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On the Inverse Problems for Conformable Fractional Integro-Dirac Differential System with Parameter Dependent Boundary Conditions

Hediye Dilara Tel[1,a], Baki Keskin[1,b,*]

[1] Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University, Sivas, Türkiye.
*Corresponding author

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ABSTRACT

This study considers a conformable fractional Dirac-type integral differential system, focusing on its mathematical properties and practical implications. Asymptotic formulas have been derived for the solutions, eigenvalues, and nodes of the problem, providing a deeper understanding of the behavior of the system under varying conditions. These asymptotic results form the basis for analyzing the spectral characteristics and node distribution of the system. In addition, an algorithm is developed that effectively solves the inverse nodal problem and reconstructs the system coefficients from the nodal data.

Keywords: Conformable fractional Dirac system, intego-differential operators, inverse nodal problems.

[a] dilara_5820@hotmail.com.tr https://orcid.org/0000-0003-1139-6146 [b] bkeskin@cumhuriyet.edu.tr https://orcid.org/0000-0003-1689-8954

Introduction

The Dirac operator is a mathematical operator that appears in quantum mechanics and quantum field theory. It was introduced by the physicist Paul Dirac in 1928 as a way to describe the behavior of electrons in relativistic quantum mechanics [1]. The first important and comprehensive results regarding these operators were discussed in Levitans's work [2]. Inverse problems for Dirac operators have been addressed and studied in detail by many researchers (see for example [3-5]).

The fractional calculus has gained considerable attention in various scientific disciplines due to its wide range of applications and its effectiveness in dealing with complex systems. Fractional calculus extends the traditional integer-order calculus to include derivatives and integrals of non-integer orders [6-9]. In 2014, Khalil et al. presented a new but easy definition of the fractional derivative, called the compatible fractional derivative [10]. The new definition seems to be a natural extension of the traditional differentiation and seems to agree with known fractional derivatives on polynomials (up to a constant multiple).

This derivative was defined as follows:

Let $f:[0,\infty)\to\mathbb{R}$ be a given function. The conformable fractional derivative of f of order α is:

$$D_x^\alpha f(x)=\lim_{\epsilon\to 0}\frac{f(x+\epsilon x^{1-\alpha})-f(x)}{\epsilon},\ D_x^\alpha f(0)=\lim_{x\to 0^+}D_x^\alpha f(x),$$

for all $x>0$, $\alpha\in(0,1]$. If this limit exist and finite at x_0, we say f is α $-$differentiable at x_0. If f is differentiable, then $D_x^\alpha f(x)=x^{1-\alpha}f'(x)$.

In the past few years, fractional calculus has been investigated by several author ([11], [12] and references therein). In recent years,there has been a growing interest among scholars in exploring fractional generalizations of well-known mathematical problems, including those related to Sturm-Liouville, diffusion and Dirac operators [13-21].

The nodal set consists of points where the eigenfunction vanishes. In 1988, the concept of the inverse nodal problem for the Sturm-Liouville operator was first discussed by McLaughlin [22], and later Hald and McLaughlin showed that it was sufficient to know only the nodal points to determine the potential function with more general boundary conditions [23].Yang proposed a solution in 1997 to reconstruct the potential and the boundary condition of the Sturm-Liouville operator from its nodes. [24]. Inverse nodal problems continue to be studied by many researchers [25-34].

The inverse nodal problem for Dirac operators involves determining the coefficients of the Dirac operator and other parameters of the problem from the knowledge of the nodal set of the corresponding eigenfunctions. For certain types of Dirac operators with various boundary conditions, it has been demonstrated that a dense subset of the nodal points of the eigenfunctions alone is sufficient to uniquely determine the coefficients of the Dirac operator [35-37].

Eigenvalue problems with eigenvalue-dependent boundary conditions is an important application area in applied sciences. Fulton's [38-39], studies and the references in this study can be cited as examples of studies conducted on this subject until 1980. The most recent examples of its applications in physics can be found in [40]. We refer to [41-42] and references therein regarding studies in this field.

Nowadays, studies on the integro-differential operator have gained significant popularity and interest by many authors and have gained an important place in the literature [43-46].The inverse nodal problem for Dirac type integro-differential operators was first considered in [47]. It is shown in this study that the coefficients of the operator can be determined by using nodal points. In [48], the authors have addressed a similar problem where the boundary conditions depend linearly on the spectral parameter.

The conformable fractional derivative was first discussed in [10-11]. Some other definitions and basic properties can be found in these works.

Main Results

Consider the following BVP $L(\theta,\beta,p(x),q(x))$:

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} D_x^{\alpha}Y(x)+\begin{pmatrix}p(x) & 0\\ 0 & r(x)\end{pmatrix}Y(x)+\int_0^x M(x,t)Y(t)d_{\alpha}t=\lambda Y \ , x\in(0,\pi) \quad (1)$$

with

$$(\lambda\cos\theta+a)y_1(0)+(\lambda\sin\theta+b)y_2(0)=0 \quad (2)$$

$$(\lambda\cos\beta+c)y_1(\pi)+(\lambda\sin\beta+d)y_2(\pi)=0 \quad (3)$$

where, $M(x,t)=\begin{pmatrix}M_{11}(x,t) & M_{12}(x,t)\\ M_{21}(x,t) & M_{22}(x,t)\end{pmatrix}$, $Y(x)=\begin{pmatrix}y_1(x)\\ y_2(x)\end{pmatrix}$ and $p(x),q(x)$, and $M_{ij}(x,t)$ $(i,j=1,2)$ are real-valued conformable fractional differentiable functions and $x^{\alpha-1}p(x)$ and $x^{\alpha-1}r(x)$ are continuous on $(0,\pi)$, $0\le\theta,\beta<\pi$ are real numbers, λ is the spectral parameter.

Let $\varphi(x,\lambda)=(\varphi_1(x,\lambda),\varphi_2(x,\lambda))^T$ be the solution of (1) satisfying $\varphi(0,\lambda)=(\lambda\sin\theta+b,-\lambda\cos\theta-a)^T$. $\varphi(x,\lambda)$ satisfies the following asymptotic relations for $|\lambda|\to\infty$,

$$\begin{aligned}
\varphi_1(x,\lambda)&=\lambda\sin\left(\theta+\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)+a\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\\
&+b\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)+\frac{1}{2}\mu(x)\sin\left(\theta+\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\\
&+\frac{a}{2\lambda}\mu(x)\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)+\frac{b}{2\lambda}\mu(x)\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\\
&+\frac{1}{2}\mu(0)\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)-\theta\right)+\frac{a}{2\lambda}\mu(0)\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right) \qquad (4)\\
&-\frac{b}{2\lambda}\mu(0)\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)-\frac{1}{2}\cos\left(\theta+\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\int_0^x\mu^2(t)d_{\alpha}t\\
&-\frac{a}{2\lambda}\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\int_0^x\mu^2(t)d_{\alpha}t+\frac{b}{2\lambda}\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\int_0^x\mu^2(t)d_{\alpha}t\\
&-\frac{1}{2}K(x)\sin\left(\theta+\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)-\frac{a}{2\lambda}K(x)\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\\
&-\frac{b}{2\lambda}K(x)\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)+\frac{1}{2}L(x)\cos\left(\theta+\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\\
&+\frac{a}{2\lambda}L(x)\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)-\frac{b}{2\lambda}L(x)\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)+o\left(\frac{e^{|\tau|\frac{x^{\alpha}}{\alpha}}}{\lambda}\right)
\end{aligned}$$

$$\begin{aligned}
\varphi_2(x,\lambda)&=-\lambda\cos\left(\theta+\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)-a\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\\
&+b\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)+\frac{1}{2}\mu(x)\cos\left(\theta+\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\\
&+\frac{a}{2\lambda}\mu(x)\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)-\frac{b}{2\lambda}\mu(x)\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\\
&-\frac{1}{2}\mu(0)\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)-\theta\right)-\frac{a}{2\lambda}\mu(0)\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\\
&-\frac{b}{2\lambda}\mu(0)\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)-\frac{1}{2}\sin\left(\theta+\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\int_0^x\mu^2(t)d_{\alpha}t \qquad (5)\\
&-\frac{a}{2\lambda}\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\int_0^x\mu^2(t)d_{\alpha}t-\frac{b}{2\lambda}\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\int_0^x\mu^2(t)d_{\alpha}t\\
&+\frac{1}{2}K(x)\cos\left(\theta+\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)+\frac{a}{2\lambda}K(x)\cos\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)\\
&-\frac{b}{2\lambda}K(x)\sin\left(\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)+\frac{1}{2}L(x)\sin\left(\theta+\lambda\frac{x^{\alpha}}{\alpha}-\rho(x)\right)
\end{aligned}$$

$+\frac{a}{2\lambda}L(x)\sin\left(\lambda\frac{x^\alpha}{\alpha}-\rho(x)\right)+\frac{b}{2\lambda}L(x)\cos\left(\lambda\frac{x^\alpha}{\alpha}-\rho(x)\right)+o\left(\frac{e^{|\tau|\frac{x^\alpha}{\alpha}}}{\lambda}\right)$

uniformly in $x\in[0,\pi]$, where, $\mu(x)=\frac{1}{2}(p(x)-r(x)), \rho(x)=\frac{1}{2}\int_0^x (p(t)+r(t))d_\alpha t$,
$K(x)=\int_0^x (M_{11}(t,t)-M_{22}(t,t))d_\alpha t, L(x)=\int_0^x (M_{12}(t,t)-M_{21}(t,t))d_\alpha t$
and $\tau=Im\lambda$.
$\Delta(\lambda)$ is called the characteristic function of L and defined by as follows

$$\Delta(\lambda)=\varphi_1(\pi,\lambda)(\lambda\cos\beta+c)+\varphi_2(\pi,\lambda)(\lambda\sin\beta+d), \tag{6}$$

The zeros $\{\lambda_n\}_{n\in\mathbb{Z}}$ of $\Delta(\lambda)$ coincide with the eigenvalues of the problem L. Using (4) and (5), we get

$$\begin{aligned}\Delta(\lambda)&=\lambda^2\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\rho(\pi)-\beta+\theta\right)+\lambda a\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\rho(\pi)-\beta\right)\\&+b\lambda\cos\left(\lambda\frac{\pi^\alpha}{\alpha}-\rho(\pi)-\beta\right)+\frac{\mu(x)}{2}\lambda\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\rho(\pi)+\beta+\theta\right)\\&+\frac{\mu(0)}{2}\lambda\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\rho(\pi)-\beta-\theta\right)-\frac{1}{2}\lambda\cos\left(\lambda\frac{\pi^\alpha}{\alpha}-\rho(\pi)-\beta+\theta\right)\int_0^\pi \mu^2(t)d_\alpha t\\&-\frac{1}{2}\lambda K(\pi)\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\rho(\pi)-\beta+\theta\right)+\frac{1}{2}\lambda L(\pi)\cos\left(\lambda\frac{\pi^\alpha}{\alpha}-\rho(\pi)-\beta+\theta\right)\\&+c\lambda\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\rho(\pi)+\theta\right)-d\lambda\cos\left(\lambda\frac{\pi^\alpha}{\alpha}-\rho(\pi)+\theta\right)+o\left(\frac{e^{|\tau|\frac{\pi^\alpha}{\alpha}}}{\lambda^3}\right),\end{aligned} \tag{7}$$

for sufficiently large $|\lambda|$.
By using $\Delta(\lambda_n)=0$, we get

$$\lambda_n=\frac{(n-1)\pi\alpha}{\pi^\alpha}+\frac{(\rho(\pi)+\beta-\theta)\alpha}{\pi^\alpha}+\frac{D}{(n-1)\pi^\alpha}+o\left(\frac{1}{n}\right)n\geq 2 \tag{8}$$

and

$$\lambda_n=\frac{(n+1)\pi\alpha}{\pi^\alpha}+\frac{(\rho(\pi)+\beta-\theta)\alpha}{\pi^\alpha}+\frac{D}{(n+1)\pi^\alpha}+o\left(\frac{1}{n}\right)n\leq -2$$

for sufficiently large n,

where, $D=a\sin\theta-b\cos\theta-\frac{\mu(x)}{2}\sin 2\beta+\frac{\mu(0)}{2}\sin 2\theta+\frac{1}{2}\int_0^\pi \mu^2(t)d_\alpha t-\frac{L(\pi)}{2}-c\sin\beta+d\cos\beta$

Lemma 1 *For sufficiently large n, the first component $\varphi_1(x,\lambda_n)$ of the eigenfunction $\varphi(x,\lambda_n)$ has exactly $n-2$ nodes $\{x_n^j: j=0,1,\cdots,n-3\}$ in the interval $(0,\pi)$: $0<x_n^0<x_n^1<\ldots<x_n^{n-3}<\pi$.* The numbers $\{x_n^j\}$ satisfy the following asymptotic formula:

$$\begin{aligned}\left(x_n^j\right)^\alpha&=\frac{j\pi^\alpha}{n}-\frac{j\pi^\alpha}{n}\frac{\rho(\pi)+\beta-\theta}{n\pi}+\rho\left(x_n^j\right)\frac{\pi^{\alpha-1}}{n}-\theta\frac{\pi^{\alpha-1}}{n}-\rho\left(x_n^j\right)\frac{\rho(\pi)+\beta-\theta}{n^2}\pi^{\alpha-2}\\&+\theta\frac{\rho(\pi)+\beta-\theta}{n^2}\pi^{\alpha-2}+T\frac{\pi^{2\alpha-2}}{2n^2\alpha}+\frac{\pi^{2\alpha-2}}{2n^2\alpha}\int_0^{x_n^j x}\mu^2(t)d_\alpha t-\frac{\pi^{2\alpha-2}}{2n^2\alpha}L\left(x_n^j\right)\\&\div\\&-T\frac{\rho(\pi)+\beta-\theta}{n^3\alpha}\pi^{2\alpha-3}-\frac{\rho(\pi)+\beta-\theta}{n^3\alpha}\int_0^{x_n^j x}\mu^2(t)d_\alpha t-\frac{\rho(\pi)+\beta-\theta}{n^3\alpha}L\left(x_n^j\right)+o\left(\frac{1}{n^3}\right).\end{aligned} \tag{9}$$

Where, $T=2a\sin\theta-2b\cos\theta+\mu(0)\sin 2\theta$

Proof. From (4), the following equation is valid

$$\begin{aligned}\varphi_1(x,\lambda_n)&=\lambda_n\sin\left(\theta+\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)+a\sin\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)\\&+b\cos\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)+\frac{1}{2}\mu(x)\sin\left(\theta+\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)\\&+\frac{a}{2\lambda_n}\mu(x)\sin\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)+\frac{b}{2\lambda_n}\mu(x)\cos\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)\\&+\frac{1}{2}\mu(0)\sin\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)-\theta\right)+\frac{a}{2\lambda_n}\mu(0)\sin\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)\end{aligned}$$

$$-\frac{b}{2\lambda_n}\mu(0)\cos\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)-\frac{1}{2}\cos\left(\theta+\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)\int_0^x \mu^2(t)d_\alpha t$$

$$-\frac{a}{2\lambda_n}\cos\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)\int_0^x \mu^2(t)d_\alpha t+\frac{b}{2\lambda_n}\sin\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)\int_0^x \mu^2(t)d_\alpha t$$

$$-\frac{1}{2}K(x)\sin\left(\theta+\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)-\frac{a}{2\lambda_n}K(x)\sin\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)$$

$$-\frac{b}{2\lambda_n}K(x)\cos\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)+\frac{1}{2}L(x)\cos\left(\theta+\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)$$

$$+\frac{a}{2\lambda_n}L(x)\cos\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)-\frac{b}{2\lambda_n}L(x)\sin\left(\lambda_n\frac{x^\alpha}{\alpha}-\rho(x)\right)+o\left(\frac{1}{\lambda_n}\right)$$

for sufficiently large n. If we put $\varphi_1\left(\left(x_n^j\right)^\alpha,\lambda_n\right)=0$, we get

$$\lambda_n\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)+a\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\cos\theta$$

$$-a\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\sin\theta+b\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\cos\theta$$

$$+b\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\sin\theta+\frac{1}{2}\mu(x_n^j)\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)$$

$$+\frac{a}{2\lambda_n}\mu(x_n^j)\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\cos\theta-\frac{a}{2\lambda_n}\mu(x_n^j)\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\sin\theta$$

$$+\frac{b}{2\lambda_n}\mu(x_n^j)\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\cos\theta+\frac{b}{2\lambda_n}\mu(x_n^j)\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\sin\theta$$

$$+\frac{1}{2}\mu(0)\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\cos2\theta-\frac{1}{2}\mu(0)\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\sin2\theta$$

$$+\frac{a}{2\lambda_n}\mu(0)\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\cos\theta-\frac{a}{2\lambda_n}\mu(0)\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\sin\theta$$

$$-\frac{b}{2\lambda_n}\mu(0)\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\cos\theta-\frac{b}{2\lambda_n}\mu(0)\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\sin\theta$$

$$-\left(\frac{1}{2}\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)+\frac{a}{2\lambda_n}cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)cos\theta\right)\int_0^{x_n^jx}\mu^2(t)d_\alpha t$$

$$-\left(\frac{a}{2\lambda_n}\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\sin\theta-\frac{b}{2\lambda_n}sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)cos\theta\int_0^{x_n^jx}\mu^2(t)d_\alpha t\right.$$

$$-\frac{b}{2\lambda_n}cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)sin\theta\int_0^{x_n^jx}\mu^2(t)d_\alpha t-\frac{1}{2}K(x_n^j)\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)$$

$$-\frac{a}{2\lambda_n}K(x_n^j)\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\cos\theta+\frac{a}{2\lambda_n}K(x_n^j)\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\sin\theta$$

$$-\frac{b}{2\lambda_n}K(x_n^j)\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\cos\theta-\frac{b}{2\lambda_n}K(x)\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\sin\theta$$

$$+\frac{1}{2}L(x_n^j)\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)+\frac{a}{2\lambda_n}L(x_n^j)\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\cos\theta$$

$$+\frac{a}{2\lambda_n}L(x_n^j)\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\sin\theta-\frac{b}{2\lambda_n}L(x_n^j)\sin\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\cos\theta$$

$$+\frac{b}{2\lambda_n}L(x_n^j)\cos\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x_n^j)+\theta\right)\sin\theta+o\left(\frac{1}{\lambda_n}\right)=0$$

$$\lambda_n\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)+a\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\cos\theta-a\sin\theta$$

$$+b\cos\theta+b\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\sin\theta+\frac{1}{2}\mu(x)\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)$$

$$+\frac{a}{2\lambda_n}\mu(x)\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\cos\theta-\frac{a}{2\lambda_n}\mu(x)\sin\theta+\frac{b}{2\lambda_n}\mu(x)\cos\theta$$

$$+\frac{b}{2\lambda_n}\mu(x)\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\sin\theta+\frac{1}{2}\mu(0)\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\cos2\theta-\frac{1}{2}\mu(0)\sin2\theta$$

$$+\frac{a}{2\lambda_n}\mu(0)\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\cos\theta-\frac{a}{2\lambda_n}\mu(0)\sin\theta$$

$$-\frac{b}{2\lambda_n}\mu(0)\cos\theta-\frac{b}{2\lambda_n}\mu(0)\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\sin\theta-\frac{1}{2}\int_0^{x_n^j}\mu^2(t)d_\alpha t-\frac{a}{2\lambda_n}cos\theta\int_0^{x_n^j}\mu^2(t)d_\alpha t$$

$$-\frac{a}{2\lambda_n}\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)sin\theta\int_0^{x_n^j}\mu^2(t)d_\alpha t+\frac{b}{2\lambda_n}\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)cos\theta\int_0^{x_n^j}\mu^2(t)d_\alpha t$$

$$-\frac{b}{2\lambda_n}sin\theta\int_0^{xx_n^j}\mu^2(t)d_\alpha t-\frac{1}{2}K(x)\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)-\frac{a}{2\lambda_n}K(x)\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\cos\theta$$

$$+\frac{a}{2\lambda_n}K(x)\sin\theta-\frac{b}{2\lambda_n}K(x)\cos\theta+\frac{1}{2}L(x)+\frac{a}{2\lambda_n}L(x)\cos\theta-\frac{b}{2\lambda_n}K(x)\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\sin\theta$$

$$+\frac{b}{2\lambda_n}L(x_n^j)\sin\theta+o\left(\frac{1}{\lambda}\right)+\frac{a}{2\lambda_n}L(x)\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\sin\theta$$

$$-\frac{b}{2\lambda_n}L(x)\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\cos\theta=0$$

$$\tan\left(\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(x)+\theta\right)\left\{1+\frac{a}{\lambda_n}\cos\theta+\frac{b}{\lambda_n}\sin\theta\right.$$
$$\left.+\frac{1}{2\lambda_n}\mu(x_n^j)+\frac{1}{2\lambda_n}\mu(0)-\frac{1}{2\lambda_n}Kx_n^j)+o\left(\frac{1}{\lambda_n}\right)\right\}=$$
$$\frac{a}{\lambda_n}\sin\theta-\frac{b}{\lambda_n}\cos\theta+\frac{1}{2\lambda_n}\mu(0)\sin2\theta+\frac{1}{2\lambda_n}\int_0^{x_n^jx}\mu^2(t)d_\alpha t-\frac{1}{2\lambda_n}L(x_n^j)+o\left(\frac{1}{\lambda_n}\right)$$

Taylor's expansion formula gives,

$$\lambda_n\frac{\left(x_n^j\right)^\alpha}{\alpha}-\rho(\left(x_n^j\right))+\theta$$
$$=j\pi+\frac{1}{2\lambda_n}\left(2a\sin\theta-2b\cos\theta+\mu(0)\sin2\theta+\int_0^{x_n^jx}\mu^2(t)d_\alpha t-L\left(x_n^j\right)\right)$$
$$+o\left(\frac{1}{\lambda_n}\right)$$

or

$$\left(x_n^j\right)^\alpha=\alpha\lambda_n^{-1}\left(\rho(\left(x_n^j\right))-\theta+j\pi\right.$$
$$\left.+\frac{1}{2\lambda_n}\left(2a\sin\theta-2b\cos\theta+\mu(0)\sin2\theta+\int_0^{x_n^j}\mu^2(t)d_\alpha t-L\left(x_n^j\right)\right)\right)+o\left(\frac{1}{\lambda_n}\right)$$

We arrive (9) by using the asymptotic formula

$$\lambda_n^{-1}=\frac{\pi^\alpha}{n\pi\alpha}-\frac{(\rho(\pi)+\beta-\theta)\pi^\alpha}{n^2\pi^2\alpha}+o\left(\frac{1}{n^2}\right)$$

Let X be the set of nodal points. For each fixed $x\in(0,\pi)$ and $\alpha\in(0,1]$, choose a sequence $\left(x_n^j\right)\subset X$ such that x_n^j converges to x. Then the following limits are exist and finite:

$$\lim_{|n|\to\infty}\left(\left(x_n^j\right)^\alpha-\frac{j\pi^\alpha}{n}\right)n\pi=-x(\rho(\pi)-\theta+\beta)+\rho(x)\pi^\alpha-\theta\pi^\alpha=f(x)$$

where

$$f(x) = -x(\rho(\pi) - \theta + \beta) + \frac{1}{2}\pi^{\alpha} \int_0^x [p(t) + r(t)]d_{\alpha}t - \theta\pi^{\alpha} \quad (10)$$

and

$$\lim_{n\to\infty} \left(\left(x_n^j\right)^{\alpha} - \frac{j\pi^{\alpha}}{n} + \frac{j\pi^{\alpha}}{n}\frac{\rho(\pi)+\beta-\theta}{n\pi} - \frac{\rho\left(x_n^j\right)\pi^{\alpha-1}}{n} + \frac{\theta\pi^{\alpha-1}}{n} \right) n^2 = g(x),$$

where

$$g(x) = -\rho(x)(\beta-\theta)\pi^{\alpha-2} + \theta(\beta-\theta)\pi^{\alpha-2} + \frac{\pi^{2\alpha-2}}{2\alpha} + \int_0^x \mu^2(t)d_{\alpha}t + \frac{\pi^{2\alpha-2}}{2\alpha}L(x) + T\frac{\pi^{2\alpha-2}}{2\alpha} \quad (11)$$

Therefore, proof of the following theorem is clear. Let $\mu(\pi) = 0$, and X be the dense subset of the nodal points.

Theorem 1 *Given the set X uniquely determines the coefficients θ and β of the problem L and if $L(x)$ is known, the potential $\Omega(x)$ a.e. on $(0,\pi)$ can be also determined by X . Moreover, $p(x)$ and $r(x)$, θ and β can be reconstructed as follows*

Step-1: For each fixed $x \in (0,\pi)$ and $\alpha \in (0,1]$, choose $\left(x_n^{j(n)}\right) \subset X$ such that $\left(x_n^{j(n)}\right) \to x$ as n→ ∞;

Step-2: Find $f(x)$ from (10) and calculate

$$\theta = -f(0)\pi^{-\alpha}$$

$$\beta = \frac{f(0)-f(\pi)-f(0)\pi^{1-\alpha}}{\pi}$$

$$D_x^{\alpha}\rho(x) = (D_x^{\alpha}f(x) - \theta + \beta)\pi^{-\alpha}$$

Step-3: From (11), find $g(x)$ and calculate

$$\mu^2(x) = (D_x^{\alpha}g(x) + (D_x^{\alpha}f(x) - \theta + \beta)(\beta-\theta)\pi^{-2})\frac{2\alpha}{\pi^{2\alpha-2}} + D_x^{\alpha}L(x) \quad (12)$$

Step-4: From (10) and (11) calculate

$$p(x) = \frac{D_x^{\alpha}f(x)}{\pi^{\alpha}} + \frac{f(0)-f(\pi)-f(0)\pi^{1-\alpha}}{\pi^{1+\alpha}} + f(0) + 2\sqrt{\frac{2\alpha}{\pi^{2\alpha-2}}(D_x^{\alpha}g(x) + D_x^{\alpha}\rho(x)(\beta-\theta)\pi^{\alpha-2}) + D_x^{\alpha}L(x)}$$

$$r(x) = \frac{D_x^{\alpha}f(x)}{\pi^{\alpha}} + \frac{f(0)-f(\pi)-f(0)\pi^{1-\alpha}}{\pi^{1+\alpha}} + f(0) - 2\sqrt{\frac{2\alpha}{\pi^{2\alpha-2}}(D_x^{\alpha}g(x) + D_x^{\alpha}\rho(x)(\beta-\theta)\pi^{\alpha-2}) + D_x^{\alpha}L(x)}$$

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Conflicts of interest

There are no conflicts of interest in this work.

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