_n}\right\} \subset K_1 \cup K_2 .$$

Hence, $\alpha x + \beta y \in c^I(M,q)^{M_\lambda}$. Thus $c^I(M,q)^{M_\lambda}$ is a linear space. We can prove that $c_0^I(M,q)^{M_\lambda}$, $m^I(M,q)^{M_\lambda}$ and $m_0^I(M,q)^{M_\lambda}$are linear spaces with similar techniques.

Theorem 2 Let M be an Orlicz function. Then, $c_0^I(M,q)^{M_\lambda} \subset c^I(M,q)^{M_\lambda} \subset l_\infty(M,q)^{M_\lambda}$.

Proof. The inclusion $c_0^I(M,q)^{M_\lambda} \subset c^I(M,q)^{M_\lambda}$ is obvious. Let $x \in c^I(M,q)^{M_\lambda}$. Then, there exist $L \in \mathbb{C}$ and $\rho > 0$ such that

$$I - \lim_n M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k - L\right|}{\rho}\right)^{q_n} = 0.$$

We have

$$M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k\right|}{2\rho}\right)^{q_n} \leq \frac{1}{2}M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k - L\right|}{\rho}\right)^{q_n} + \frac{1}{2}M\left(\frac{|L|}{\rho}\right)^{q_n}.$$

If we take supremum over n on both sides, we get $x \in l_\infty(M,q)^{M_\lambda}$. Hence we obtain

$$c_0^I(M,q)^{M_\lambda} \subset c^I(M,q)^{M_\lambda} \subset l_\infty(M,q)^{M_\lambda}.$$

This completes the proof of the theorem.

Theorem 3 Let M be an Orlicz function and let $q = (q_n)$ be a bounded sequence of positive real numbers. Then, $l_\infty(M,q)^{M_\lambda}$ is a paranormed space with paranorm defined by

$$g(x) = inf_{n\geq 1}\left\{\rho^{\frac{q_n}{H}} > 0 : sup_{n\in\mathbb{N}} M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k\right|}{\rho}\right)^{q_n} \leq 1 \; for\; some\; \rho > 0\right\},$$

where $H = max\{1, sup_n q_n\}$.

Proof. It is clear that $g(x) = g(-x)$. Since $M(0) = 0$, we get $g(\theta) = 0$. Let us take $x, y \in l_\infty(M,q)^{M_\lambda}$ and denote

$$K(x) = \left\{\rho > 0 : sup_n M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k\right|}{\rho}\right)^{q_n} \leq 1\right\} \text{ and } K(y) = \left\{\rho > 0 : sup_n M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}y_k\right|}{\rho}\right)^{q_n} \leq 1\right\}.$$

Let $\rho_1 \in K(x)$ and $\rho_2 \in K(y)$. If $\rho = \rho_1 + \rho_2$, then we have

$$sup_{n\in\mathbb{N}} M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}(x_k+y_k)\right|}{\rho}\right) \leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right) sup_{n\in\mathbb{N}} M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k\right|}{\rho_1}\right) + \left(\frac{\rho_2}{\rho_1+\rho_2}\right) sup_{n\in\mathbb{N}} M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}y_k\right|}{\rho_2}\right)$$

which in terms give us, $sup_{n\in\mathbb{N}} M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}(x_k+y_k)\right|}{\rho_1+\rho_2}\right)^{q_n} \leq 1$ and

$$g(x+y) = inf\left\{(\rho_1+\rho_2)^{\frac{q_n}{H}} > 0 : \rho_1 \in K(x), \rho_2 \in K(y)\right\}$$

$$\leq inf\left\{(\rho_1)^{\frac{q_n}{H}} > 0 : \rho_1 \in K(x)\right\} + inf\left\{(\rho_2)^{\frac{q_n}{H}} > 0 : \rho_2 \in K(y)\right\}$$

$$= g(x) + g(y).$$

Let (σ^s) be a sequence of scalars with $\sigma^s \longrightarrow \sigma$, where $\sigma, \sigma^s \in \mathbb{C}$ and let $(x^s), x \in l_\infty(M,q)^{M_\lambda}$ be such that $g(x^s - x) \longrightarrow 0$ as $s \longrightarrow \infty$. To prove that $g(\sigma^s x^s - \sigma s) \longrightarrow 0$ as $s \longrightarrow \infty$. Let

$$A=\left\{\rho_s>0: sup_{n\in\mathbb{N}}M\left(\frac{\left|\sum_{k=0}^{n}\lambda_{n-k}x^s\right|}{\rho_s}\right)^{q_n}\leq 1\right\} \text{ and } B=\left\{\rho_s'>0: sup_{n\in\mathbb{N}}M\left(\frac{\left|\sum_{k=0}^{n}\lambda_{n-k}(x^s-x)\right|}{\rho_s'}\right)^{q_n}\leq 1\right\}.$$

If $\rho_s\in A$ and $\rho_s'\in B$, then we observe that

$$M\left(\frac{\left|\sum_{k=0}^{n}\lambda_{n-k}(\sigma^s x^s-\sigma x)\right|}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|}\right)\leq M\left(\frac{\left|\sum_{k=0}^{n}\lambda_{n-k}(\sigma^s x^s-\sigma x^s)\right|}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|}+\frac{\left|\sum_{k=0}^{n}\lambda_{n-k}(\sigma x^s-\sigma x)\right|}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|}\right)$$

$$\leq \frac{|\sigma^s-\sigma|\rho_s}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|}M\left(\frac{\left|\sum_{k=0}^{n}\lambda_{n-k}x^s\right|}{\rho_s}\right)+\frac{|\sigma|\rho_s'}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|}M\left(\frac{\left|\sum_{k=0}^{n}\lambda_{n-k}(x^s-x)\right|}{\rho_s'}\right).$$

From the above inequality, it follows that

$$sup_{n\in\mathbb{N}}M\left(\frac{\left|\sum_{k=0}^{n}\lambda_{n-k}(\sigma^s x^s-\sigma x)\right|}{\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|}\right)^{q_n}\leq 1$$

and consequently,

$$g(\sigma^s x^s-\sigma x)\leq inf\left\{(\rho_s|\sigma^s-\sigma|+\rho_s'|\sigma|)^{\frac{q_n}{H}}>0:\rho_s\in A,\ \rho_s'\in B\right\}$$

$$\leq(|\sigma^s-\sigma|)^{\frac{q_n}{H}}inf\left\{(\rho_s)^{\frac{q_n}{H}}>0:\rho_s\in A\right\}+(|\sigma|)^{\frac{q_n}{H}}inf\left\{(\rho_s')^{\frac{q_n}{H}}>0:\rho_s'\in B\right\}$$

$$\leq max\left\{1,|\sigma^s-\sigma|^{\frac{q_n}{H}}\right\}g(x^s)+max\left\{1,|\sigma|^{\frac{q_n}{H}}\right\}g(x^s-x).$$

Hence by our assumption, the right hand side tends to 0 as $s\longrightarrow\infty$. This completes the proof.

Theorem 4 Let M_1 and M_2 be Orlicz functions that satisfy the Δ_2 -condition. Then, for $Z=c^I,c_0^I,m^I,m_0^I$,

1. $Z(M_2,q)^{M_\lambda}\subseteq Z(M_1\circ M_2)^{M_\lambda}$,

2. $Z(M_1,q)^{M_\lambda}\cap Z(M_2,q)^{M_\lambda}\subseteq Z(M_1+M_2,q)^{M_\lambda}$.

Proof. (1) Let $x\in c_0^I(M,q)^{M_\lambda}$. Then, there exist $\rho>0$ such that

$$I-\lim_n M_2\left(\frac{\sum_{k=0}^{n}\lambda_{n-k}x_k}{\rho}\right)^{q_n}=0. \tag{4}$$

Let $\varepsilon>0$ and choose δ with $0<\delta<1$ such that $M_1(t)<\varepsilon$ for $0\leq t\leq\delta$. Write $y_n=M_2\left(\frac{\left|\sum_{k=0}^{n}\lambda_{n-k}x_k\right|}{\rho}\right)^{q_n}$ and consider $\lim_{n\in\mathbb{N}}M_1(y_n)=\lim_{\substack{y_n\leq\delta\\ n\in\mathbb{N}}}M_1(y_n)+\lim_{\substack{y_n>\delta\\ n\in\mathbb{N}}}M_1(y_n)$. Since M_1 is an Orlicz function, we have

$$\lim_{\substack{y_n\leq\delta\\ n\in\mathbb{N}}}M_1(y_n)\leq M_1(2)\lim_{\substack{y_n\leq\delta\\ n\in\mathbb{N}}}(y_n) \tag{5}$$

For $y_n>\delta$ we have $y_n<\frac{y_n}{\delta}<1+\frac{y_n}{\delta}$.

Since M_1 is nondecreasing and convex, it follows that

$$M_1(y_n) < M_1\left(1+\frac{y_n}{\delta}\right) < \frac{1}{2}\frac{M_1(2y_n)}{\delta}.$$

Since M_1 satisfies Δ_2-condition we have

$$M_1(y_n) < \frac{1}{2}K\frac{y_n}{\delta}M_1(2) + \frac{1}{2}K\frac{y_n}{\delta}M_1(2) = K\frac{y_n}{\delta}M_1(2).$$

Hence,

$$\lim_{\substack{y_n>\delta \\ n\in\mathbb{N}}} M_1(y_n) \le max(1, K\delta^{-1}M_1(2)) \lim_{\substack{y_n>\delta \\ n\in\mathbb{N}}}(y_n) \tag{6}$$

From (4), (5) and (6), we have $x=(x_k)\in c_0^I(M_1\circ M_2\,, q)^{M_\lambda}$. Thus $c_0^I(M_2\,, q)^{M_\lambda}\subseteq c_0^I(M_1\circ M_2\,, q)^{M_\lambda}$.

We can prove the other cases similarly.

(2) Let $x\in c_0^I(M_1\,, q)^{M_\lambda}\cap c_0^I(M_2\,, q)^{M_\lambda}$ then there exist $\rho>0$ such that

$$I-\lim_n M_1\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k\right|}{\rho}\right)^{q_n} = 0 \text{ and } I-\lim_n M_2\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k\right|}{\rho}\right)^{q_n} = 0\,.$$

The result is obtained by the following equality

$$\lim_{n\in\mathbb{N}}(M_1+M_2)\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k\right|}{\rho}\right)^{q_n} = \lim_{n\in\mathbb{N}} M_1\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k\right|}{\rho}\right)^{q_n} + \lim_{n\in\mathbb{N}} M_2\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k\right|}{\rho}\right)^{q_n}.$$

Corollary1 Let M be an Orlicz function which satisfies Δ_2-condition. Then $Z(q)^{M_\lambda}\subseteq Z(M,q)^{M_\lambda}$ holds for $Z=c^I, c_0^I, m^I, m_0^I$.

Theorem 5 The spaces $c_0^I(M,q)^{M_\lambda}$ and $m_0^I(M,q)^{M_\lambda}$are solid.

Proof. We will prove for the space $c_0^I(M,q)^{M_\lambda}$. For $m_0^I(M,q)^{M_\lambda}$, the proof shall be similar. Let $x\in c_0^I(M,q)^{M_\lambda}$, then there exists $\rho>0$ such that

$$I-\lim_n M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k\right|}{\rho}\right)^{q_n} = 0\,.$$

Let α_k be a sequence of scalars such that $|\alpha_k|\le 1$, $\forall\, k\in\mathbb{N}$. Then the result follows from the following inequality

$$\lim_n M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}\alpha_k x_k\right|}{\rho}\right)^{q_n} \le \lim_n M\left(\frac{\left|\sum_{k=0}^n \lambda_{n-k}x_k\right|}{\rho}\right)^{q_n} = 0$$

and this completes the proof.

Corollary 2 The spaces $c_0^I(M,q)^{M_\lambda}$ and $m_0^I(M,q)^{M_\lambda}$ are monotone.

REFERENCES

[1]. Birnbaum Z., Orlicz W. Uber die Verallgemeinerung des Begriffes der zueinander Konjugierten Potenzen, Studia Mathematica, 1931; 3: 1-67.

[2]. Krasnosel'skii M.A., Rutickii Y.B. Convex Functions and Orlicz Spaces, P. Noordhoff Ltd., Groningen, Netherlands, 1961.

[3]. Lindberg, K., On subspaces of Orlicz sequence spaces, Studia Mathematica, 1973; 45 (2):119-146.

[4]. Lindenstrauss J., Tzafriri L. On Orlicz Sequence Spaces, Israel Journal of Mathematics, 1971; 10 (3): 379-390.

[5]. Parashar S.D., Choudhary B. Sequence Spaces Defined by Orlicz Function, Indian Journal Pure and Applied Mathematics, 1994; 24 (4): 419-428.

[6]. Kostyrko P., Wilczynski W., Salat T. I-convergence, Real Analysis Exchange, 2000; 26 (2): 669-686.

[7]. Tripathy B.C., Hazarika B. Some I-convergent sequence spaces defined by Orlicz functions, Acta Mathematicae Applicatae Sinica, 2011; 27 (1): 149-154.

[8]. Savas E. On some new sequence spaces in 2-normed spaces using ideal convergence and an Orlicz function, Journal of Inequalities and Applications, 2010; Article ID 482392, 8 pages.

[9]. Mursaleen M., Sharma S.K. Spaces of Ideal Sequences, Hindawi, 2014; Artical ID 134534, 6 pages.

[10]. Savaş E., Das P. A generalized statistical convergence via ideals, Application Letters, 2011; 24 (6): 826-830.

[11]. Salat T., Tripathy B.C., Ziman M. On some properties of I-convergence, Tatra Mountains Mathematical Publications, 2004; 28: 279-286.

[12]. Natarajan P.N. On the (M,λ_n) method of summability, Analysis, 2013; {33}: 51-56.

[13]. Aral N.D., Küçükaslan M. On M_λ-statistical Convergence, Journal of Mathematical Analysis, 2016; 7 (2): 37-46.

[14]. Kamrhan P.K., Gupta M. Sequence spaces and series Marcel Dekker, New York, 1981.

[15]. Maddox I.J. Elements of Functional Analysis Cambridge Univ. Press, Cambridge, 1970.