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Darboux Frame of Pancake Curve in Euclidean 3-Space

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ABSTRACT

In this paper, we focus on a rare curve, "The pancake curve" which was named by Robert Ferréol. We investigate the Darboux frame of the curve, and we calculate the curvatures: geodesic curvature, normal curvature, and geodesic torsion. We discuss two cases, for instance, pancake curve as an intersection of Plücker's conoid and unit cylinder, and the intersection of a hyperbolic paraboloid and unit cylinder. Moreover, we give the relations between the Frenet and Darboux curvatures. Finally, we give some examples related to the Frenet and Darboux frame and curvatures.

Keywords: Plücker's conoid, Hyperbolic paraboloid, Darboux frame, Unit cylinder, Pancake curve

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Introduction

The curve theory has continued to extend for centuries. There are many well-known planar and space curves in Euclidean space, and in other spaces. Today, especially in geometry and related branches, the focus is on space curves in 3-dimension and in higher dimensions. While most of the curves in space look well-known, some of them are encountered in the literature rarely. For example, one of them is the "pancake curve" which takes place at the intersection of such two surfaces: (a) Plücker's conoid and unit cylinder, (b) hyperbolic paraboloid and unit cylinder, and (c) parabolic cylinder and unit cylinder. Here, we mean by "unit cylinder" that it is a circular cylinder with a unit radius. We note that this curve was first named the "pancake curve" by Robert Ferréol [1]. A quotation from [1] can be given as follows: "As far as I know, this curve doesn't have any name of its own. However, it is closely related to a famous item of 19th-century mathematics, the cylindroid surface, discovered by William Kingdon Clifford during his research into the theory of screws. The equation of the cylindroid in 3D Cartesian coordinates commonly reads $z=\frac{x^2-y^2}{x^2+y^2}$. Turning the whole thing thru 90 deg about the z axis yields $z=\frac{2xy}{x^2+y^2}$, and there you are: your curve is the intersection of this cylindroid and the unit cylinder about the z axis. This is generic: cylindroid and cylinder with common axis always intersect in this kind of space curve". In fact, the author mentions Plücker's conoid by saying "cylindroid".

On the other hand, pancake curve has many interesting properties and applications. The pancake curve is a special case of cylindrical sine wave; therefore if we make it roll on a plane, the contact point describes a sinusoid. The projection on $x\circ y$ plane is a circle; the projections on $x\circ z$ and $y\circ z$ planes are isometric lemniscates of Gerono [1]. The projections on the planes passing by Oz are the besaces. The projections on the planes passing by Oz give a portion of parabola and an ovoid quartic. The projections on the planes containing Oy give a circle and the piriform quartic [1]. In real life, pancake curve has a lot of applications in several areas. For example, in the food industry, the edges of potato chips (pringles) form a pancake curve. Furthermore, in architecture and design, it has many applications for various aims like constructing decorative lightings and buildings.

In this study, we consider the pancake curve as an intersection of two surfaces such as: (a) Plücker's conoid and unit cylinder, and (b) hyperbolic paraboloid and unit cylinder. We investigate the Darboux frame, and the curve-surface curvatures. Moreover, in Theorem 3.8, the relation between Frenet and Darboux frame of pancake curve is given. In the last section, some examples are given, and the Frenet and Darboux frame of the pancake curve are illustrated.

Preliminaries

It is well known that Euclidean space is furnished by the following metric:

$$d(x,y)=\left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{1/2},$$

where $x=(x_1,x_2,\dots,x_n)$, and $y=(y_1,y_2,\dots,y_n)$. The dot product of two vectors in $n-$dimensional Euclidean space is given by

$$x \bullet y = \sum_{i=1}^{n} x_i y_i.$$

Let $\psi: I \to \mathbb{R}^3$ be a regular curve in $\mathbb{R}^3$ Euclidean space (i.e. $\|\psi'\|$ is nowhere zero), where I is an interval in $\mathbb{R}$. The curve ψ is called a Frenet curve of rank three if $\psi'(s), \psi''(s), \psi^{(3)}(s)$ are linear independent and $\psi'(s), \psi''(s), \psi^{(3)}(s), \psi^{(4)}(s)$ are no longer linear independent for all s in I. In this case, ψ lies in $3-$dimensional Euclidean subspace of $\mathbb{R}^3$. For each unit speed Frenet curve of rank three, there occur an associated orthonormal $3-$frame field $\{T, N, B\}$ along ψ, the Frenet $3-$frame, and functions $\kappa, \tau: I \to \mathbb{R}$, the Frenet curvatures such that

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

where $T' = dT/ds, \kappa$ and τ is called the curvature and the torsion of the curve. The principal normal vector of the curve $N(s) = T'(s)/\|T'(s)\|$, and the binormal vector is cross product of unit tangent and unit normal vector, $B = T \times N$. Let us consider the ψ lies on surface M in $\mathbb{R}^3$. Then, instead of the Frenet frame field, we can consider the Darboux frame field $\{\mathbb{T}_\psi = T, \mathbb{V}_\psi, \mathbb{N}_\psi\}$ where $\mathbb{T}_\psi$ is the unit tangent vector field of ψ, $\mathbb{N}_\psi$ is surface normal which restricted to ψ, and $\mathbb{V}_\psi = \mathbb{N}_\psi \times \mathbb{T}_\psi$. Here we note that ψ is a unit-speed curve. The derivative formulas of the Darboux frame field are given by

$$\begin{pmatrix} \mathbb{T}'_\psi \\ \mathbb{V}'_\psi \\ \mathbb{N}'_\psi \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbb{T}_\psi \\ \mathbb{V}_\psi \\ \mathbb{N}_\psi \end{pmatrix}, \tag{1}$$

where κ_n, κ_g and τ_g are the normal curvature of the surface in the direction of $\mathbb{T}_\psi$, the geodesic curvature and the geodesic torsion of the curve ψ, respectively [2].

From Eq. (1) we have,

$$\kappa_g = \mathbb{T}'_\psi \bullet \mathbb{V}_\psi, \quad \kappa_n = \mathbb{T}'_\psi \bullet \mathbb{N}_\psi, \quad \tau_g = \mathbb{V}'_\psi \bullet \mathbb{N}_\psi.$$

On the other hand, for the non-unit speed ψ curve, it is written by [3], such that

$$\kappa_g = \frac{1}{|\psi'|}\mathbb{T}'_\psi \bullet \mathbb{V}_\psi, \ \kappa_n = \frac{1}{|\psi'|}\mathbb{T}'_\psi \bullet \mathbb{N}_\psi,$$
$$\tau_g = \frac{1}{|\psi'|}\mathbb{V}'_\psi \bullet \mathbb{N}_\psi. \tag{2}$$

Here, ψ is geodesic, asymptotic or principal curve if and only if either κ_g, τ_g or κ_n vanishes everywhere respectively [4]. For some of the studies on the Frenet and Darboux frames in various spaces, see [5-10].

The Intersection of Plücker's Conoid and Unit Cylinder

In this section, we consider the pancake curve as an intersection of such two surfaces: Plücker's conoid and unit cylinder (see Figure 1).

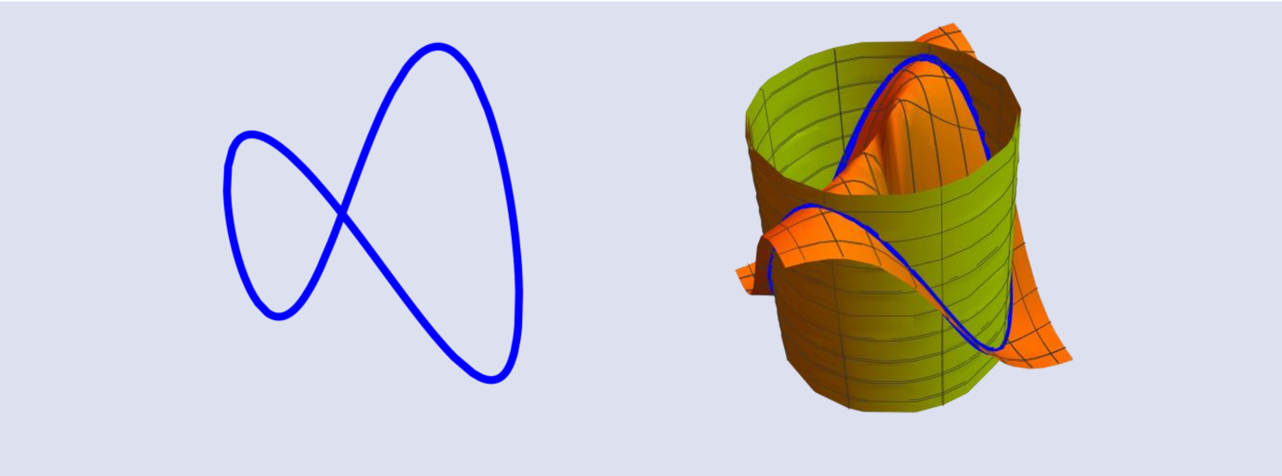


Figure 1 Pancake curve as an intersection of Plücker's conoid and unit cylinder.

In our calculations and illustrations, we utilized *Mathematica*. Here, note that we investigate the Darboux frame of the pancake curve on Plücker's conoid. According to [11], we can give the Plücker's conoid surface by

$$\Upsilon(u, v) = \left(u, v, \frac{2uv}{u^2 + v^2}\right). \tag{3}$$

The unit normal $\vec{n}$ on the surface $\Upsilon(u, v)$ is given by

$$(\vec{n} \circ \Upsilon)(u, v) = \frac{\Upsilon_u \times \Upsilon_v}{|\Upsilon_u \times \Upsilon_v|} = \frac{1}{\sqrt{\mu^2 + \frac{4\lambda^2}{\mu}}}\left(\frac{2\lambda v}{\mu}, -\frac{2\lambda u}{\mu}, \mu\right),$$

where $\lambda = u^2 - v^2$ and $\mu = u^2 + v^2$.

Since the curve is the intersection of Plücker's conoid and the unit cylinder, we can consider first and second components of the intersection curve as $u = coss$ and $v = sins$, respectively. Then, the third component must be $\frac{2uv}{u^2+v^2} = \frac{2sinscoss}{sin^2 s + cos^2 s} = sin(2s)$. Hence, we calculate the equation of the intersection curve as follows:

$$\alpha(s)=(coss, sins, sin(2s)). \quad (4)$$

Obviously, α satisfies the equations of both Plücker's conoid and unit cylinder. Now, we can give our calculations with the help of Eq. (4). First of all, $|\alpha'(s)|=\sqrt{1+4cos^2(2s)}$, and α is not unit-speed curve. By Eq. (4) we have the unit tangent vector,

$$\mathbb{T}_\alpha(s)=\frac{\alpha\prime(s)}{|\alpha\prime(s)|}\frac{1}{\sqrt{3+2cos(4s)}}(-sins, coss, 2cos(2s)). \quad (5)$$

Since $\mathbb{N}_\alpha$ is the surface normal restricted to α,

$$\mathbb{N}_\alpha(s)=(\vec{n}\circ\alpha)(s)=\vec{n}(\Upsilon(coss, sins))$$
$$=\frac{1}{\sqrt{3+2cos(4s)}}(2\,cos(2s)\,sins, -coss-cos(3s), 1). \quad (6)$$

By Eqs. (5) and (6), we calculate the binormal vector as follows:

$$\mathbb{V}_\alpha(s)=\mathbb{N}_\alpha(s)\times\mathbb{T}_\alpha(s)=(-coss, -sins, 0). \quad (7)$$

Hence, we have the Darboux frame $\{\mathbb{T}_\alpha, \mathbb{V}_\alpha, \mathbb{N}_\alpha\}$ the pancake curve on Plücker's conoid. Now we can give our results about the curvatures.

Theorem 3.1 The geodesic curvature of the pancake curve on Plücker's conoid is

$$\kappa_g(s)=\frac{1}{3+2\,cos(4s)}.$$

Proof. Since α(s) is a non-unit speed curve, from Eqs. (2), (5) and (7) we obtain,

$$\mathbb{T}'_\alpha(s)=\left(\frac{-3(coss+cos(3s))+cos(5s)}{(3+2cos(4s))^{3/2}}, \frac{-3sins+3sin(3s)+sin(5s)}{(3+2cos(4s))^{3/2}}, -\frac{4sin(2s)}{(3+2cos(4s))^{3/2}}\right)$$

and

$$\kappa_g(s)=\mathbb{T}'_\alpha(s)\bullet\mathbb{V}_\alpha(s)=\frac{1}{3+2\,cos(4s)},$$

which completes the proof.

Theorem 3.2 The geodesic torsion of the pancake curve on Plücker's conoid is

$$\tau_g(s)=\frac{2cos(2s)}{3+2\,cos(4s)}.$$

Proof. Since α(s) is non-unit speed curve, from Eqs. (2), (6) and (7) we obtain

$$\mathbb{V}'_\alpha(s)=(sins, -coss, 0),$$

and

$$\tau_g(s)=\frac{1}{|\alpha'(s)|}\mathbb{V}'_\alpha(s)\bullet\mathbb{N}_\alpha(s)=\frac{2\,cos(2s)}{3+2\,cos(4s)},$$

which is intended.

Theorem 3.3 The normal curvature of the Plücker's conoid in the direction of $\mathbb{T}_\alpha$ is given by

$$\kappa_n(s)=-\frac{4(2\,sin(2s)+sin(6s))}{(3+2\,cos(4s))^{5/2}}.$$

Proof. Since α(s) is non-unit speed curve, from Eqs. (2), (5) and (6) we obtain

$$\kappa_n(s)=\frac{1}{|\alpha'(s)|}\mathbb{T}'_\alpha(s)\bullet\mathbb{N}_\alpha(s)$$
$$=-\frac{4(2\,sin(2s)+sin(6s))}{(3+2\,cos(4s))^{5/2}},$$

which completes our proof.

Corollary 3.4 The pancake curve is a non-geodesic curve on Plücker's conoid.

Proof. It follows from Theorem 3.1 that $\kappa_g(s)$ never vanishes, and obviously α is non-geodesic.

Corollary 3.5 The piece of pancake curve is an asymptotic curve on Plücker's conoid if and only if

$$s=\frac{\pi n}{2}-\frac{\pi}{4}, n\in\mathbb{Z}.$$

Corollary 3.6 The piece of pancake curve is a principal curve on Plücker's conoid if and only if

$$s=\frac{\pi n}{2}, n\in\mathbb{Z}.$$

Corollary 3.7 The trace of binormal vector $\mathbb{V}_\alpha$ represents a circle.

Theorem 3.8 There is the following equation between the curvatures of Frenet and Darboux frames of the pancake curve on Plücker's conoid:

$$\kappa^2\tau+3\tau_g{\kappa_g}^2=0,$$

where κ_g is the geodesic curvature and τ_g is the geodesic torsion of the pancake curve on Plücker's conoid.

Proof. By Frenet derivative formulas we calculate the Frenet curvature and torsion as follows:

$$\kappa(s)=T'(s)\bullet N(s)=\frac{\sqrt{11-6\cos(4s)}}{(3+2\,cos(4s))^{3/2}},$$

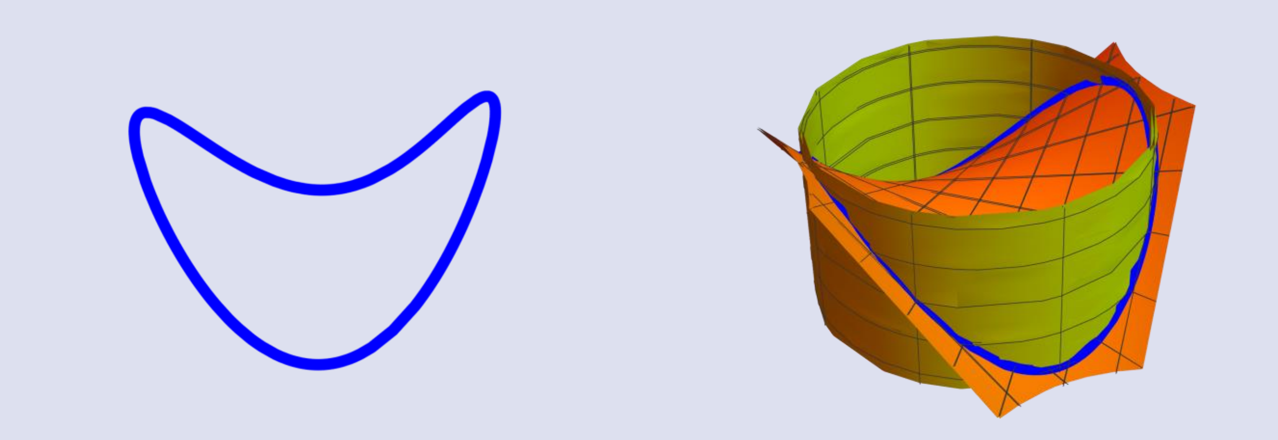
and

$$\tau(s) = N'(s) \bullet B(s) = \frac{6cos2s}{-11 + 6\cos(4s)}.$$

By straightforward calculations, we obtain the equation.

The Hyperbolic Paraboloid and Unit Cylinder

In this section, we consider the pancake curve as an intersection of such two surfaces: hyperbolic paraboloid, and unit cylinder (see Figure 2).



Figüre 2. Pancake curve as an intersection of hyperbolic paraboloid and unit cylinder.

Here, note that we investigate the Darboux frame of the pancake curve on hyperbolic paraboloid. According to [1], we can give the hyperbolic paraboloid surface by

$$\Psi(u, v) = (u, v, uv). \tag{8}$$

The unit normal $\vec{n}$ on the surface $\Psi(u, v)$ is given by

$$(\vec{n} \circ \Psi)(u, v) = \frac{\Psi_u \times \Psi_v}{|\Psi_u \times \Psi_v|} = \frac{1}{\sqrt{1 + u^2 + v^2}}(-v, -u, 1).$$

Since the curve is intersection of hyperbolic paraboloid, and unit cylinder, we can consider first and second components of the curve as $u = coss$ and $v = sins$, respectively. Then, third component of the intersection curve must be
$uv = coss\ sins = \frac{sin(2s)}{2}$. Hence, we calculate the equation of the intersection curve as follows:

$$\beta(s) = \left(coss, sins, \frac{sin(2s)}{2}\right). \tag{9}$$

Now, we can give our calculations with the help of Eq. (9). First of all, $|\beta'(s)| = \sqrt{1 + cos^2(2s)}$, and β is not unit-speed curve. By Eq. (9) we have the unit tangent vector,

$$\mathbb{T}_\beta(s) = \frac{\beta'(s)}{|\beta'(s)|} = \frac{1}{\sqrt{1 + cos^2(2s)}}(-sins, coss, cos(2s)). \tag{10}$$

Since $\mathbb{N}_\beta$ is the surface normal restricted to β,

$$\mathbb{N}_\beta(s) = (\vec{n} \circ \beta)(s) = \vec{n}\big(\Psi(coss, sins)\big) = \left(\frac{-sins}{\sqrt{2}}, \frac{-coss}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \tag{11}$$

By Eqs. (10) and (11), we calculate the binormal vector as follows:

$$\mathbb{V}_\beta(s) = \mathbb{N}_\beta(s) \times \mathbb{T}_\beta(s) = \frac{1}{\sqrt{3 + cos(4s)}}(-2cos^3s, -2sin^3s, -sin(2s)). \tag{12}$$

Hence, we have the Darboux frame of β as $\{\mathbb{T}_\beta, \mathbb{V}_\beta, \mathbb{N}_\beta\}$. Now we can give our results about the curvatures.

Theorem 4.1 The geodesic curvature of the pancake curve on hyperbolic paraboloid is

$$\kappa_g(s) = \frac{5 - cos(4s)}{(3 + cos(4s))^{3/2}}.$$

Proof. Since β(s) is non-unit speed curve, from Eqs. (2), (10) and (12) we obtain

$$\mathbb{T}'_\beta(s) = \left(\frac{-6coss - 3cos(3s) + cos(5s)}{\sqrt{2}(3 + cos(4s))^{3/2}}, \frac{-6sins + 3sins(3s) + sin(5s)}{\sqrt{2}(3 + cos(4s))^{3/2}}, -\frac{2sin(2s)}{(1 + cos^2(2s))^{3/2}}\right),$$

and,

$$\kappa_g(s) = \frac{1}{|\beta'(s)|}\mathbb{T}'_\beta(s) \bullet \mathbb{V}_\beta(s) = \frac{5 - cos(4s)}{(3 + cos(4s))^{3/2}},$$

which completes the proof.

Theorem 4.2 The geodesic torsion of the pancake curve on hyperbolic paraboloid is

$$\tau_g(s) = -\frac{2cos(2s)}{3 + cos(4s)}.$$

Proof. Since β(s) is non-unit speed curve, from Eqs. (2), (11) and (12) we obtain

$$\mathbb{V}'_\beta(s) = \left(\frac{cos^2 s(18\,sins - 5\sin(3s) + \sin(5s))}{(3 + \cos(4s))^{3/2}}, -\frac{(18\,coss + 5\cos(3s) + \cos(5s))sin^2 s}{(3 + \cos(4s))^{3/2}}, -\frac{8\cos(2s)}{(3 + \cos(4s))^{3/2}}\right),$$

and

$$\tau_g(s) = \frac{1}{|\beta'(s)|}\mathbb{V}'_\beta(s) \bullet \mathbb{N}_\beta(s) = -\frac{2cos(2s)}{3 + cos(4s)},$$

which is intended.

Theorem 4.3 The normal curvature of the hyperbolic paraboloid in the direction of $\mathbb{T}_\beta$ is given by

$$\kappa_n(s) = -\frac{\sqrt{2}sin(2s)}{3 + cos(4s)}.$$

Proof. Since β(s) is non-unit speed curve, from Eqs. (2), (10) and (11) we obtain

$$\kappa_n(s) = \frac{1}{|\beta'(s)|}\mathbb{T}'_\beta(s) \bullet \mathbb{N}_\beta(s) = -\frac{\sqrt{2}sin(2s)}{3 + cos(4s)},$$

which completes our proof.

Corollary 4.4 Pancake curve is a non-geodesic curve on hyperbolic paraboloid.

Proof. By definition of cosine function $-1 \le cos(4s) \le 1$. It follows from Theorem 4.1 that $\kappa_g(s) = 5 - cos(4s)$ never vanishes, and obviously β is non-geodesic.

Other results are similar to *Corollary 3.5* and *Corollary 3.6*.

Corollary 4.5 The trace of normal vector restricted to β represents a circle in space.

Examples

Example 5.1 Let us consider the Darboux frame of the curve which is given by Eqs. (4)-(7) and lies on Plücker's conoid. Let us take the point P as $P = \alpha(\pi/3) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$ on the curve. Then, the Darboux frame of the curve at point P is given as follows:

$$\mathbb{T}_\alpha\left(\frac{\pi}{3}\right) = \left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{2}\right),$$

$$\mathbb{V}_\alpha\left(\frac{\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right),$$

$$\mathbb{N}_\alpha(\pi/3) = \left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2}\right).$$

On the other hand, the curvatures at point P are calculated as

$$\kappa_g(\pi/3) = \frac{1}{2}, \quad \tau_g(\pi/3) = -\frac{1}{2}, \quad \kappa_n(\pi/3) = -\frac{\sqrt{6}}{2}.$$

The illustration of Darboux frame of this curve is given by Figure 3.

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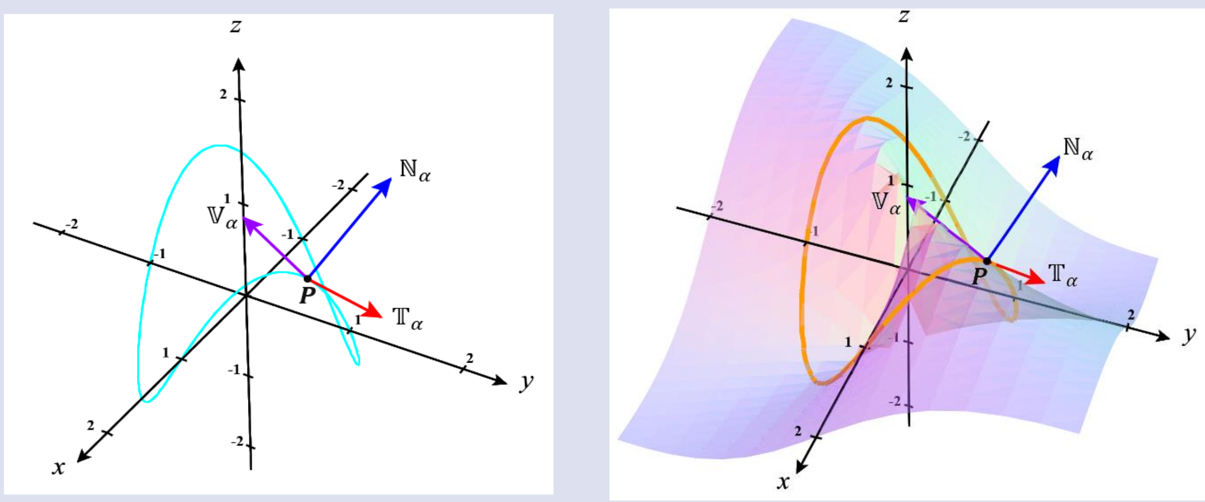


Figüre 3. Darboux frame of α at point $P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$.

Example 5.2 Let us consider the Darboux frame of the curve which is given by Eqs. (9)-(12) and lies on hyperbolic paraboloid. Let us take the point Q as $Q = \beta(\pi/6) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{4}\right)$ on the curve. Then, the Darboux frame of the curve at point Q is given as follows:

$$\mathbb{T}_\beta\left(\frac{\pi}{6}\right) = \left(-\frac{\sqrt{5}}{5}, \frac{\sqrt{15}}{5}, \frac{\sqrt{5}}{5}\right),$$

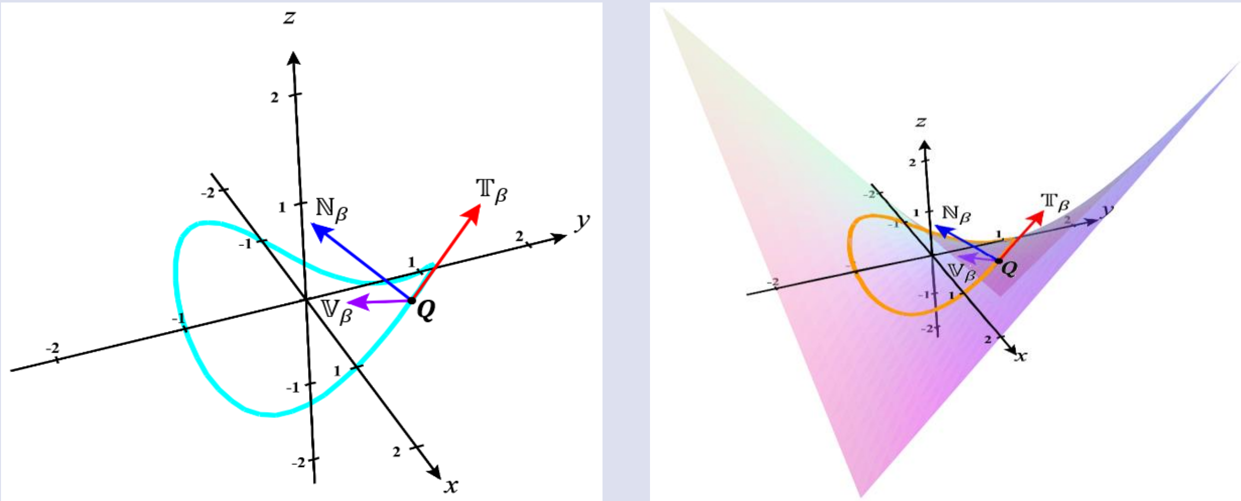
$$\mathbb{V}_\beta\left(\frac{\pi}{6}\right) = \left(-\frac{3\sqrt{30}}{20}, -\frac{\sqrt{10}}{20}, -\frac{\sqrt{30}}{10}\right),$$

$$\mathbb{N}_\beta(\pi/6) = \left(-\frac{\sqrt{2}}{4}, -\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{2}\right).$$

On the other hand, the curvatures at point Q are calculated as

$$\kappa_g(\pi/6) = \frac{11\sqrt{10}}{25}, \quad \tau_g(\pi/6) = -\frac{2}{5}, \quad \kappa_n(\pi/6) = -\frac{\sqrt{6}}{5}.$$

The illustration of Darboux frame of this curve is given by Figure 4.



Figüre 4. Darboux frame of β at point $Q\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{4}\right)$.

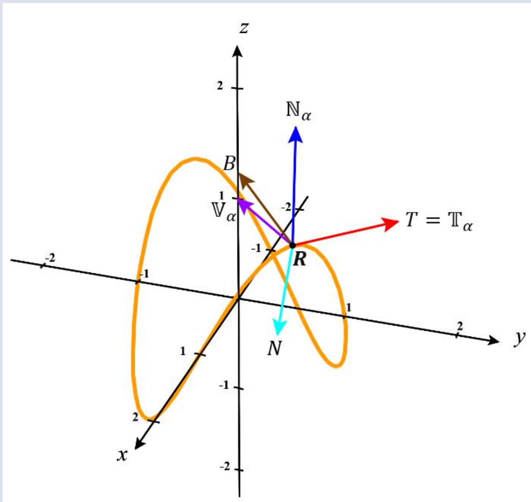
Example 5.3 Let the space curve is given by Eq. (4), $\alpha(s) = (coss, sins, sin(2s))$. At point $R = \alpha(\pi/4) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$, the Frenet and Darboux frame of this curve is given by

$$T\left(\frac{\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right),$$
$$N\left(\frac{\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2\sqrt{17}}, -\frac{\sqrt{2}}{2\sqrt{17}}, -\frac{4}{\sqrt{17}}\right),$$
$$B(\pi/4) = \left(-\frac{2\sqrt{2}}{\sqrt{17}}, -\frac{2\sqrt{2}}{\sqrt{17}}, \frac{1}{\sqrt{17}}\right),$$

and,

$$\mathbb{T}_\alpha\left(\frac{\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right),$$
$$\mathbb{V}_\alpha(\pi/4) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right), \quad \mathbb{N}_\alpha(\pi/4) = (0,0,1).$$

The picture of these frames at point R is given by Figure 5.



Figüre 5. Frenet and Darboux frame of the curve α at point $R\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right)$.

Conclusion and Recommendations

This study initiates a new curve and its Darboux frame on Plücker's conoid and hyperbolic paraboloid surfaces. With the help of Darboux frame, the curvatures of pancake curves on surfaces are calculated. Moreover, a relation between the Frenet and Darboux curvatures of the pancake curve is obtained. While our focus was Plücker's conoid and hyperbolic paraboloid surfaces, one can examine the Darboux frame of pancake curves on a parabolic cylinder. Moreover, the relation between the pancake curve and the other curves (cylindrical sine wave, lemniscates of Gerono, besaces, ovoid quartic, etc.) can be investigated. According to current knowledge, there is not enough paper related to the curve except on the website of Robert Ferréol. Therefore, this paper will be instructive for further studies about the curve.

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Conflict of interest

There are no conflicts of interest in this work.

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