Some New Integral Inequalities for n-Times Differentiable Godunova-Levin Functions

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Abstract: In this work, by using an integral identity together with the Hölder integral inequality we establish several new inequalities for *n*-times differentiable Godunova-Levin functions

Keywords: Convex function, Godunova-Levin function, Hölder Integral inequality.

n-kere Türevlenebilen Godunova-Levin Fonksiyonları için Bazı Yeni İntegral Eşitsizlikler

Özet: Bu çalışmada, Hölder integral eşitsizliği ile birlikte bir integral eşitliği kullanılarak n-kere türevlenebilen Godunova-Levin Fonksiyonları için bir kaç yeni eşitsizlik bulunmuştur.

Anahtar Kelimeler: Konveks fonksiyon, Godunova-Levin fonksiyonu, Hölder İntegral eşitsizliği.

1. INTRODUCTION

Theory of convex functions plays an important role in different fields of pure and applied sciences. Recently much attention has been given to theory of convex functions by many researchers. In this paper, by using the Hölder integral inequality, we establish some new inequalities for functions whose nth derivatives in absolute value are convex functions. For some inequalities, generalizations and applications concerning convexity see [5-12,

21]. Recently, in the literature there are so many papers about n-times differentiable functions on several kinds of convexities. In references [2-4, 11, 14, 17, 19, 20], readers can find some results about this issue. Many papers have been written by a number of mathematicians concerning inequalities for different classes of convex and Godunova-Levin functions see for instance the recent papers [1, 13, 15, 16, 18] and the references within these papers.

Definition 1.1: A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

Definition 1.2: A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be Godunova-Levin function, if

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}$$

where $\forall x, y \in I, t \in (0,1)$.

Throughout this paper we will use the following notations and conventions. Let $J = [0, \infty) \subset \mathbb{R} = (-\infty, +\infty)$, and $a, b \in J$ with $0 < a < b$ and

$$A(a,b) = \frac{a+b}{2}, \qquad L_p(a,b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, a \neq b, p \in \mathbb{R}, \qquad p \neq -1,0$$

be the arithmetic, geometric, identic, harmonic, logarithmic, generalized logarithmic mean for $a, b > 0$ respectively.

For we obtain the main results we will use the following Lemma [14].

Lemma 1.1: Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be n-times differentiable mapping on I° for $n \in \mathbb{N}$ and $f^{(n)} \in L[a,b]$, where $a, b \in I^{\circ}$ with $a < b$, we have the identity

$$\sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx = \frac{(-1)^{n+1}}{n!} \int_a^b x^n f^{(n)}(x)dx.$$

2. MAIN RESULTS

Theorem 2.1. For $\forall n \in \mathbb{N}$; let $f: I \subset (0, \infty) \to \mathbb{R}$ be n-times differentiable function on I° and $a, b \in I^{\circ}$ with $a < b$. If $\left|f^{(n)}\right|^q$ for $q > 1$ is Godunova-Levin function on $[a,b]$, then the following inequality holds:

$$\left|\sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right|$$

$$\leq \frac{1}{n!}(b-a)^{\frac{3}{q}} C^{\frac{1}{p}}(a,b,n,p) A^{\frac{1}{q}} \left(\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right)$$

Where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < 2$ and $C(a,b,n,p) = \int_a^b \frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}} dx$.

Proof. If $\left|f^{(n)}\right|^q$ for $q > 1$ is Godunova-Levin function on $[a,b]$, using Lemma1.1, the Hölder integral inequality and

$$\left|f^{(n)}(x)\right|^q = \left|f^{(n)}\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right)\right|^q \leq \frac{\left|f^{(n)}(b)\right|^q}{\frac{x-a}{b-a}} + \frac{\left|f^{(n)}(a)\right|^q}{\frac{b-x}{b-a}},$$

$$\left|f^{(n)}(x)\right|^q \leq \frac{b-a}{x-a}\left|f^{(n)}(b)\right|^q + \frac{b-a}{b-x}\left|f^{(n)}(a)\right|^q$$

$$(x-a)(b-x)\left|f^{(n)}(x)\right|^q \leq (b-a)(b-x)\left|f^{(n)}(b)\right|^q + (b-a)(x-a)\left|f^{(n)}(a)\right|^q$$

we have

$$\left|\sum_{k=0}^{n-1} (-1)^k \left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right| \leq \frac{1}{n!}\int_a^b x^n \left|f^{(n)}(x)\right|dx$$

$$\leq \frac{1}{n!}\left(\int_a^b \frac{x^{np}}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)(b-x)|f^{(n)}(x)|^q dx\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left(\int_a^b \left\{(b-a)(b-x)|f^{(n)}(b)|^q + (b-a)(x-a)|f^{(n)}(a)|^q\right\}dx\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left((b-a)|f^{(n)}(b)|^q\left(bx-\frac{x^2}{2}\right)\Big|_a^b + (b-a)|f^{(n)}(a)|^q\left(\frac{x^2}{2}-ax\right)\Big|_a^b\right)^{\frac{1}{q}}$$

$$= \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left[(b-a)|f^{(n)}(b)|^q\frac{(b-a)^2}{2} + (b-a)|f^{(n)}(a)|^q\frac{(b-a)^2}{2}\right]^{\frac{1}{q}}$$

$$= \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)\left[|f^{(n)}(b)|^q\frac{(b-a)^3}{2} + |f^{(n)}(a)|^q\frac{(b-a)^3}{2}\right]^{\frac{1}{q}}$$

$$= \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)(b-a)^{\frac{3}{q}}\left[\frac{|f^{(n)}(b)|^q + |f^{(n)}(a)|^q}{2}\right]^{\frac{1}{q}}$$

$$= \frac{1}{n!}C^{\frac{1}{p}}(a,b,n,p)(b-a)^{\frac{3}{q}}A^{\frac{1}{q}}\left(|f^{(n)}(a)|^q,|f^{(n)}(b)|^q\right)$$

It is stated that the improper integral $C(a,b,n,p)$ is convergent for $1 < p < 2$.

Corollary 2.1. Under the conditions Theorem 2.1 for $n = 1$ we have the following inequality:

$$\left|\frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq C^{\frac{1}{p}}(a,b,1,p)(b-a)^{\frac{3}{q}-1}A^{\frac{1}{q}}(|f'(a)|^q,|f'(b)|^q).$$

Theorem 2.2. For $n \in \mathbb{N}$; let $f:(0,\infty)\subset\mathbb{R}\to\mathbb{R}$ be n-times differentiable function and $0 \leq a < b$. If $|f^{(n)}|^q \in L[a,b]$ and $|f^{(n)}|^q$ for $q > 1$ is Godunova-Levin function on $[a,b]$, then the following inequality holds:

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right| \leq \frac{1}{n!}(b-a)^{\frac{2}{q}}D^{\frac{1}{p}}(a,b,n,p)$$

$$\times \left[|f^{(n)}(b)|^q\{bL_n^n(a,b) - L_{n+1}^{n+1}(a,b)\} + |f^{(n)}(a)|^q\{L_{n+1}^{n+1}(a,b) - aL_n^n(a,b)\}\right]^{\frac{1}{q}}$$

Where $\frac{1}{p}+\frac{1}{q}=1$, $1 < p < 2$ and $D(a,b,n,p) = \int_a^b \frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx$.

Proof. From Lemma1.1 and Hölder integral inequality, we obtain

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right) - \int_a^b f(x)dx\right| \leq \frac{1}{n!}\int_a^b x^n|f^{(n)}(x)|dx$$

$$\leq \frac{1}{n!}\left(\int_a^b \frac{x^n}{(x-a)^{p-1}(b-x)^{p-1}}dx\right)^{\frac{1}{p}}\left(\int_a^b x^n(b-x)(x-a)|f^{(n)}(x)|^q dx\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{n!} D^{\frac{1}{p}}(a,b,n,p) \left[\int_a^b x^n \left[(b-a)(b-x)\left|f^{(n)}(b)\right|^q + (b-a)(x-a)\left|f^{(n)}(a)\right|^q \right] dx \right]^{\frac{1}{q}}$$

$$\leq \frac{1}{n!} D^{\frac{1}{p}}(a,b,n,p)$$

$$\times \left[(b-a)\left|f^{(n)}(b)\right|^q \left\{ b\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right) - \left(\frac{b^{n+2}-a^{n+2}}{n+2}\right) \right\} \right.$$

$$\left. + (b-a)\left|f^{(n)}(a)\right|^q \left\{ \left(\frac{b^{n+2}-a^{n+2}}{n+2}\right) - a\left(\frac{b^{n+1}-a^{n+1}}{n+1}\right) \right\} \right]^{\frac{1}{q}}$$

$$\leq \frac{1}{n!} D^{\frac{1}{p}}(a,b,n,p)$$

$$\times \left[(b-a)^2\left|f^{(n)}(b)\right|^q \left\{ b\left(\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right) - \left(\frac{b^{n+2}-a^{n+2}}{(b-a)(n+2)}\right) \right\} \right.$$

$$\left. + (b-a)^2\left|f^{(n)}(a)\right|^q \left\{ \left(\frac{b^{n+2}-a^{n+2}}{(b-a)(n+2)}\right) - a\left(\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right) \right\} \right]^{\frac{1}{q}}$$

$$\leq \frac{1}{n!}(b-a)^{\frac{2}{q}} D^{\frac{1}{p}}(a,b,n,p)$$

$$\times \left[\left|f^{(n)}(b)\right|^q \{ bL_n^n(a,b) - L_{n+1}^{n+1}(a,b) \} + \left|f^{(n)}(a)\right|^q \{ L_{n+1}^{n+1}(a,b) - aL_n^n(a,b) \} \right]^{\frac{1}{q}}$$

It is stated that the improper integral $D(a,b,n,p)$ is convergent for $1 < p < 2$.

Corollary 2.2. Under the conditions Theorem 2.2 for $n = 1$ we have the following inequality:

$$\left| \frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \left(\frac{1}{6}\right)^{\frac{1}{q}}(b-a)^{\frac{3}{q}-1} D^{\frac{1}{p}}(a,b,1,p)[(b+2a)|f'(b)|^q + (2b+a)|f'(a)|^q]^{\frac{1}{q}}.$$

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