Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.38-4 (2017) 619-625

Archimedean Copula Estimation Parameter with Kendall Distribution Function

Ayşe METIN KARAKAS[1]*, Murat KARAKAS[1], Mine DOGAN[1]

[1]*Department of Statistics, Faculty of Art and Science, Bitlis Eren University*

Received: 09.03.2017; Accepted: 30.05.2017 *http://dx.doi.org/10.17776/csj.348292*

Abstract: In the literature, up to now, it is common that for Gumbel, Clayton and Frank calculated Kendall Distribution function and to the extent those applications have been made. In this paper, we made Kendall Distribution function calculation for Ali Mikhail Haq and Joe and in relation that simulation study. We generated dependent gamma distribution. For dependency between these variables we used Archimedean copula. In connection with this, we define basic properties of copulas and their nonparametric method. In this study, to explain the relationship between the variables, five Archimedean copula families were used; Gumbel, Clayton, Frank Joe and Ali Mikhail Haq. We obtained nonparametric estimation of these copula families parameters and the suitable Archimedean copula family for this data set.

Keywords: Copula Function, Archimedean Copula, Kendall Tau, Kendall Distribution Function

Kendall Dağılım Fonksiyonu ile Archimedean Copula Parametre Tahmini

Özet: Literatürde şimdiye kadar Gumbel Clayton ve Frank arşimedyan copula aileleri için Kendall dağılım fonksiyonu hesaplanmış ve bununla ilgili uygulamalar yapılmıştır.Bu makalede Ali Mikhail ve Joe için Kendall dağılım fonksiyonu hesaplayarak simülasyon çalışması yaptık. Gamma dağılımından bağımlı iki değişken ürettik.Bu değişkenler arasındaki bağımlılık yapısı için arşimedyan copula kullandık. Bununla bağlantılı olarak copulanın temel özelliklerini ve parametrik olmayan method tanımladık. Bu çalışmada değişkenler arasındaki bağımlılık yapısını açıklamak için beş copula ailesi Gumbel, Clayton, Frank Joe ve Ali Mikhail Haq ailesi kullanıldı.Bu copula ailelerinin parametrik olmayan tahmini ve veri seti için uygun copula ailesini elde ettik.

Anahtar Kelimeler: Copula fonksiyonu, Arşimedyan Copula, Kendall Tau, Kendall dağılım fonksiyonu.

1. INTRODUCTION

Copulas were first introduced in the context of theory metric spaces. The statistical properties and applications of copulas has been developing in recent years. In 1959 A.Sklar ıntroduced the general notions of a copula (1981 By.B. Schweizer and E.F.Wolff) [6]. A copula function ~~is~~ links univariate marginal to their multivariate distribution. Using Copula function, we model connection between random variables. Copula function is analyzing the dependence structure and it provides degree of dependence structure. Copula is continuous transformation and invariant under increasing. Copulas can be used for modeling dependence in several applied fields such as econometric, finance and actuarial studies. Archimedean copula defines us to reduce the study of multivariate copula to a single univariate function In this article explores for Gumbel, Clayton and Frank calculated Kendall Distribution function and to the extent that applications have

* Corresponding author. *Email address: aysekarakas5767@gmail.com*

been made. We made $K(u)$ function calculation for Ali Mikhail Haq and Joe and in relation that simulation study. Throughout the paper we work bivariate Archimedean copulas; Clayton, Gumbel and Frank, Joe and Ali Mikhail Haq.

2. COPULA THEORY

The copula is defined as $C:[0,1]^2 \to [0,1]$ which provides the following conditions

- ✓ $C(u,0)=C(0,u)=0$ and $C(u,1)=C(1,u)=u, \forall u \in [0,1]$.
- ✓ $(u_1,u_2,v_1,v_2) \in [0,1]^4$, such that $u_1 \le u_2, v_1 \le v_2$

$$C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1) \ge 0.$$

Ultimately, for twice differentiable and 2-increasing property can be replaced by the condition

$$c(u,v)=\frac{\partial^2 C(u,v)}{\partial u \partial v} \ge 0 \qquad (2.1)$$

where $c(u,v)$ is the copula density. In the following, for n-uniform random $U_1,U_2,...,U_n$ variables, the joint distribution function C is defined

$$C(u_1,u_2,\dots,u_n,\theta)=P(U_1 \le u_1, U_2 \le u_2, \dots U_n \le u_n).$$

Here θ is dependence parameter. [1,2,3,4,5,6,7,8]

2.1. Sklar Theorem

Let X and Y be random variables with continuous distribution functions F_X and F_Y which are uniformly distributed on the interval $[0,1]$. Then, there is a copula such that for all $x, y \in R$,

$$F_{XY}(X,Y)=C(F_X(X),F_Y(Y)) . \qquad (2.2)$$

The copula C for (X,Y) is the joint distribution function for the pair $F_X(X), F_Y(Y)$ provided F_X and F_Y continuous [1, 5, 7, 8].

2.2. Archimedean Copula

Let φ define a function $\phi:[0,1] \to [0,\infty]$ which is continuous and provides:

- ✓ $\phi(1)=0,\ \phi(0)=\infty.$
- ✓ For all $t \in (0,1)$, $\phi'(t)<0$, φ is decreasing, for all $t \in (0,1)$ $\phi''(t) \ge 0$, φ is convex.

ϕ has an inverse $\phi^{-1}:[0,\infty] \to [0,1]$, which has the same properties out of $\phi^{(-1)}(0)=1$ and $\phi^{(-1)}(\infty)=0$. The Archimedean Copula is defined by

$$C(u,v)=\phi^{(-1)}[\phi(u)+\phi(v)]. \qquad (2.3)$$

Formatınıza uygunsa bu denklemler ortalansın

[2,3, 7,8]

2.3. Gumbel Copula

This Archimedean copula is defined with the help of generator function $\phi(t)=(-lnt)^\theta$, $\theta \ge 1$;

$$C_\theta(u,v)=\exp\left(-[(-\ln u)^\theta+(-\ln v)^\theta]^{1/\theta}\right); 0 \le u,v \le 1 \qquad (2.4)$$

Where θ is the copula parameter restricted to $[1,\infty)$ [2].

2.4. Clayton Copula

This Archimedean copula is defined with the help of generator function $\phi(t)=\dfrac{t^{-\theta}-1}{\theta}$,

$\theta \in [-1,\infty)/\{0\}$

$$C_\theta(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{1/\theta}. \quad (2.5)$$

Where θ is the copula parameter restricted to $(0,\infty)$ [2].

2.5. Frank Copula

This Archimedean copula is defined with the help of generator function;

$$\phi(t) = -\ln\frac{-e^{-\theta t}-1}{e^{-\theta}-1}, \theta\in R/\{0\};$$

$$C_\theta(u,v) = -\frac{1}{\theta}\ln\left(1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{(e^{-\theta}-1)}\right) \quad (2.6)$$

Where θ is the copula parameter restricted to $(0,\infty)$ [2].

2.6. Ali Mikhail Haq Copula

This Archimedean copula is defined with the help of generator function $\varphi(t) = \ln[1-\theta(1-t)]/t$

$$C_\theta(u,v) = \frac{uv}{1-\theta(1-u)(1-v)} \quad (2.7)$$

Where θ is the copula parameter restricted to $[-1,1]$ [2].

2.7. Joe Copula

This Archimedean copula is defined with the help of generator function $\varphi(t) = -\ln\left[1-(1-t)^\theta\right]$

$$C_\theta(u,v) = 1-\left[(1-u)^\theta + (1-v)^\theta - ((1-u)^\theta(1-v)^\theta\right]^{1/\theta}$$
(2.8)

Where θ is the copula parameter restricted to $[1,\infty]$ [2].

2.8. The Nonparametric Estimation

The Archimedean Copula submits that each copula has statement connections its parameters to related Kendal Tau and Spearman Rho. In this study only the relationships contains Kendall Tau that given in table.

Table1. The link between Archimedean copulas and Kendall Tau.

Family	Range of θ	τ
Gumbel	$\theta\in[1,\infty)$	$\frac{\theta-1}{\theta}$
Clayton	$\theta\in[0,\infty)$	$\frac{\theta}{\theta+2}$
Frank	$\theta\in(-\infty,\infty)$	$1-\frac{4}{\theta}[1-D_1(\theta)]$
Ali Mikhail Haq	$\theta\in[-1,1]$	$\frac{3\theta-2}{3\theta}-\frac{2(1-\theta)^2\ln(1-\theta)}{3\theta^2}$
Joe	$\theta\in[1,\infty)$	$1+\frac{4}{\theta}D_J(\theta)$

Here D is debye functions.

$$D_J(\theta) = \int_{t=0}^{1}\frac{\left[\ln(1-t^\theta)\right](1-t^\theta)}{t^{\theta-1}}dt \ [2].$$

2.9. Kendall Distribution Function and Properties

Genest and Rivest [3] proposed a nonparametric method for estimating the dependence function of a pair of random variables for Archimedean copula. The problem of emphasizing a probability model for independent observations $(x_1,y_1),\ldots,(x_n,y_n)$ from a bivariate non Gaussian distribution function $H(X,Y)$ can be simplified by denoted H and its marginals of F_X and F_Y and its associated dependence function C. C is the association copula with generator ϕ and Kendall Distribution function is defined by

$$K(u) = \Pr\{C(U_1,\ldots,U_n)\le u\}$$

Genest and Rivest give that if C is Archimedean copula, estimation of Archimedean copula is uniquely determined by function on the interval $(0,1)$;

$$K(u)=u-\frac{\phi(u)}{\phi'(u)} \qquad (2.9)$$

a nonparametric estimation of K is given by

$$K_n(u)=\sum_{j=1}^{n} I\{U_j \le u\}/n+1 \quad .$$

To define the generator function ϕ, we take the steps; estimate Kendall Tau correlation coefficient using the non-parametric estimation and nonparametric estimation of K. For $K_n(u)$ nonparametric estimation of, $K(u)$

i) The nonparametric estimation of Archimedean copula using Kendall Tau correlation coefficient

ii) Define the pseudo-observations

$$U_i=F_n(X_i,Y_i)$$
$$=\sum_{j=1}^{n} I\left[\left\{X_j \le X_i, Y_j \le Y_i\right\}\right]/n+1$$

$i=1,2,\ldots,n$

$$K_n(u)=\frac{(U_i \le u)}{n+1}=\frac{number\ of\ U_i \le u}{n+1}$$

iii) Construct a parametric estimation of K

$$K(u)=u-\frac{\phi(u)}{\phi'(u)}$$

iv) Freez and Valdez the selection of Archimedean copula that fits the data better can be done by minimizing a distance

$$\int\left[K_{\phi_n}(u)-K_n(u)\right]^2 dK_n(u) \qquad (2.10)$$

[2,3,8]

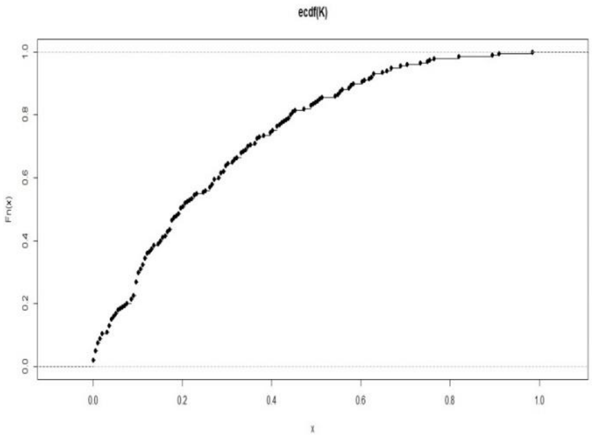


Figure 1. Kendall Distribution Function.

Accordingly, following table give distribution functions of Archimedean copulas.

Table 2. Kendall Distribution functions of Archimedean copulas.

Family	Generator $\phi(u)$	Generator first derivative $\phi'(u)$	The distribution function $K(u)=u-\frac{\phi(u)}{\phi'(u)}$
Gumbel	$(-\ln(u))^{\theta}$	$-\theta(\ln u)^{\theta-1}\frac{1}{u}$	$u-\frac{(u\ln u)}{\theta}$
Clayton	$u^{-\theta}-1$	$-\theta u^{-\theta-1}$	$u-\frac{(u^{\theta+1}-u)}{\theta}$
Frank	$-\ln\left(\frac{e^{-\theta u}-1}{e^{-\theta}-1}\right)$	$\frac{\theta}{1-e^{\theta u}}$	$u-\frac{\ln\frac{e^{-\theta u}-1}{e^{-\theta}-1}}{\theta}(e^{-\theta u}-1)$
Ali Mikhail Haq	$\ln[1-\theta(1-u)]/u$	$\frac{\theta u-\ln[1-\theta(1-u)][1-\theta(1-u)]}{u^2[1-\theta(1-u)]}$	$u-\frac{\ln[1-\theta(1-u)]u[1-\theta(1-u)]}{\theta u-\ln[1-\theta(1-u)][1-\theta(1-u)]}$
Joe	$-\ln[1-(1-u)^{\theta}]$	$\left[\frac{\theta(1-u)^{\theta-1}}{[1+(1-u)^{\theta}}\right]$	$u-\frac{\ln[1-(1-u)^{\theta}][1-(1-u)^{\theta}]}{\theta(1-u)^{\theta-1}}$

3.APPLICATION

The first section, the method that has been suggested by Genest is given. In this section, Pearson correlation coefficient alternative measures of dependence called rank correlation. We focus on Kendall Tau that has been nonparametric measures of dependence .We have seen that the pairwise correlations are all positive. Namely, Kendall Tau value is positive. This study consists of estimation of Archimedean copula. Genest and Mackay simplified method and lead to estimate the parameters of Archimedean copula that focus on Kendall Tau statistics. This study, up to now, in previous studies, in the literature it is common that for Gumbel, Clayton and Frank calculated Kendall Distribution function and to the extent that applications have been made. We made Kendall Distribution function calculation for Ali Mikhail Haq and Joe and in relation that simulation study. In this simulation study, we generated dependent gamma distribution $X \sim Gam(1,1)$ and $Y \sim Gam(1,1)$. Here $n = 200$ data were used. We calculated Kendall Tau value in such that 0,042 for (X,Y) .Using this that is shown parameters of copulas calculated and results are given in table 3.

Table 3. Non parametric estimation of Archimedean copula.

Dependency parameter	Gumbel	Clayton	Frank	Ali Mikhail Haq	Joe
$\hat{\theta}$	1,0438	0,0876	0,378	0,1802	1,076

Finally, this study consists of fitting a suitable copula to the data. The results of estimations are given in table 4.

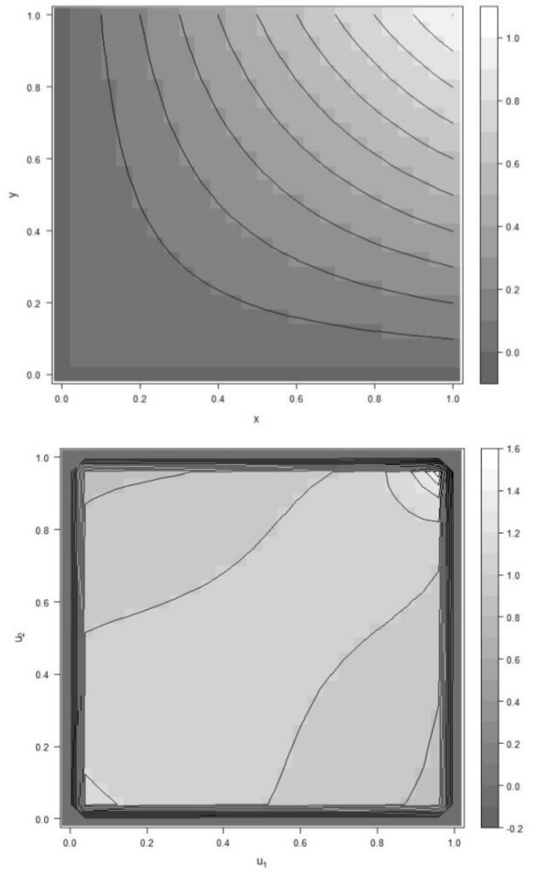
Table 4. Fitting a suitable copula the data.

Pairs	Gumbel	Clayton	Frank	Ali Mikhail Haq	Joe
(X,Y)	0,000135	0,000183	0,000281	0,000149	0,000038

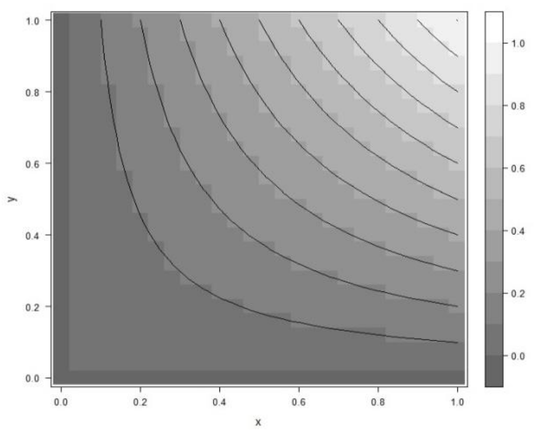
4. CONCLUSION

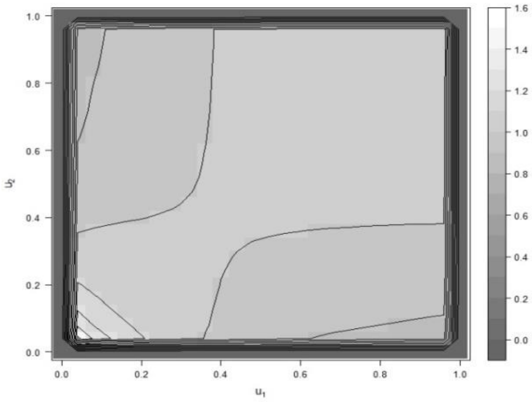
In this paper, we modeled the dependence structure between $X \sim Gam(1,1)$ and $Y \sim Gam(1,1)$ using Archimedean copula. According to table 4, $K_n(u)$ the nonparametric estimation of $K(u)$ is calculated by using pseudo-observations, and using table 2, for Gumbel, Clayton, Frank, Ali Mikhail Haq and Joe respectively $K_G(u)$, $K_C(u)$, $K_F(u)$, $K_{AMH}(u)$ and $K_J(u)$ values calculated. In table 4 $K_n(u)$ value comparedburayı düzeltemedim. $K_G(u)$ $K_C(u)$, $K_F(u)$, $K_{AMH}(u)$ and $K_J(u)$. Consequently, using the square distance measure, with 0,000038 value, Joe copula gives better fits than Gumbel Clayton, Frank, Ali Mikhail Haq and Joe.

5. FIGURES

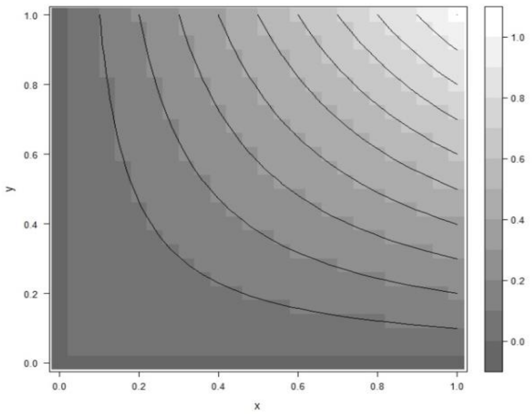


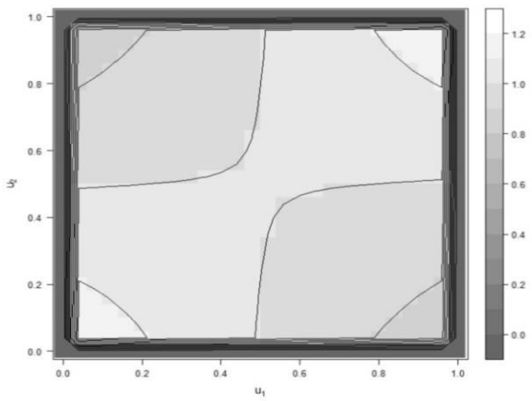
Figures 2. For Gumbel Copula $\theta = 1,0438$, respectively dimensional pobability and density function.



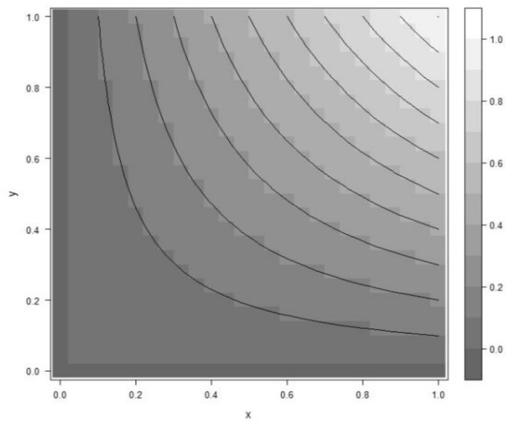


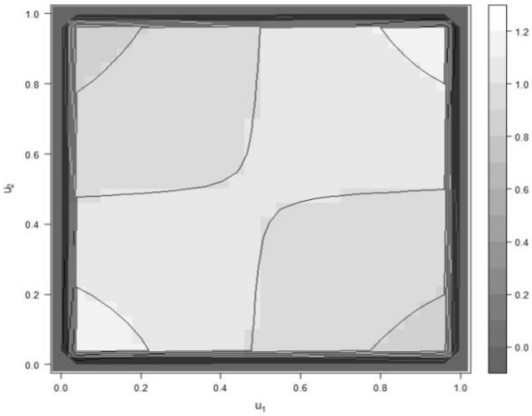
Figures 3. For Clayton Copula $\theta = 0,0876$, respectively two dimensional pobability and density function.



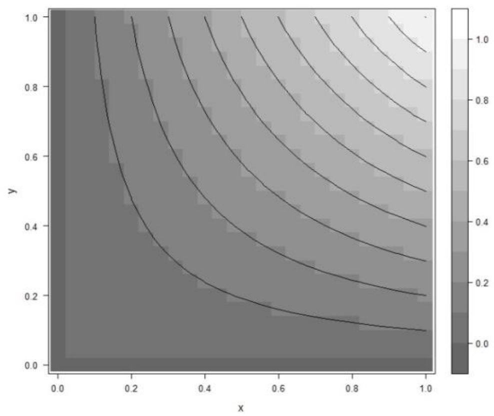


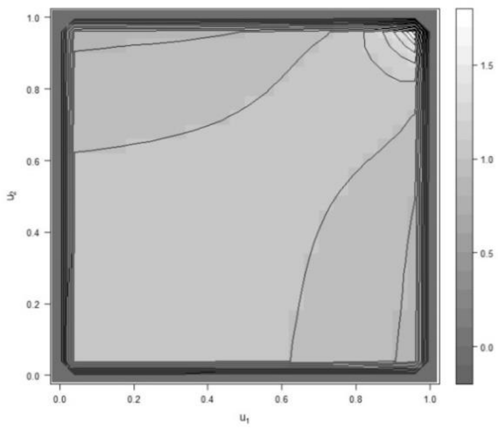
Figures 4. For Frank Copula $\theta = 0,378$, respectively two dimensional pobability and density function.





Figures 5. For Ali Mikhail Haq Copula $\theta = 0{,}1802$, respectively two dimensional pobability and density function.





Figures 6. For Joe Copula $\theta = 1{,}076$, respectively two dimensional pobability and density function.

REFERENCES

[1]. Sklar. Fonctions de Repartition a n Dimensions et Leurs Marges., Publications de I'lnstitut de Statistique de I'Universite de Paris, 1959, vol 8: pp. 229-231

[2]. Nelsen R. An Introduction to Copulas. Springer-Verlag, New York 1999.

[3]. Genest L.P. Rivest. Statistical inference procedures for bivariate Archimedean copulas. Journal of the American Statistical Association 1993, 88: (423) 1034-1043.

[4]. Genest L.P., MacKay J. The joy of copulas: bivariate distributions with uniform marginal. The American Statisticien 1986 40: 280-283.

[5]. Frees E.W. Valdez E.A. Understanding relationships using copulas. North American Actuarial Journal1998. 2:1-25.

[6]. Schweizer B., Wolff E.F. On nonparametric measures of dependence for random variables. Annals of Statistics 1981, 9:879-885.

[7]. Metin A, Çalık S.,Copula Function and Application with Economic Data.. Turkish Journal of Science and Technology. 2012, Vol 7: No 2, 199-204.

[8]. Naifar N. Modeling dependence structure with Archimedean copulas and applications to the iTraxx CDS index. Journal of Computational and Applied Mathematics. 2010 235: 2459-2466.