Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.38-4 (2017) 626-638

Some New Time Scales Weighted Ostrowski-Grüss Type Inequalities

Adnan TUNA

Nigde Omer Halisdemir University, Art and Science Faculty, Mathematics Department, 51240, Niğde / TURKEY

Received: 24.02.2017; Accepted: 21.06.2017 *http://dx.doi.org/10.17776/csj.349330*

Abstract: In this paper, we obtain some new weighted Ostrowski-Grüss type inequalities on time scales. We also give some other interesting inequalities as special cases.

2010 Mathematics Subject Classifications : 26D15; 26E70.

Keywords: Montgomery Identitiy, Ostrowski-Grüss Type Inequalities, Time Scales.

Zaman Skalasında Bazı Yeni Ağırlıklı Ostrowski-Grüss Tipi Eşitsizlikler

Özet: Bu makalede, zaman skalasında bazı yeni ağırlıklı Ostrowski-Grüss tipi eşitsizlik elde edilmiştir. Buna ilaveten özel durumlar olarak bazı diğer ilgili eşitsizlikler verilmiştir.

Anahtar Kelimeler: Montgomery Eşitliği, Ostrowski-Grüss Tipi Eşitsizlikler, Zaman Skalası.

1. INTRODUCTION

In 1938, Ostrowski [11] proved the following classical Ostrowski inequality.

Theorem 1.1 Let $f:[a,b]\to R$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f':(a,b)\to R$ is bounded in (a,b). Then for any $x\in[a,b]$, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left(\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)\|f'\|_\infty, \tag{1}$$

where $\|f'\|_\infty:=\sup_{x\in(a,b)}|f'(x)|<\infty$. This inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

Using the Montgomery identity on time scales, Bohner and Matthews [4] obtained the time scales Ostrowski inequality.

* Corresponding author. *Email address:* *atuna@ohu.edu.tr*

The following inequality is Ostrowski-Grüss type integral inequality given by Dragomir and Wang [5] in 1997.

Theorem 1.2 Let $f:[a,b]\to R$ be continuous on $[a,b]$ and differentiable in (a,b). If $\gamma\le f'(x)\le\Gamma$ for all $x\in[a,b]$ for some $\gamma,\Gamma\in R$, then, for all $x\in[a,b]$, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\le\frac{1}{4}(b-a)(\Gamma-\gamma), \tag{2}$$

In order to unify continuous and discrete analysis, S. Hilger [6] introduced the time scales theory in 1988. For some Ostrowski-Grüss type inequalities on time scales, see the papers [8,9,12,13] where further references are provided.

In the present paper, we give the weighted Ostrowski-Grüss type inequalities on time scales and apply our results to the continuous, discrete and quantum calculus cases.

2. GENERAL DEFINITIONS

We briefly introduce the time scales elements in this section and refer the reader to Hilger's Ph.D. thesis [6], the books [2,3,7], and the survey [1] for proofs and further details.

Definition 2.1 A time scale T is a nonempty closed subset of R.

We assume that T has the topology that is inherited from the standard topology on R. The interval $[a,b]$ in T means the set $[a,b]:=\{t\in T: a\le t\le b\}$ for the points $a<b$ in T.

Definition 2.2 The forward jump operator $\sigma:T\to T$ by $\sigma(t)=\inf\{s\in T: s>t\}$, while the backward jump operator $\rho:T\to T$ is defined by $\rho(t)=\sup\{s\in T: s<t\}$.

Definition 2.3 A point $t\in T$ is called right-scattered, left-scattered, right-dense and left-dense if $\sigma(t)>t$, $\rho(t)<t$, $\sigma(t)=t$ and $\rho(t)=t$, respectively. Points that are both right-dense and left-dense are called dense.

Definition 2.4 The graininess function $\mu:T\to[0,\infty)$ is defined by $\mu(t)=\sigma(t)-t$ for $t\in T$. The set T^k is defined as follows: if T has a left-scattered maximum m, then $T^k=T-\{m\}$; otherwise, $T^k=T$.

If $T=R$, then $\mu(t)=0$, and when $T=Z$, we have $\mu(t)=1$.

Definition 2.5 Let $f:T\to R$ and fix $t\in T^k$. Then we define $f^\Delta(t)\in R$ to be the number (provided it exists) with the property that for any given $\varepsilon>0$ there exists a neighborhood U of t such that

$$\left|f(\sigma(t))-f(s)-f^\Delta(t)[\sigma(t)-s]\right|\le\varepsilon|\sigma(t)-s|, \qquad \forall s\in U.$$

We call $f^\Delta(t)$ the delta derivative of $f(t)$ at t.

If $T=R$, then $f^\Delta(t)=\frac{df(t)}{dt}$. In the case $T=Z$, $f^\Delta(t)=\Delta f(t)=f(t+1)-f(t)$, that is, is the usual forward difference operator.

Theorem 2.6 Assume $f,g:T\to R$ are differentiable at $t\in T^k$. Then the product $fg:T\to R$ is differentiable at t with

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t).$$

Definition 2.7 The function $f:T\to R$ is said to be rd-continuous (denote $f\in C_{rd}(T,\mathrm{R})$) on T provided it is continuous at all right-dense points $t\in T$ and its left-sided limits exist at all left-dense points $t\in T$.

It follows from [2,Theorem 1.74] that every rd-continuous function has an anti-derivative.

Definition 2.8 Let $f\in C_{rd}(T,\mathrm{R})$. Then $F:T^k\to R$ is called a delta-antiderivative of f on T if it satisfies $F^{\Delta}(t)=f(t)$ for all $t\in T^k$. In this case we define the Cauchy integral

$$\int_a^b f(s)\Delta s=F(b)-F(a),\ \ a,b\in T.$$

Theorem 2.9 Let f,g be rd-continuous, $a,b,c\in T$ and $\alpha,\beta\in R$. Then

(1) $\int_a^b[\alpha f(t)+\beta g(t)]\Delta t=\alpha\int_a^b f(t)\Delta t+\beta\int_a^b g(t)\Delta t,$

(2) $\int_a^b f(t)\Delta t=-\int_b^a f(t)\Delta t,$

(3) $\int_a^b f(t)\Delta t=\int_a^c f(t)\Delta t+\int_c^b f(t)\Delta t,$

(4) $\int_a^b f(t)g^{\Delta}(t)\Delta t=(fg)(b)-(fg)(a)-\int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t,$

Theorem 2.10 If f is Δ-integrable on $[a,b]$, then so is $|f|$, and

$$\left|\int_a^b f(t)\Delta t\right|\le\int_a^b|f(t)|\Delta t.$$

Definition 2.11 Let $h_k:T^2\to R,\ k\in N_0$, be defined by $h_0(t,s)=1$, for all $s,t\in T$ and then recursively by

$h_{k+1}(t,s)=\int_s^t h_k(\tau,s)\Delta\tau$, for all $s,t\in T$.

3. MAIN RESULTS

For the proof our main results, we will need the following lemma due to Nwaeze [10].

Lemma 3.1 (A Weighted generalized Montgomery Identity**)** Let $v:[a,b]\to[0,\infty)$ be rd-continuous and positive and $w:[a,b]\to R$ be differentiable such that $w^{\Delta}(t)=v(t)$ on $[a,b]$. Suppose also that $a,b,t,x\in T$, $a<b$, $f:[a,b]\to R$ is differentiable, and that ψ is a function of $[0,1]$ into $[0,1]$. Then we have

$$\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(x)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\right]\int_a^b v(t)\Delta t$$
$$=\int_a^b v(t)f(\sigma(t))\Delta t+\int_a^b K(x,t)f^{\Delta}(t)\Delta t, \tag{3}$$

where

$$K(x,t)=\begin{cases} w(t)-\left(w(a)+\psi(\lambda)\dfrac{w(b)-w(a)}{2}\right), & t\in[a,x),\\ w(t)-\left(w(a)+(1+\psi(1-\lambda))\dfrac{w(b)-w(a)}{2}\right), & t\in[x,b]. \end{cases} \tag{4}$$

The following inequalities are the weighted Ostrowski-Grüss type inequalities on time scales.

Theorem 3.2 Let $v:[a,b]\to[0,\infty)$ be rd-continuous and positive and $w:[a,b]\to R$ be differentiable such that $w^{\Delta}(t)=v(t)$ on $[a,b]$. Suppose that $a,b,t,x\in T$, $a<b$, $f:[a,b]\to R$ is differentiable, and that ψ is a function of $[0,1]$ into $[0,1]$. Then for all $x\in[a,b]$, we have

$$\left|\frac{1}{b-a}\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(x)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\right]\int_a^b v(t)\Delta t\right.$$
$$\left.-\frac{1}{b-a}\int_a^b v(t)f(\sigma(t))\Delta t-\frac{f(b)-f(a)}{(b-a)^2}\int_a^b K(x,t)\Delta t\right|$$
$$\le\left[\frac{1}{b-a}\int_a^b K^2(x,t)\Delta t-\left(\frac{1}{b-a}\int_a^b K(x,t)\Delta t\right)^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\int_a^b\left(f^{\Delta}(t)\right)^2\Delta t-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \tag{5}$$

where

$$K(x,t)=\begin{cases} w(t)-\left(w(a)+\psi(\lambda)\dfrac{w(b)-w(a)}{2}\right), & t\in[a,x),\\ w(t)-\left(w(a)+(1+\psi(1-\lambda))\dfrac{w(b)-w(a)}{2}\right), & t\in[x,b].\end{cases}$$

Proof. We have

$$\frac{1}{b-a}\int_a^b K(x,t)f^{\Delta}(t)\Delta t-\left(\frac{1}{b-a}\int_a^b K(x,t)\Delta t\right)\left(\frac{1}{b-a}\int_a^b f^{\Delta}(t)\Delta t\right)$$

$$=\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left(K(x,t)-K(x,s)\right)\left(f^{\Delta}(t)-f^{\Delta}(s)\right)\Delta t\Delta s, \tag{6}$$

From the hypothesis of Lemma 3.1, we have (see also [10])

$$\int_a^b K(x,t)f^{\Delta}(t)\Delta t=\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(x)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\right]\int_a^b v(t)\Delta t$$
$$-\int_a^b v(t)f(\sigma(t))\Delta t, \tag{7}$$

and

$$\frac{1}{b-a}\int_a^b f^{\Delta}(t)\Delta t=\frac{f(b)-f(a)}{b-a}, \tag{8}$$

Using the Cauchy-Schwartz inequality, we have

$$\left|\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left(K(x,t)-K(x,s)\right)\left(f^{\Delta}(t)-f^{\Delta}(s)\right)\Delta t\Delta s\right|$$

$$\leq\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left(K(x,t)-K(x,s)\right)^2\Delta t\Delta s\right)^{\frac{1}{2}}\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left(f^{\Delta}(t)-f^{\Delta}(s)\right)^2\Delta t\Delta s\right)^{\frac{1}{2}}, \tag{9}$$

However

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left(K(x,t)-K(x,s)\right)^2\Delta t\Delta s=\frac{1}{b-a}\int_a^b K^2(x,t)\Delta t-\left(\frac{1}{b-a}\int_a^b K(x,t)\Delta t\right)^2, \tag{10}$$

and

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left(f^{\Delta}(t)-f^{\Delta}(s)\right)^2\Delta t\Delta s=\frac{1}{b-a}\int_a^b\left(f^{\Delta}(t)\right)^2\Delta t-\left(\frac{1}{b-a}\int_a^b f^{\Delta}(t)\Delta t\right)^2, \tag{11}$$

Using (6)-(11), then the proof of inequality (5) is completed.

Corollary 3.3 In case of the $T=R$ in Theorem 3.2, we have

$$\left|\frac{1}{b-a}\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(x)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\right]\int_a^b v(t)dt\right.$$

$$\left.-\frac{1}{b-a}\int_a^b v(t)f(t)dt-\frac{f(b)-f(a)}{(b-a)^2}\int_a^b K(x,t)dt\right|$$

$$\leq\left[\frac{1}{b-a}\int_a^b K^2(x,t)dt-\left(\frac{1}{b-a}\int_a^b K(x,t)dt\right)^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\int_a^b (f'(t))^2dt-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \tag{12}$$

where $w'(t)=v(t)$ on $[a,b]$ and

$$K(x,t)=\begin{cases} w(t)-\left(w(a)+\psi(\lambda)\dfrac{w(b)-w(a)}{2}\right), & t\in[a,x),\\ w(t)-\left(w(a)+(1+\psi(1-\lambda))\dfrac{w(b)-w(a)}{2}\right), & t\in[x,b].\end{cases}$$

Corollary 3.4 In case of the $T=Z$ in Theorem 3.2, we have

$$\left|\frac{1}{b-a}\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(x)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2}\right]\sum_{t=a}^{b-1} v(t)\right.$$

$$\left.-\frac{1}{b-a}\sum_{t=a}^{b-1} v(t)f(t+1)-\frac{f(b)-f(a)}{(b-a)^2}\sum_{t=a}^{b-1} K(x,t)\right|$$

$$\leq\left[\frac{1}{b-a}\sum_{t=a}^{b-1} K^2(x,t)-\left(\frac{1}{b-a}\sum_{t=a}^{b-1} K(x,t)\right)^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\sum_{t=a}^{b-1} (\Delta f(t))^2-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \tag{13}$$

where $v(t)=w(t+1)-w(t)$ on $[a,b-1]$ and

$$K(x,t)=\begin{cases} w(t)-\left(w(a)+\psi(\lambda)\dfrac{w(b)-w(a)}{2}\right), & t\in[a,x-1),\\ w(t)-\left(w(a)+(1+\psi(1-\lambda))\dfrac{w(b)-w(a)}{2}\right), & t\in[x,b-1].\end{cases}$$

Corollary 3.5 Let $T=q^{N_0}$, with $q>1$, $a=q^m$ and $b=q^n$ with $m<n$ in Theorem 3.2. Then we have

$$\left|\frac{1}{q^n-q^m}\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(x)+\frac{\psi(\lambda)f(q^m)+(1-\psi(1-\lambda))f(q^n)}{2}\right]\int_{q^m}^{q^n} v(t)d_qt\right.$$

$$-\frac{1}{q^n-q^m}\int_{q^m}^{q^n} v(t) f(qt)\,d_q t-\frac{f(q^n)-f(q^m)}{(q^n-q^m)^2}\sum_{k=m}^{n-1}K(x,q^k)\Bigg|$$

$$\leq\left[\frac{1}{q^n-q^m}\sum_{k=m}^{n-1}K^2(x,q^k)-\left(\frac{1}{q^n-q^m}\sum_{k=m}^{n-1}K(x,q^k)\right)^2\right]^{\frac{1}{2}}$$

$$\times\left[\frac{1}{q^n-q^m}\int_{q^m}^{q^n}\left(f^{\Delta}(t)\right)^2 d_q t-\left(\frac{f(q^n)-f(q^m)}{q^n-q^m}\right)^2\right]^{\frac{1}{2}}, \tag{14}$$

where $v(t)=\dfrac{w(qt)-w(t)}{(q-1)t}$ on $\left[q^m,q^n\right]$ and

$$K(x,t)=\begin{cases} w(q^k)-\left(w(q^m)+\psi(\lambda)\dfrac{w(q^n)-w(q^m)}{2}\right), & q^k\in\left[q^m,x\right),\\ w(q^k)-\left(w(q^m)+\left(1+\psi(1-\lambda)\right)\dfrac{w(q^n)-w(q^m)}{2}\right), & q^k\in\left[x,q^n\right].\end{cases}$$

Corollary 3.6 If we take $w(t)=t$ in the Theorem 3.2, then we get

$$\Bigg|\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(x)+\frac{\psi(\lambda)f(a)+\left(1-\psi(1-\lambda)\right)f(b)}{2}$$

$$-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t-\frac{f(b)-f(a)}{(b-a)^2}\left[-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)+h_2\left(x,a+\psi(\lambda)\frac{b-a}{2}\right)\right.$$

$$\left.-h_2\left(x,a+\left(1+\psi(1-\lambda)\right)\frac{b-a}{2}\right)+h_2\left(b,a+\left(1+\psi(1-\lambda)\right)\frac{b-a}{2}\right)\right]\Bigg|$$

$$\leq\left[\frac{1}{b-a}\left(\int_a^x\left[t-\left(a+\psi(\lambda)\frac{b-a}{2}\right)\right]^2\Delta t+\int_x^b\left[t-\left(a+\left(1+\psi(1-\lambda)\right)\frac{b-a}{2}\right)\right]^2\Delta t\right)\right.$$

$$-\frac{1}{(b-a)^2}\left(-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)+h_2\left(x,a+\psi(\lambda)\frac{b-a}{2}\right)\right.$$

$$\left.\left.-h_2\left(x,a+\left(1+\psi(1-\lambda)\right)\frac{b-a}{2}\right)+h_2\left(b,a+\left(1+\psi(1-\lambda)\right)\frac{b-a}{2}\right)\right)^2\right]^{\frac{1}{2}}$$

$$\times\left[\frac{1}{b-a}\int_a^b\left(f^{\Delta}(t)\right)^2\Delta t-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \tag{15}$$

for all $\lambda\in[0,1]$ such that $a+\psi(\lambda)\frac{b-a}{2}$ and $a+(1+\psi(1-\lambda))\frac{b-a}{2}$ are in T, $x\in\left[a+\psi(\lambda)\frac{b-a}{2},a+(1+\psi(1-\lambda))\frac{b-a}{2}\right]\cap T$.

Corollary 3.7 If we consider $\psi(\lambda)=\lambda$ in the Corollary 3.6, then we have

$$\Bigg|(1-\lambda)f(x)+\lambda\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t$$
$$-\frac{f(b)-f(a)}{(b-a)^2}\left[-h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(x,a+\lambda\frac{b-a}{2}\right)\right.$$
$$\left.-h_2\left(x,b-\lambda\frac{b-a}{2}\right)+h_2\left(b,b-\lambda\frac{b-a}{2}\right)\right]\Bigg|$$
$$\leq\left[\frac{1}{b-a}\left(\int_a^x\left[t-\left(a+\lambda\frac{b-a}{2}\right)\right]^2\Delta t+\int_x^b\left[t-\left(b-\lambda\frac{b-a}{2}\right)\right]^2\Delta t\right)\right.$$
$$-\frac{1}{(b-a)^2}\left(-h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(x,a+\lambda\frac{b-a}{2}\right)\right.$$
$$\left.\left.-h_2\left(x,b-\lambda\frac{b-a}{2}\right)+h_2\left(b,b-\lambda\frac{b-a}{2}\right)\right)^2\right]^{\frac{1}{2}}\left[\frac{1}{b-a}\int_a^b\left(f^{\Delta}(t)\right)^2\Delta t-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}, \tag{16}$$

for all $\lambda\in[0,1]$ such that $a+\lambda\frac{b-a}{2}$ and $b-\lambda\frac{b-a}{2}$ are in T, $x\in\left[a+\lambda\frac{b-a}{2},b-\lambda\frac{b-a}{2}\right]\cap T$.

Proposition 3.8 In case of the $\lambda=0$ in the Corollary 3.7, we obtain

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t-\frac{f(b)-f(a)}{(b-a)^2}\left[h_2(x,a)-h_2(x,b)\right]\right|$$
$$\leq\left[\frac{1}{b-a}\left(\int_a^x(t-a)^2\Delta t+\int_x^b(b-t)^2\Delta t\right)-\frac{1}{(b-a)^2}\left(h_2(x,a)-h_2(x,b)\right)^2\right]^{\frac{1}{2}}$$
$$\times\left[\frac{1}{b-a}\int_a^b\left(f^{\Delta}(t)\right)^2\Delta t-\left(\frac{f(b)-f(a)}{b-a}\right)^2\right]^{\frac{1}{2}}. \tag{17}$$

Proposition 3.9 In case of the $\lambda = \frac{1}{2}$ in the Corollary 3.7, we get

$$\left| \frac{1}{2} f(x) + \frac{f(a)+f(b)}{4} - \frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t \right.$$

$$\left. - \frac{f(b)-f(a)}{(b-a)^2}\left[-h_2\left(a,\frac{3a+b}{4}\right) + h_2\left(x,\frac{3a+b}{4}\right) - h_2\left(x,\frac{a+3b}{4}\right) + h_2\left(b,\frac{a+3b}{4}\right)\right]\right|$$

$$\leq \left[\frac{1}{b-a}\left(\int_a^x \left(t-\frac{3a+b}{4}\right)^2 \Delta t + \int_x^b \left(t-\frac{a+3b}{4}\right)^2 \Delta t \right)\right.$$

$$\left. - \frac{1}{(b-a)^2}\left(-h_2\left(a,\frac{3a+b}{4}\right) + h_2\left(x,\frac{3a+b}{4}\right) - h_2\left(x,\frac{a+3b}{4}\right) + h_2\left(b,\frac{a+3b}{4}\right)\right)^2 \right]^{\frac{1}{2}}$$

$$\times \left[\frac{1}{b-a}\int_a^b \left(f^{\Delta}(t)\right)^2 \Delta t - \left(\frac{f(b)-f(a)}{b-a}\right)^2 \right]^{\frac{1}{2}}, \tag{18}$$

for all $\lambda \in [0,1]$ such that $\frac{3a+b}{4}$ and $\frac{a+3b}{4}$ are in T, $x \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right] \cap T$.

Theorem 3.10 Let $v:[a,b] \to [0,\infty)$ be rd-continuous and positive and $w:[a,b] \to R$ be differentiable such that $w^{\Delta}(t) = v(t)$ on $[a,b]$. Suppose that $a,b,t,x \in T$, $a<b$, $f:[a,b] \to R$ is differentiable function such that there exist constant $\gamma, \Gamma \in R$, with $\gamma \leq f^{\Delta}(x) \leq \Gamma$, $x \in [a,b]$ and that ψ is a function of $[0,1]$ into $[0,1]$. Then for all $x \in [a,b]$, we have

$$\left| \left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2} f(x) + \frac{\psi(\lambda) f(a) + (1-\psi(1-\lambda)) f(b)}{2} \right] \int_a^b v(t)\Delta t \right.$$

$$\left. - \int_a^b v(t) f(\sigma(t))\Delta t - \frac{\gamma+\Gamma}{2}\int_a^b K(x,t)\Delta t \right| \leq \frac{\Gamma-\gamma}{2}\int_a^b |K(x,t)|\Delta t, \tag{19}$$

where

$$K(x,t) = \begin{cases} w(t) - \left(w(a) + \psi(\lambda)\frac{w(b)-w(a)}{2}\right), & t \in [a,x), \\ w(t) - \left(w(a) + (1+\psi(1-\lambda))\frac{w(b)-w(a)}{2}\right), & t \in [x,b]. \end{cases}$$

Proof. From Lemma 3.1, we have

$$\int_a^b K(x,t) f^{\Delta}(t)\Delta t = \left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2} f(x) + \frac{\psi(\lambda) f(a) + (1-\psi(1-\lambda)) f(b)}{2}\right]\int_a^b v(t)\Delta t$$
$$-\int_a^b v(t) f(\sigma(t))\Delta t, \tag{20}$$

Let $C = \frac{\gamma+\Gamma}{2}$. Using (20), we get

$$\int_a^b K(x,t)\left(f^{\Delta}(t)-C\right)\Delta t = \left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2} f(x) + \frac{\psi(\lambda) f(a) + (1-\psi(1-\lambda)) f(b)}{2}\right]\int_a^b v(t)\Delta t$$
$$-\int_a^b v(t) f(\sigma(t))\Delta t - \frac{\gamma+\Gamma}{2}\int_a^b K(x,t)\Delta t. \tag{21}$$

Taking absolute value, we get

$$\left|\int_a^b K(x,t)\left(f^{\Delta}(t)-C\right)\Delta t\right| \le \frac{\Gamma-\gamma}{2}\int_a^b |K(x,t)|\Delta t, \tag{22}$$

From (20)-(22), we get the desired result.

Corollary 3.11 In case of the $T = R$ in Theorem 3.10, we have

$$\left|\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2} f(x) + \frac{\psi(\lambda) f(a) + (1-\psi(1-\lambda)) f(b)}{2}\right]\int_a^b v(t)dt\right.$$
$$\left.-\int_a^b v(t) f(t)dt - \frac{\gamma+\Gamma}{2}\int_a^b K(x,t)dt\right| \le \frac{\Gamma-\gamma}{2}\int_a^b |K(x,t)|dt, \tag{23}$$

where $w'(t) = v(t)$ on $[a,b]$ and

$$K(x,t) = \begin{cases} w(t) - \left(w(a) + \psi(\lambda)\frac{w(b)-w(a)}{2}\right), & t \in [a,x), \\ w(t) - \left(w(a) + (1+\psi(1-\lambda))\frac{w(b)-w(a)}{2}\right), & t \in [x,b]. \end{cases}$$

Corollary 3.12 In case of the $T = Z$ in Theorem 3.10, we have

$$\left|\left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2} f(x) + \frac{\psi(\lambda) f(a) + (1-\psi(1-\lambda)) f(b)}{2}\right]\sum_{t=a}^{b-1} v(t)\right.$$
$$\left.-\sum_{t=a}^{b-1} v(t) f(t+1) - \frac{\gamma+\Gamma}{2}\sum_{t=a}^{b-1} K(x,t)\right| \le \frac{\Gamma-\gamma}{2}\sum_{t=a}^{b-1} |K(x,t)|, \tag{24}$$

where $v(t) = w(t+1) - w(t)$ on $[a,b-1]$ and

$$K(x,t)=\begin{cases} w(t)-\left(w(a)+\psi(\lambda)\dfrac{w(b)-w(a)}{2}\right), & t\in[a,x-1), \\ w(t)-\left(w(a)+(1+\psi(1-\lambda))\dfrac{w(b)-w(a)}{2}\right), & t\in[x,b-1]. \end{cases}$$

Corollary 3.13 Let $T=q^{N_0}$, with $q>1$, $a=q^m$ and $b=q^n$ with $m<n$ in Theorem 3.10. Then we have

$$\left\| \left[\frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(x)+\frac{\psi(\lambda)f(q^m)+(1-\psi(1-\lambda))f(q^n)}{2}\right]\int_{q^m}^{q^n}v(t)d_qt \right.$$

$$\left. -\int_{q^m}^{q^n}v(t)f(qt)d_qt-\frac{\gamma+\Gamma}{2}\sum_{k=m}^{n-1}K(x,q^k)\right| \le \frac{\Gamma-\gamma}{2}\sum_{k=m}^{n-1}\left|K(x,q^k)\right|, \tag{25}$$

where $v(t)=\dfrac{w(qt)-w(t)}{(q-1)t}$ on $[q^m,q^n]$ and

$$K(x,t)=\begin{cases} w(q^k)-\left(w(q^m)+\psi(\lambda)\dfrac{w(q^n)-w(q^m)}{2}\right), & q^k\in[q^m,x), \\ w(q^k)-\left(w(q^m)+(1+\psi(1-\lambda))\dfrac{w(q^n)-w(q^m)}{2}\right), & q^k\in[x,q^n]. \end{cases}$$

Corollary 3.14 If we take $w(t)=t$ in the Theorem 3.10, then we obtain

$$\left| \frac{1+\psi(1-\lambda)-\psi(\lambda)}{2}f(x)+\frac{\psi(\lambda)f(a)+(1-\psi(1-\lambda))f(b)}{2} \right.$$

$$-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t-\frac{\gamma+\Gamma}{2(b-a)}\left[-h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)+h_2\left(x,a+\psi(\lambda)\frac{b-a}{2}\right)\right.$$

$$\left.\left.-h_2\left(x,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right]\right|$$

$$\le\frac{\Gamma-\gamma}{2(b-a)}\left[h_2\left(a,a+\psi(\lambda)\frac{b-a}{2}\right)+h_2\left(x,a+\psi(\lambda)\frac{b-a}{2}\right)\right.$$

$$\left.+h_2\left(x,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)+h_2\left(b,a+(1+\psi(1-\lambda))\frac{b-a}{2}\right)\right], \tag{26}$$

for all $\lambda\in[0,1]$ such that $a+\psi(\lambda)\dfrac{b-a}{2}$ and $a+(1+\psi(1-\lambda))\dfrac{b-a}{2}$ are in T,

$x\in\left[a+\psi(\lambda)\dfrac{b-a}{2},a+(1+\psi(1-\lambda))\dfrac{b-a}{2}\right]\cap T$.

Corollary 3.15 If we take $\psi(\lambda)=\lambda$ in the Corollary 3.14, then we get

$$\left|(1-\lambda) f(x)+\lambda \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right.$$
$$\left.-\frac{\gamma+\Gamma}{2(b-a)}\left[-h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(x,a+\lambda\frac{b-a}{2}\right)-h_2\left(x,b-\lambda\frac{b-a}{2}\right)+h_2\left(b,b-\lambda\frac{b-a}{2}\right)\right]\right|$$
$$\leq\frac{\Gamma-\gamma}{2(b-a)}\left[h_2\left(a,a+\lambda\frac{b-a}{2}\right)+h_2\left(x,a+\lambda\frac{b-a}{2}\right)+h_2\left(x,b-\lambda\frac{b-a}{2}\right)+h_2\left(b,b-\lambda\frac{b-a}{2}\right)\right], \tag{27}$$

for all $\lambda\in[0,1]$ such that $a+\lambda\frac{b-a}{2}$ and $b-\lambda\frac{b-a}{2}$ are in T, $x\in\left[a+\lambda\frac{b-a}{2},b-\lambda\frac{b-a}{2}\right]\cap T$.

Proposition 3.16 In case of the $\lambda=0$ in the Corollary 3.15, we get

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t-\frac{\gamma+\Gamma}{2(b-a)}\left[h_2(x,a)-h_2(x,b)\right]\right|\leq\frac{\Gamma-\gamma}{2(b-a)}\left(h_2(x,a)+h_2(x,b)\right), \tag{28}$$

Proposition 3.17 In case of the $\lambda=\frac{1}{2}$ in the Corollary 3.15, we obtain

$$\left|\frac{1}{2} f(x)+\frac{f(a)+f(b)}{4}-\frac{1}{b-a}\int_a^b f(\sigma(t))\Delta t\right.$$
$$\left.-\frac{\gamma+\Gamma}{2(b-a)}\left[-h_2\left(a,\frac{3a+b}{4}\right)+h_2\left(x,\frac{3a+b}{4}\right)-h_2\left(x,\frac{a+3b}{4}\right)+h_2\left(b,\frac{a+3b}{4}\right)\right]\right|$$
$$\leq\frac{\Gamma-\gamma}{2(b-a)}\left[h_2\left(a,\frac{3a+b}{4}\right)+h_2\left(x,\frac{3a+b}{4}\right)+h_2\left(x,\frac{a+3b}{4}\right)+h_2\left(b,\frac{a+3b}{4}\right)\right] \tag{29}$$

for all $\lambda\in[0,1]$ such that $\frac{3a+b}{4}$ and $\frac{a+3b}{4}$ are in T, $x\in\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]\cap T$.

REFERENCES

[1]. Agarwal R., Bohner M., Peterson A., Inequalities on time scales: a survey, Math. Inequal. Appl., 2001; 4, 535-557.

[2]. Bohner M., Peterson A., Dynamic Equations on Time Scales. An Introduction with Applications, Birkhäuser Boston, Inc., Boston, MA, 2001.

[3]. Bohner M., Peterson A., Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003.

[4]. Bohner M., Matthews T., Ostrowski inequalities on time scales, JIPAM. J. Inequal. Pure Appl. Math., 2008; 9, Article 6, 8 pp.

[5]. S.S. Dragomir and S. Wang, An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numarical quadrature rules, Comput. Math. Appl., 1997; 33 (11), pp. 16-20.

[6]. Hilger S., Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph.D. thesis, Universität Würzburg, Würzburg, Germany, 1988.

[7]. Lakshmikantham V., Sivasundaram S., Kaymakcalan B., Dynamic systems on measure chains, Mathematics and its Applications, 370, Kluwer Academic Publishers Group, Dordrecht, 1996.

[8]. Liu W., Tuna A., Jiang Y., On weighted Ostrowski type, Trapezoid type, Grüss type and Ostrowski-Grüss like inequalities on time scales, Applicable Analysis, 2014; Volume 93, Issue 3, 551-571.

[9]. Liu W., Tuna A., Jiang Y., New weighted Ostrowski and Ostrowski-Grüss type inequalities on time scales, Annals of the Alexandru Ioan Cuza University--Mathematics, 2014; Volume LX, Issue 1, 57-76.

[10]. Nwaeze E. R., A new weiğhted Ostrowski type inequality on arbitrary time scale, Journal of King Saud University-Science, 2017; volume 29, number 1, pp. 230-234.

[11]. Ostrowski A., Uber die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, Comment. Math. Helv., 1937; 10, 226-227.

[12]. Tuna A., Daghan D., Generalization of Ostrowski and Ostrowski-Grüss type inequalities on time scales, Comput. Math. Appl., 2010; 60, 803-811.

[13]. Tuna A., Jiang Y., Liu W.J., Weighted Ostrowski, Ostrowski-Grüss and Ostrowski-Čebyšev Type Inequalities on Time Scales, Publ. Math. Debrecen, 2012; 81, 81-10