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A Note on the Sequence Space $b_p^{r,s}(G)$

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Abstract: In this study, we define the sequence space $b_p^{r,s}(G)$ derived by the composition of the Binomial matrix and generalized difference(double band) matrix and show that the space $b_p^{r,s}(G)$ is linearly isomorphic to the space l_p, where $1 \leq p < \infty$. Furthermore, we mention some inclusion relations and give Schauder basis of the space $b_p^{r,s}(G)$. Moreover, we determine α-, β- and γ-duals of the space $b_p^{r,s}(G)$. Lastly, we characterize some matrix classes related to the space $b_p^{r,s}(G)$.

MSC: 40C05;40H05;46B45

***Keywords*:** Matrix Transformation, Matrix Domain, Schauder Basis, α-, β- and γ-Duals

$b_p^{r,s}(G)$ Dizi Uzayı Üzerine Bir Not

Özet: Bu çalışmada, Binom ve genelleştirilmiş fark(ikili band) matrislerinin kompozisyonu ile türetilen $b_p^{r,s}(G)$ dizi uzayı tanımlandı ve $b_p^{r,s}(G)$ uzayının $1 \leq p < \infty$ durumlarında l_p uzayına lineer olarak izomorfik olduğu gösterildi. Ayrıca, bazı kapsama bağıntılarından bahsedildi ve $b_p^{r,s}(G)$ uzayının Schauder bazı verildi. Bundan başka, $b_p^{r,s}(G)$ uzayının α-, β- ve γ-dualleri belirlendi. Son olarak, $b_p^{r,s}(G)$ uzayı ile ilgili bazı matris sınıfları karakterize edildi.

MKS: 40C05;40H05;46B45

***Anahtar Kelimeler*:** Matris Dönüşümü, Etki Alanı, Schauder Bazı, α-, β- ve γ-Dualleri

1. INTRODUCTION

A sequence space is a vector subspace of w which becomes a vector space under pointwise addition and scalar multiplication, where w is a set of all real(or complex) valued sequences. The symbols l_∞, c, c_0 and l_p represent the classical sequence spaces of all bounded, convergent, null and absolutely p-summable sequences, respectively, where $1 \leq p < \infty$.

A Banach sequence space is called a BK-space provided each of the maps $p_n: X \longrightarrow \mathbb{C}$ defined by $p_n(x) = x_n$ is continuous for all $n \in \mathbb{N}$[1]. By considering this notion, one can say that l_∞, c and c_0 are BK-spaces with their usual sup-norm defined by $\|x\|_\infty = \sup_{k\in\mathbb{N}}|x_k|$ and l_p is a BK-space with its p-norm defined by

$$\|x\|_p = \left(\sum_{k=0}^{\infty}|x_k|^p\right)^{\frac{1}{p}}$$

where $1 \leq p < \infty$. For simplicity, the summation without limits runs from 0 to ∞ in the rest of the paper.

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Let $A = (a_{nk})$ be an infinite matrix of complex entries, X and Y be two sequence spaces and $x = (x_k) \in w$. Then, the A-transform of x is defined by

$$(Ax)_n = \sum_k a_{nk} x_k$$

and is assumed to be convergent for all $n \in \mathbb{N}$, the class of all infinite matrices from X into Y is defined by

$$(X:Y) = \{A = (a_{nk}): Ax \in Y \; for \; all \; x \in X\}$$

and the matrix domain of $A = (a_{nk})$ in X is defined by

$$X_A = \{x = (x_k) \in w: Ax \in X\}$$

which is also a sequence space[2].

We write bs and cs for the sets of all bounded and convergent series, which are defined by means of the matrix domain of the summation matrix $S = (s_{nk})$ such that $bs = (l_\infty)_S$ and $cs = c_S$, respectively, where $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1 \quad , & 0 \le k \le n \\ 0 \quad , & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

An infinite matrix $A = (a_{nk})$ is called a triangle provided the entries $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n, k \in \mathbb{N}$. A triangle matrix has an inverse which is unique and a triangle. Unless stated otherwise, any term with negative subscript is assumed to be zero.

The method constructing a new sequence space by means of the matrix domain of an infinite matrix has recently been used by many authors : $(l_p)_{N_q}$ and c_{N_q} in [3], X_p and X_∞ in [4], $l_\infty(\Delta), c_0(\Delta)$ and $c(\Delta)$ in [5], $l_\infty(\Delta^2), c_0(\Delta^2)$ and $c(\Delta^2)$ in [6], e_0^r and e_c^r in [7], e_p^r and e_∞^r in [8] and [9], $e_0^r(\Delta)$, $e_c^r(\Delta)$ and $e_\infty^r(\Delta)$ in [10], $e_0^r(\Delta^m)$, $e_c^r(\Delta^m)$ and $e_\infty^r(\Delta^m)$ in [11], $e_0^r(B^{(m)})$, $e_c^r(B^{(m)})$ and $e_\infty^r(B^{(m)})$ in [12], $\hat{l}_\infty, \hat{c}, \hat{c}_0$ and $\hat{l}_p$ in [13].

2. THE SEQUENCE SPACE $b_p^{r,s}(G)$

In this chapter, we speak of the previous studies of Binomial matrix and Euler matrix, and define the sequence space $b_p^{r,s}(G)$. Moreover, we prove that the sequence space $b_p^{r,s}(G)$ is linearly isomorphic to the sequence space l_p and is not a Hilbert space except the case $p = 2$, where $1 \le p < \infty$. Furthermore, we mention some inclusion relations.

The usage of matrix domain of the Euler matrix was first motivated by Altay, Başar and Mursaleen in [7], [8] and [9]. They constructed the Euler sequence spaces e_0^r, e_c^r, e_∞^r and e_p^r as follows:

$$e_0^r = \left\{x = (x_k) \in w: \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k = 0\right\},$$

$$e_c^r = \left\{x = (x_k) \in w: \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k \;\; exists\right\},$$

$$e_\infty^r = \left\{x = (x_k) \in w: \sup_{n\in\mathbb{N}} \left|\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k\right| < \infty\right\}$$

and

$$e_p^r = \left\{x = (x_k) \in w: \sum_n \left|\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k\right|^p < \infty\right\}$$

where $1 \le p < \infty$, $0 < r < 1$ and the Euler matrix of order r is defined by

$$e_{nk}^{r} = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^{k} & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

Thereafter, Altay and Polat improved Altay, Başar and Mursaleen's work by defining the sequence spaces $e_0^r(\Delta)$, $e_c^r(\Delta)$ and $e_\infty^r(\Delta)$ in [10] as follows:

$$e_0^r(\Delta) = \left\{x = (x_k) \in w \colon \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}(1-r)^{n-k}r^k(x_k - x_{k-1}) = 0\right\},$$

$$e_c^r(\Delta) = \left\{x = (x_k) \in w \colon \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}(1-r)^{n-k}r^k(x_k - x_{k-1}) \ \ exists\right\}$$

and

$$e_\infty^r(\Delta) = \left\{x = (x_k) \in w \colon \sup_{n\in\mathbb{N}} \left|\sum_{k=0}^{n} \binom{n}{k}(1-r)^{n-k}r^k(x_k - x_{k-1})\right| < \infty\right\}$$

Quite recently, Bişgin has generalized Altay, Başar and Mursaleen's works by defining the Binomial sequence spaces $b_0^{r,s}, b_c^{r,s}, b_\infty^{r,s}$ *and* $b_p^{r,s}$ in [14] and [15] as follows:

$$b_0^{r,s} = \left\{x = (x_k) \in w \colon \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k}r^k x_k = 0\right\},$$

$$b_c^{r,s} = \left\{x = (x_k) \in w \colon \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k}r^k x_k \ \ exists\right\},$$

$$b_\infty^{r,s} = \left\{x = (x_k) \in w \colon \sup_{n\in\mathbb{N}} \left|\frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k}r^k x_k\right| < \infty\right\}$$

and

$$b_p^{r,s} = \left\{x = (x_k) \in w \colon \sum_{n} \left|\frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k}r^k x_k\right|^p < \infty\right\}$$

where $1 \le p < \infty$ and the Binomial matrix $B^{r,s} = \left(b_{nk}^{r,s}\right)$ is defined by

$$b_{nk}^{r,s} = \begin{cases} \dfrac{1}{(s+r)^n}\binom{n}{k} s^{n-k}r^k & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, $r, s \in \mathbb{R}$ and $s.r > 0$. Here, if we take $r + s = 1$, we obtain the Euler matrix of order r.

By considering the Binomial matrix and generalized difference matrix $G = (g_{nk})$, we define the sequence space $b_p^{r,s}(G)$ by

$$b_p^{r,s}(G) = \left\{ x = (x_k) \in w : \sum_n \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (u x_k + v x_{k-1}) \right|^p < \infty \right\}$$

where $1 \le p < \infty$ and generalized difference matrix $G = (g_{nk})$ is defined by

$$g_{nk} = \begin{cases} u & , \quad k = n \\ v & , \quad k = n-1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $n, k \in \mathbb{N}$ and $u, v \in \mathbb{R} \backslash \{0\}$. Here, we would like to touch on a point, if we take $u = 1$ and $v = -1$, we obtain the difference matrix Δ. So, generalized difference matrix generalizes the difference matrix [13].

If we use the domain of the generalized difference matrix, we define the sequence space $b_p^{r,s}(G)$ by

$$b_p^{r,s}(G) = \left(b_p^{r,s}\right)_G \tag{2.1}$$

Also, by constructing a matrix $T^{r,s} = \left(t_{nk}^{r,s}\right)$ so that

$$t_{nk}^{r,s} = \begin{cases} \dfrac{s^{n-k-1} r^k}{(s+r)^n} \left[us \binom{n}{k} + vr \binom{n}{k+1} \right] & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, we redefine the sequence space $b_p^{r,s}(G)$ by aid of the $T^{r,s} = \left(t_{nk}^{r,s}\right)$ matrix as follows:

$$b_p^{r,s}(G) = \left(l_p\right)_{T^{r,s}} \tag{2.2}$$

So, for given $x = (x_k) \in w$, the $T^{r,s}$-transform of x is defined by

$$y_k = (T^{r,s} x)_k = \frac{1}{(s+r)^k} \sum_{i=0}^{k} \binom{k}{i} s^{k-i} r^i (u x_i + v x_{i-1}) \tag{2.3}$$

or

$$y_k = (T^{r,s} x)_k = \frac{1}{(s+r)^k} \sum_{i=0}^{k} \left[us \binom{k}{i} + vr \binom{k}{i+1} \right] s^{k-i-1} r^i x_i \tag{2.4}$$

for all $k \in \mathbb{N}$.

Theorem 2.1

The sequence space $b_p^{r,s}(G)$ is a BK-space with its norm defined by

$$\|x\|_{b_p^{r,s}(G)} = \|T^{r,s} x\|_p = \left(\sum_{k=0}^{\infty} |(T^{r,s} x)_k|^p \right)^{\frac{1}{p}}$$

where $1 \le p < \infty$.

Proof. It is known that l_p is a BK-space according to its p-norm and (2.2) holds. Also, the matrix $T^{r,s} = \left(t_{nk}^{r,s}\right)$ is a triangle. By combining these results and Theorem 4.3.12 of Wilansky [2], we deduce that the sequence space $b_p^{r,s}(G)$ is a BK-space, where $1 \le p < \infty$. This completes the proof.

Theorem 2.2

The sequence space $b_p^{r,s}(G)$ is linearly isomorphic to the sequence space l_p, where $1 \leq p < \infty$.

Proof. Let L be a transformation such that $L: b_p^{r,s}(G) \longrightarrow l_p$, $L(x) = T^{r,s}x$. Then, we should show that L is a linear bijection. The linearity of L and $x = \theta$ whenever $Tx = \theta$ are clear. So, L is injective.

Now, let us define a sequence $x = (x_k)$ such that

$$x_k = \frac{1}{u}\sum_{j=0}^{k}\left[\sum_{i=j}^{k}\binom{i}{j}\left(-\frac{v}{u}\right)^{k-i}(-s)^{i-j}(r+s)^{j}r^{-i}\right]y_j$$

for all $k \in \mathbb{N}$, where $y = (y_k) \in l_p$ and $1 \leq p < \infty$. Then, we have

$$\begin{aligned} ux_k + vx_{k-1} &= \sum_{j=0}^{k}\left[\sum_{i=j}^{k}\binom{i}{j}\left(-\frac{v}{u}\right)^{k-i}(-s)^{i-j}(r+s)^{j}r^{-i}\right]y_j \\ &\quad - \sum_{j=0}^{k-1}\left[\sum_{i=j}^{k-1}\binom{i}{j}\left(-\frac{v}{u}\right)^{k-i}(-s)^{i-j}(r+s)^{j}r^{-i}\right]y_j \\ &= \sum_{j=0}^{k}\binom{k}{j}(-s)^{k-j}(r+s)^{j}r^{-k}y_j \end{aligned}$$

and so

$$\begin{aligned} \|x\|_{b_p^{r,s}(G)} &= \|T^{r,s}x\|_p \\ &= \left(\sum_{n=0}^{\infty}|(T^{r,s}x)_n|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=0}^{\infty}\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^{k}(ux_k + vx_{k-1})\right|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=0}^{\infty}\left|\frac{1}{(s+r)^n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}r^{k}\sum_{j=0}^{k}\binom{k}{j}(-s)^{k-j}(r+s)^{j}r^{-k}y_j\right|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{n=0}^{\infty}|y_n|^p\right)^{\frac{1}{p}} \\ &= \|y\|_p < \infty. \end{aligned}$$

Therefore, L is norm preserving and $x = (x_n) \in b_p^{r,s}(G)$ for all $y = (y_k) \in l_p$, namely L is surjective. As a consequence, L is a linear bijection as desired. This completes the proof.

Theorem 2.3

The sequence space $b_p^{r,s}(G)$ is not a Hilbert space in circumstances $p \neq 2$, where $1 \leq p < \infty$.

Proof. Let us take $p = 2$. One can say from the Theorem 2.1 that the sequence space $b_2^{r,s}(G)$ is a BK-space with its norm defined by

$$\|x\|_{b_2^{r,s}(G)} = \|T^{r,s}x\|_2 = \left(\sum_{k=0}^{\infty} |(T^{r,s}x)_k|^2\right)^{\frac{1}{2}}$$

which is also generated by an inner product such that

$$\|x\|_{b_2^{r,s}(G)} = \langle T^{r,s}x, T^{r,s}x\rangle^{\frac{1}{2}}.$$

So, $b_2^{r,s}(G)$ is a Hilbert space.

On the other hand, assuming that $p \in [1,\infty)\backslash\{2\}$, we define two sequences $y = (y_k)$ and $z = (z_k)$ as follows:

$$y_k = \frac{1}{u}\sum_{i=0}^{k}\left(-\frac{v}{u}\right)^{k-i}\left(-\frac{s}{r}\right)^{i-1}\frac{-s+i(r+s)}{r}$$

and

$$z_k = \frac{1}{u}\sum_{i=0}^{k}\left(-\frac{v}{u}\right)^{k-i}\left(-\frac{s}{r}\right)^{i-1}\frac{-s-i(r+s)}{r}$$

for all $k \in \mathbb{N}$. Then we get

$$\|y+z\|_{b_p^{r,s}(G)}^2 + \|y-z\|_{b_p^{r,s}(G)}^2 = 8 \neq 2^{\frac{2}{p}+2} = 2\left[\|y\|_{b_p^{r,s}(G)}^2 + \|z\|_{b_p^{r,s}(G)}^2\right].$$

Therefore, the norm of the sequence space $b_p^{r,s}(G)$ does not satisfy the parallelogram equality, namely the norm can not be generated by an inner product. As a consequence, the sequence space $b_p^{r,s}(G)$ is not a Hilbert space in circumstances $p \neq 2$, where $1 \leq p < \infty$. This completes the proof.

Theorem 2.4

The inclusion $l_p(G) \subset b_p^{r,s}(G)$ strictly holds, where $1 \leq p < \infty$.

Proof. We give the proof of theorem for $1 < p < \infty$. In case of $p = 1$, the proof can be given by using a similar way.

For a given arbitrary sequence $x = (x_k) \in l_p(G)$, from the definition of the sequence space $l_p(G)$, we have

$$\sum_k |ux_k + vx_{k-1}|^p < \infty$$

where $1 < p < \infty$. Also, by considering the Hölder's inequality, we write

$$|(T^{r,s}x)_k|^p=\left|\frac{1}{(s+r)^k}\sum_{j=0}^{k}\binom{k}{j}s^{k-j}r^j\left(ux_j+vx_{j-1}\right)\right|^p$$

$$\leq\left(\frac{1}{|s+r|^k}\right)^p\left[\sum_{j=0}^{k}\left[\left(\binom{k}{j}|s|^{k-j}|r|^j\right)^{\frac{1}{q}}\right]\left[\left(\binom{k}{j}|s|^{k-j}|r|^j\right)^{\frac{1}{p}}\left|ux_j+vx_{j-1}\right|\right]\right]^p$$

$$\leq\left(\frac{1}{|s+r|^k}\right)^p\left[\left(\sum_{j=0}^{k}\binom{k}{j}|s|^{k-j}|r|^j\right)^{p-1}\times\left(\sum_{j=0}^{k}\binom{k}{j}|s|^{k-j}|r|^j\left|ux_j+vx_{j-1}\right|^p\right)\right]$$

$$=\frac{1}{|s+r|^k}\sum_{j=0}^{k}\binom{k}{j}|s|^{k-j}|r|^j\left|ux_j+vx_{j-1}\right|^p$$

$$=\sum_{j=0}^{k}\binom{k}{j}\left|\frac{s}{s+r}\right|^k\left|\frac{r}{s}\right|^j\left|ux_j+vx_{j-1}\right|^p$$

where $1<p<\infty$. Then we obtain

$$\sum_k|(T^{r,s}x)_k|^p\leq\sum_k\sum_{j=0}^{k}\binom{k}{j}\left|\frac{s}{s+r}\right|^k\left|\frac{r}{s}\right|^j\left|ux_j+vx_{j-1}\right|^p$$

$$=\sum_j\left|ux_j+vx_{j-1}\right|^p\sum_{k=j}^{\infty}\binom{k}{j}\left|\frac{s}{s+r}\right|^k\left|\frac{r}{s}\right|^j$$

$$=\left|\frac{s+r}{s}\right|\sum_j\left|ux_j+vx_{j-1}\right|^p$$

where $1<p<\infty$. If we connect this result and comparison test, we bring to a conclusion that $T^{r,s}x\in l_p$, namely $x=(x_k)\in b_p^{r,s}(G)$. This gives us that $l_p(G)\subset b_p^{r,s}(G)$.

Let us define a sequence $z=(z_k)$ such that $z_k=\frac{(-1)^k}{u-v}\left[1-\left(\frac{v}{u}\right)^{k+1}\right]$ for all $k\in\mathbb{N}$ and $u\neq v$. Then, one can see that $Gz=\left((-1)^k\right)\notin l_p$ and $T^{r,s}z=\left(\left(\frac{s-r}{s+r}\right)^k\right)\in l_p$, namely $z=(z_k)\notin l_p(G)$ and $z=(z_k)\in b_p^{r,s}(G)$. This shows us that the inclusion $l_p(G)\subset b_p^{r,s}(G)$ is strict. This completes the proof.

Theorem 2.5

The inclusion $b_p^{r,s}(G)\subset b_q^{r,s}(G)$ strictly holds in case of $1\leq p<q<\infty$.

Proof. It is known that the inclusion $l_p\subset l_q$ holds in case of $1\leq p<q<\infty$. Let us take an arbitrary sequence $x=(x_k)\in b_p^{r,s}(G)$. Then, we have $T^{r,s}x\in l_p$. By combining these two facts, we write $T^{r,s}x\in l_q$, namely $x=(x_k)\in b_q^{r,s}(G)$. This shows us that the inclusion $b_p^{r,s}(G)\subset b_q^{r,s}(G)$ holds.

Let us consider the sequence $d=(d_k)$ defined by

$$d_k = \frac{1}{u}\sum_{j=0}^{k}\left[\sum_{i=j}^{k}\binom{i}{j}\left(-\frac{v}{u}\right)^{k-i}(-s)^{i-j}(r+s)^j r^{-i}\right](j+1)^{-\frac{1}{p}}$$

for all $k \in \mathbb{N}$. Then, it is clear that $T^{r,s}d = \left(\frac{1}{(k+1)^{\frac{1}{p}}}\right) \in l_q \backslash l_p$, namely $d = (d_k) \in b_q^{r,s}(G) \backslash b_p^{r,s}(G)$ in case of $1 \le p < q < \infty$. Therefore the inclusion $b_p^{r,s}(G) \subset b_q^{r,s}(G)$ strictly holds. This completes the proof.

Theorem 2.6

The sequence spaces $b_p^{r,s}(G)$ and $l_\infty(G)$ overlap but do not include each other, where $p \in [1,\infty)$.

Proof. Let us define three sequences $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ such that

$$x_k = \frac{(-1)^k}{u-v}\left[1-\left(\frac{v}{u}\right)^{k+1}\right], \; y_k = \frac{1}{u+v}\left[1-\left(-\frac{v}{u}\right)^{k+1}\right] \; \text{and} \; z_k = \frac{r(-1)^k}{vr-us}\left(\frac{v}{u}\right)^{k+1}\left[1-\left(\frac{us}{vr}\right)^{k+1}\right]$$

for all $k \in \mathbb{N}$, where $u - v \neq 0$, $u + v \neq 0$, $vr - us \neq 0$, $\left|\frac{s}{r}\right| > 1$. Then $Gx = \left((-1)^k\right) \in l_\infty$, $T^{r,s}x = \left(\left(\frac{s-r}{s+r}\right)^k\right) \in l_p$, $Gy = e \in l_\infty$, $T^{r,s}y = e \notin l_p$, $Gz = \left(\left(-\frac{s}{r}\right)^k\right) \notin l_\infty$ and $T^{r,s}z = (1,0,0,\dots) \in l_p$, namely $x \in l_\infty(G) \cap b_p^{r,s}(G)$, $y \in l_\infty(G) \backslash b_p^{r,s}(G)$ and $z \in b_p^{r,s}(G) \backslash l_\infty(G)$. As a consequence of these the spaces $b_p^{r,s}(G)$ and $l_\infty(G)$ overlap but do not include each other, where $p \in [1,\infty)$. This completes the proof.

3. The Schauder Basis And $\boldsymbol{\alpha-,\beta-,\gamma}$ −Duals Of The Space $\boldsymbol{b_p^{r,s}(G)}$

In this section, we determine the Schauder basis and α-, β-, γ-duals of the sequence space $b_p^{r,s}(G)$.

A sequence $y = (y_k)$ is called a Schauder basis of a normed space $(X, \|.\|_X)$, if for each $x = (x_k) \in X$, there exists a unique sequence $\lambda = (\lambda_k)$ of scalars such that

$$\lim_{m\to\infty}\left\|x - \sum_{k=0}^{m}\lambda_k y_k\right\|_X = 0.$$

Then the expansion of $x = (x_k)$ with respect to $y = (y_k)$ is written by

$$x = \sum_{k=0}^{\infty}\lambda_k y_k$$

We know from [16] of Jarrah and Malkowsky that X_A has a Schauder basis if and only if X has a Schauder basis whenever $A = (a_{nk})$ is a triangle. Also, the sequence $\left(e^{(k)}\right)$ is a Schauder basis for l_p and the matrix $T^{r,s} = \left(t_{nk}^{r,s}\right)$ is a triangle, where $e^{(k)}$ is a sequence with 1 in the k-th place and zeros elsewhere.

By combining these results, we can give next corollary.

Corollary 3.1

Let $\mu^{(k)}(r,s) = \left\{\mu_n^{(k)}(r,s)\right\}_{n\in\mathbb{N}}$ be a sequence defined by

$$\mu_n^{(k)}(r,s)=\begin{cases}\frac{1}{u}\sum_{i=k}^{n}\binom{i}{k}\left(-\frac{v}{u}\right)^{n-i}(-s)^{i-k}(r+s)^k r^{-i} & , \quad n\geq k\\ 0 & , \quad 0\leq n<k\end{cases}$$

for all fixed $k\in\mathbb{N}$. Then, the Schauder basis of the sequence space $b_p^{r,s}(G)$ is the sequence $\{\mu^{(k)}(r,s)\}_{k\in\mathbb{N}}$ and every $x=(x_k)\in b_p^{r,s}(G)$ can be uniquely written on the form

$$x=\sum_k \sigma_k\mu^{(k)}(r,s)$$

where $\sigma_k=(T^{r,s}x)_k$ for all $k\in\mathbb{N}$.

By connecting the results of Theorem 2.1 and Corollary 3.1 , one more result can be given.

Corollary 3.2

The sequence space $b_p^{r,s}(G)$ is separable.

A set defined by

$$M(X,Y)=\{y=(y_k)\in w:\ xy=(x_ky_k)\in Y\ \text{ for all } x=(x_k)\in X\}$$

is called the multiplier space of the sequence spaces X and Y. Then, the α-, β- and γ-duals of the sequence space X are defined by means of the multiplier space, l_1, cs and bs such that

$$X^\alpha=M(X,l_1)\ ,\ X^\beta=M(X,cs)\ \text{ and }\ X^\gamma=M(X,bs)$$

respectively.

Now, we continue with quoting lemmas from Stieglitz and Tietz [17].

Lemma 3.3 (see [17])

Let $A=(a_{nk})$ be an infinite matrix. Then, the following statements hold.

i-) $A=(a_{nk})\in(l_1:l_1)$ if and only if

$$\sup_{k\in\mathbb{N}}\sum_n|a_{nk}|<\infty \tag{3.1}$$

ii-) $A=(a_{nk})\in(l_1:l_\infty)$ if and only if

$$\sup_{n,k\in\mathbb{N}}|a_{nk}|<\infty \tag{3.2}$$

iii-) $A=(a_{nk})\in(l_1:c)$ if and only if (3.2) holds and

$$\lim_{n\to\infty}a_{nk}=a_k\ \text{ for all }\ k\in\mathbb{N} \tag{3.3}$$

Lemma 3.4 (see [17])

Let $A=(a_{nk})$ be an infinite matrix. Then, the following statements hold.

i-) $A=(a_{nk})\in(l_p:l_1)$ if and only if

$$\sup_{K\in\mathcal{F}}\sum_k\left|\sum_{n\in K}a_{nk}\right|^q<\infty \tag{3.4}$$

ii-) $A=(a_{nk})\in(l_p:l_\infty)$ if and only if

$$\sup_{n\in\mathbb{N}}\sum_k |a_{nk}|^q < \infty \tag{3.5}$$

iii-) $A = (a_{nk}) \in (l_p : c)$ if and only if (3.3) and (3.5) hold

where $\frac{1}{p}+\frac{1}{q} = 1$, $1 < p < \infty$ and $\mathcal{F}$ is the collection of all finite subset of $\mathbb{N}$.

Theorem 3.5

Let $\xi_1^{r,s}(\mathrm{G})$ and $\xi_2^{r,s}(\mathrm{G})$ be two sets defined by

$$\xi_1^{r,s}(\mathrm{G}) = \left\{a = (a_k) \in w : \sup_{K\in\mathcal{F}}\sum_k \left|\frac{1}{u}\sum_{n\in K}\sum_{i=k}^{n}\binom{i}{k}\left(-\frac{v}{u}\right)^{n-i}(-s)^{i-k}(r+s)^k r^{-i} a_n\right|^q < \infty\right\}$$

and

$$\xi_2^{r,s}(\mathrm{G}) = \left\{a = (a_k) \in w : \sup_{k\in\mathbb{N}}\sum_n \left|\frac{1}{u}\sum_{i=k}^{n}\binom{i}{k}\left(-\frac{v}{u}\right)^{n-i}(-s)^{i-k}(r+s)^k r^{-i} a_n\right| < \infty\right\}.$$

Then $\{b_1^{r,s}(G)\}^\alpha = \xi_2^{r,s}(\mathrm{G})$ and $\{b_p^{r,s}(G)\}^\alpha = \xi_1^{r,s}(\mathrm{G})$, where $1 < p < \infty$.

Proof. Consider the sequence $x = (x_n)$, which is defined by

$$x_n = \frac{1}{u}\sum_{k=0}^{n}\left[\sum_{i=k}^{n}\binom{i}{k}\left(-\frac{v}{u}\right)^{n-i}(-s)^{i-k}(r+s)^k r^{-i}\right] y_k \tag{3.6}$$

for all $n \in \mathbb{N}$. Then, for given $a = (a_n) \in w$, we write

$$a_n x_n = \sum_{k=0}^{n}\left[\frac{1}{u}\sum_{i=k}^{n}\binom{i}{k}\left(-\frac{v}{u}\right)^{n-i}(-s)^{i-k}(r+s)^k r^{-i} a_n\right] y_k = \sum_{k=0}^{n} d_{nk}^{r,s} y_k = (D^{r,s}y)_n$$

for all $n \in \mathbb{N}$. By taking into account the equality above, we observe that $ax = (a_n x_n) \in l_1$ whenever $x = (x_k) \in b_1^{r,s}(G)$ or $x = (x_k) \in b_p^{r,s}(G)$ if and only if $D^{r,s}y \in l_1$ whenever $y = (y_k) \in l_1$ or $y = (y_k) \in l_p$, respectively where $1 < p < \infty$. So, we obtain that $a = (a_n) \in \{b_1^{r,s}(G)\}^\alpha$ or $a = (a_n) \in \{b_p^{r,s}(G)\}^\alpha$ if and only if $D^{r,s} \in (l_1 : l_1)$ or $D^{r,s} \in (l_p : l_1)$, respectively, where $1 < p < \infty$. By connecting these results, Lemma 3.3(i) and Lemma 3.4(i), we deduce that

$$a = (a_n) \in \{b_1^{r,s}(G)\}^\alpha \Leftrightarrow \sup_{k\in\mathbb{N}}\sum_n \left|\frac{1}{u}\sum_{i=k}^{n}\binom{i}{k}\left(-\frac{v}{u}\right)^{n-i}(-s)^{i-k}(r+s)^k r^{-i} a_n\right| < \infty$$

and

$$a = (a_n) \in \{b_p^{r,s}(G)\}^\alpha \Leftrightarrow \sup_{K\in\mathcal{F}}\sum_k \left|\frac{1}{u}\sum_{n\in K}\sum_{i=k}^{n}\binom{i}{k}\left(-\frac{v}{u}\right)^{n-i}(-s)^{i-k}(r+s)^k r^{-i} a_n\right|^q < \infty$$

where $1 < p < \infty$. These yield us that $\{b_1^{r,s}(G)\}^\alpha = \xi_2^{r,s}(\mathrm{G})$ and $\{b_p^{r,s}(G)\}^\alpha = \xi_1^{r,s}(\mathrm{G})$, where $1 < p < \infty$. This completes the proof.

Theorem 3.6

Consider the sets $\xi_3^{r,s}(\mathrm{G})$, $\xi_4^{r,s}(\mathrm{G})$ and $\xi_5^{r,s}(\mathrm{G})$ defined by

$$\xi_3^{r,s}(\mathrm{G}) = \left\{ a = (a_k) \in w : \frac{1}{u}\sum_{j=k}^{\infty}\sum_{i=k}^{j}\binom{i}{k}\left(-\frac{v}{u}\right)^{j-i}(-s)^{i-k}(r+s)^k r^{-i} a_j \ \text{ exists for all } k \in \mathbb{N} \right\}$$

$$\xi_4^{r,s}(\mathrm{G}) = \left\{ a = (a_k) \in w : \sup_{k,n\in\mathbb{N}} \left| \frac{1}{u}\sum_{j=k}^{n}\sum_{i=k}^{j}\binom{i}{k}\left(-\frac{v}{u}\right)^{j-i}(-s)^{i-k}(r+s)^k r^{-i} a_j \right| < \infty \right\}$$

and

$$\xi_5^{r,s}(\mathrm{G}) = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \frac{1}{u}\sum_{j=k}^{n}\sum_{i=k}^{j}\binom{i}{k}\left(-\frac{v}{u}\right)^{j-i}(-s)^{i-k}(r+s)^k r^{-i} a_j \right|^q < \infty \right\}$$

where $1 < q < \infty$.

Then the following statements hold:

(I) $\{b_1^{r,s}(G)\}^\beta = \xi_3^{r,s}(\mathrm{G}) \cap \xi_4^{r,s}(\mathrm{G})$,

(II) $\left\{b_p^{r,s}(G)\right\}^\beta = \xi_3^{r,s}(\mathrm{G}) \cap \xi_5^{r,s}(\mathrm{G})$, where $1 < p < \infty$,

(III) $\{b_1^{r,s}(G)\}^\gamma = \xi_4^{r,s}(\mathrm{G})$,

(IV) $\left\{b_p^{r,s}(G)\right\}^\gamma = \xi_5^{r,s}(\mathrm{G})$, where $1 < p < \infty$.

Proof. Since the proofs of the parts (II), (III) and (IV) may be obtained by using a same way, we prove the theorem for only the part (I). Let $a = (a_n) \in w$ be arbitrarily given. Consider the sequence $x = (x_n)$ defined by the relation (3.6). Then, we write

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left[\frac{1}{u}\sum_{j=0}^{k}\sum_{i=j}^{k}\binom{i}{j}\left(-\frac{v}{u}\right)^{k-i}(-s)^{i-j}(r+s)^j r^{-i} y_j\right] a_k \\ &= \sum_{k=0}^{n}\left[\frac{1}{u}\sum_{j=k}^{n}\sum_{i=k}^{j}\binom{i}{k}\left(-\frac{v}{u}\right)^{j-i}(-s)^{i-k}(r+s)^k r^{-i} a_j\right] y_k \\ &= (V^{r,s}y)_n \end{aligned}$$

for all $n \in \mathbb{N}$, where the matrix $V^{r,s} = \left(v_{nk}^{r,s}\right)$ is defined by

$$v_{nk}^{r,s} = \begin{cases} \dfrac{1}{u}\displaystyle\sum_{j=k}^{n}\sum_{i=k}^{j}\binom{i}{k}\left(-\frac{v}{u}\right)^{j-i}(-s)^{i-k}(r+s)^k r^{-i} a_j & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. So, $ax = (a_n x_n) \in cs$ whenever $x = (x_k) \in b_1^{r,s}(G)$ if and only if $V^{r,s}y \in c$ whenever $y = (y_k) \in l_1$. This yields us that $a = (a_n) \in \{b_1^{r,s}(G)\}^\beta$ if and only if $V^{r,s} \in (l_1 : c)$. By connecting this result and Lemma 3.3 (iii), we obtain that $a = (a_n) \in \{b_1^{r,s}(G)\}^\beta$ if and only if

$$\sup_{k,n\in\mathbb{N}} \left| \frac{1}{u}\sum_{j=k}^{n}\sum_{i=k}^{j}\binom{i}{k}\left(-\frac{v}{u}\right)^{j-i}(-s)^{i-k}(r+s)^k r^{-i} a_j \right| < \infty$$

and

$$\frac{1}{u}\sum_{j=k}^{\infty}\sum_{i=k}^{j}\binom{i}{k}\left(-\frac{v}{u}\right)^{j-i}(-s)^{i-k}(r+s)^{k}r^{-i}\,a_j \text{ exists for all } k\in\mathbb{N}$$

This result shows us that $\{b_1^{r,s}(G)\}^{\beta}=\xi_3^{r,s}(\mathrm{G})\cap\xi_4^{r,s}(\mathrm{G})$. This completes the proof.

4. Some Matrix Classes

In this section, we charecterize some matrix classes related to the sequence space $b_p^{r,s}(G)$, where $1\le p<\infty$.

For simplicity in notation, we prefer to use following equality throughout the section 4.

$$h_{nk}^{r,s,G}=\frac{1}{u}\sum_{j=k}^{\infty}\sum_{i=k}^{j}\binom{i}{k}\left(-\frac{v}{u}\right)^{j-i}(-s)^{i-k}(r+s)^{k}r^{-i}\,a_{nj}$$

for all $n,k\in\mathbb{N}$.

Theorem 4.1

Given an infinite matrix $A=(a_{nk})$, the following statements hold.

(i) $A=(a_{nk})\in(b_1^{r,s}(G):l_\infty)$ if and only if

$$\sup_{k,n\in\mathbb{N}}\left|h_{nk}^{r,s,G}\right|<\infty \tag{4.1}$$

(ii) $A=(a_{nk})\in\left(b_p^{r,s}(G):l_\infty\right)$ if and only if

$$\sup_{n\in\mathbb{N}}\sum_{k}\left|h_{nk}^{r,s,G}\right|^{q}<\infty \tag{4.2}$$

$$\{a_{nk}\}_{k\in\mathbb{N}}\in\xi_5^{r,s}(\mathrm{G}) \tag{4.3}$$

where $1<p<\infty$.

Proof. Let $p\in(1,\infty)$. We take any $x=(x_k)\in b_p^{r,s}(G)$ by assuming that the conditions (4.2) and (4.3) hold. Then, it is obtained that $\{a_{nk}\}_{k\in\mathbb{N}}\in\left\{b_p^{r,s}(G)\right\}^{\beta}$. This result implies the existence of the A transform of x. From the relation (3.6), we have

$$\begin{aligned}\sum_{k=0}^{m}a_{nk}x_k&=\sum_{k=0}^{m}\left[\frac{1}{u}\sum_{j=0}^{k}\sum_{i=j}^{k}\binom{i}{j}\left(-\frac{v}{u}\right)^{k-i}(-s)^{i-j}(r+s)^{j}r^{-i}\,y_j\right]a_{nk}\\&=\sum_{k=0}^{m}\sum_{j=k}^{m}\left[\frac{1}{u}\sum_{i=k}^{j}\binom{i}{k}\left(-\frac{v}{u}\right)^{j-i}(-s)^{i-k}(r+s)^{k}r^{-i}\right]a_{nj}y_k\end{aligned} \tag{4.4}$$

By taking limit (4.4) side by side as $m\to\infty$, we obtain that

$$\sum_{k}a_{nk}x_k=\sum_{k}h_{nk}^{r,s,G}y_k\quad(n\in\mathbb{N}) \tag{4.5}$$

Then, we derive by taking l_∞-norm (4.5) side by side and by applying Hölder's inequality that

$$\|Ax\|_\infty = \sup_{n\in\mathbb{N}}\left|\sum_k h_{nk}^{r,s,G} y_k\right|$$

$$\leq \sup_{n\in\mathbb{N}}\left(\sum_k \left|h_{nk}^{r,s,G}\right|^q\right)^{\frac{1}{q}}\left(\sum_k |y_k|^p\right)^{\frac{1}{p}} < \infty$$

As a result of this, we obtain that $Ax \in l_\infty$, namely $A = (a_{nk}) \in \left(b_p^{r,s}(G): l_\infty\right)$.

Conversly, assume that $A = (a_{nk}) \in \left(b_p^{r,s}(G): l_\infty\right)$. This gives us to $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{b_p^{r,s}(G)\right\}^\beta$ for all $n \in \mathbb{N}$. Then, the necessity of (4.3) is immediate and $\left\{h_{nk}^{r,s,G}\right\}_{k,n\in\mathbb{N}}$ exists. On account of $\{a_{nk}\}_{k\in\mathbb{N}} \in \left\{b_p^{r,s}(G)\right\}^\beta$, we can see that the condition (4.5) holds and the sequences $a_n = (a_{nk})_{k\in\mathbb{N}}$ define the continuous linear functionals f_n on $b_p^{r,s}(G)$ by

$$f_n(x) = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$. Also, we know from the Theorem 2.2 that the sequence spaces $b_p^{r,s}(G)$ and l_p are norm isomorphic. By connecting this result and the condition (4.5), we obtain that

$$\|f_n\| = \left\|\left(h_{nk}^{r,s,G}\right)_{k\in\mathbb{N}}\right\|_q$$

which yields that the functionals f_n are pointwise bounded. Moreover, we derive from the Banach-Steinhaus theorem that the functionals f_n are uniformly bounded, namely there exists a constant $M > 0$ such that

$$\left(\sum_k \left|h_{nk}^{r,s,G}\right|^q\right)^{\frac{1}{q}} = \|f_n\| \leq M$$

for all $n \in \mathbb{N}$, which shows us that the condition (4.2) holds. The part (i) can be proved by using a similar method. This completes the proof.

Now, we quote a lemma from Stieglitz and Tietz [17], which is needed in the next proof.

Lemma 4.2 (see [17])

Let $A = (a_{nk})$ be an infinite matrix. Then, $A = (a_{nk}) \in \left(l_1: l_p\right)$ if and only if

$$\sup_{k\in\mathbb{N}} \sum_n |a_{nk}|^p < \infty$$

where $1 < p < \infty$.

Theorem 4.3

Let an infinite matrix $A = (a_{nk})$ be given. Then, $A = (a_{nk}) \in \left(b_1^{r,s}(G) : l_p\right)$ if and only if

$$\sup_{k\in\mathbb{N}} \sum_n \left|h_{nk}^{r,s,G}\right|^p < \infty \tag{4.6}$$

where $1 \leq p < \infty$.

Proof. Let a sequence $x=(x_k)\in b_1^{r,s}(G)$ be given. Assume that the condition (4.6) holds. Then, it is clear that $y=(y_k)\in l_1$ and $\{a_{nk}\}_{k\in\mathbb{N}}\in\{b_1^{r,s}(G)\}^{\beta}$ for all $n\in\mathbb{N}$, namely A-transform of x exists. As a result of this, the series $\sum_k h_{nk}^{r,s,G}y_k$ are absolutely convergent for all $n\in\mathbb{N}$ and $y=(y_k)\in l_1$. By applying the Minkowsky inequality to (4.5), we can write

$$\left(\sum_n|(Ax)_n|^p\right)^{\frac{1}{p}}\leq\sum_k|y_k|\left(\sum_n\left|h_{nk}^{r,s,G}\right|^p\right)^{\frac{1}{p}}$$

which yields that $Ax\in l_p$, namely $A=(a_{nk})\in\left(b_1^{r,s}(G):l_p\right)$.

Conversly, we suppose that $A=(a_{nk})\in\left(b_1^{r,s}(G):l_p\right)$, where $1\leq p<\infty$, namely $Ax\in l_p$ for all $x=(x_k)\in b_1^{r,s}(G)$. So, $\{a_{nk}\}_{k\in\mathbb{N}}\in\{b_1^{r,s}(G)\}^{\beta}$ for all $n\in\mathbb{N}$, which shows us that the relation (4.5) holds. These results give us that $H^{r,s,G}=\left(h_{nk}^{r,s,G}\right)\in\left(l_1:l_p\right)$. By combining last result and Lemma 4.2, we obtain that the condition (4.6) holds. This completes the proof.

5. CONCLUSION

The domain of Binomial matrix $B^{r,s}=\left(b_{nk}^{r,s}\right)$ in the sequence space l_p has been introduced by Bişgin in [15]. Also, the domain of generalized difference(double band) matrix $G=(g_{nk})$ in some sequence spaces was used and studied by many authors. Since $T^{r,s}=\left(t_{nk}^{r,s}\right)$ is composition of $B^{r,s}=\left(b_{nk}^{r,s}\right)$ and $G=(g_{nk})$, and $T^{r,s}=\left(t_{nk}^{r,s}\right)$ is stronger than $G=(g_{nk})$, our results are more general.

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