Cumhuriyet Science Journal

Cumhuriyet Sci. J., 45(1) (2024) 147-152
DOI: https://doi.org/10.17776/csj.1400543

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Kähler Magnetic Curves in Conformally Euclidean Schwarzschild Space

Özgür Kelekçi [1,a,*]

[1] [1]Department of Basic Sciences & Faculty of Engineering, University of Turkish Aeronautical Association Ankara, Türkiye.
*Corresponding author

Research Article

History
Received: 05/12/2023
Accepted: 04/03/2024

ABSTRACT

In this paper, we study the magnetic curves on a Kähler manifold which is conformally equivalent to Euclidean Schwarzschild space. We show that Euclidean Schwarzschild space is locally conformally Kähler and transform it into a Kähler space by applying a conformal factor coming from its Lee form. We solve Lorentz equation to find analytical expressions for magnetic curves which are compatible with the almost complex structure of the proposed Kähler manifold. We also calculate the energy of magnetic curves.

Keywords: Magnetic curve, Lorentz equation, Kähler magnetic fields, Euclidean Schwarzschild metric.

[a] *okelekci@thk.edu.tr* *https://orcid.org/0000-0001-7617-0231*

Introduction

The path of a charged particle moving on a manifold under the influence of a magnetic field is characterized by a magnetic curve. Generally, a magnetic field is defined by a closed 2-form (B) on a Riemannian manifold (M, g). The magnetic trajectories associated with the magnetic field are the curves γ on M that satisfy the Lorentz equation,

$$\nabla_{\dot{\gamma}}\dot{\gamma} = q\phi\dot{\gamma} \tag{1}$$

where ∇ is the Levi-Civita connection of g, q is a physical constant called charge, and φ is an anti-symmetric (1,1) tensor field called the Lorentz force. Magnetic field and Lorentz force are metrically equivalent such that B(X,Y)=g(φ(X),Y) , $\forall X,Y \in \mathfrak{X}$. Note that the Lorentz equation becomes geodesic equation for B=0. Anti-symmetric (1,1) tensor field in (1) becomes almost complex structure J for almost complex manifolds. Magnetic curves are generally labelled according to the studied ambient space. If the ambient space is a Kähler manifold the magnetic curves associated with it are called Kähler magnetic curves. Similarly, if the ambient space is a contact manifold, then the magnetic curves on it are called contact magnetic curves. There are also F-planar curves which generalize the magnetic curves and hence, the geodesics [1].

There has been significant amount of research on contact magnetic fields in three-dimensional model spaces of Thurston geometry [2-9]. Studies on Kähler magnetic curves in non-flat Kähler space forms were initiated by works of Adachi [10-12]. Adachi showed that every trajectory associated with a Kähler magnetic field on a complex projective space $\mathbb{CP}^n(c)$, (c > 0), is a small circle on a totally geodesic embedded 2-sphere [10]. Same author also obtained explicit expressions for magnetic curves in complex hyperbolic spaces $\mathbb{CH}^n(-c)$, (c > 0) [11]. Kalinin studied Kähler magnetic fields and their trajectories on Kähler manifolds of constant holomorphic sectional curvature [13]. In that study, Kalinin obtained differential equations deduced from Lorentz equation, but he did not provide solutions to those equations. Another line of research was initiated by studying J-trajectories in locally conformally Kähler (lcK) manifolds which are solutions of equation (1) for almost complex manifolds. Ateş, Munteanu and Nistor investigated trajectories in $\mathbb{R}\times S^3$ [14], and the results were extended to arbitrary Vaisman manifolds in [15]. Moreover, Inoguchi showed that J-trajectories in a lcK manifold with parallel anti Lee field are of osculating order at most 3 [16].

Jleli and Munteanu showed that spacelike and timelike magnetic trajectories corresponding to the para-Kähler 2-form on a para-Kähler manifold are circles [17]. Erjavec and Inoguchi studied magnetic curves on $\mathbb{H}^3\times\mathbb{R}$ with respect to the strictly almost Kähler structure and obtained explicit expressions for magnetic curves which correspond to the almost complex structure compatible with the product metric [18]. Despite the existence of these studies about Kähler magnetic curves, there are few studies which concentrate on explicit construction of Kähler magnetic curves on frequently studied Kähler spaces in physics literature. Our study aims to make a better connection in this sense and potentially provide new research problems. We chose Euclidean Schwarzschild space for this purpose. As will be shown in the next section Euclidean Schwarzschild space is not Kähler but locally conformally Kähler. This fact was briefly mentioned in [19]. Schwarzschild space and its Euclidean version have been studied extensively in physics literature. A recent paper focusing on the detailed analysis of geodesic motion in Euclidean Schwarzschild geometry is more relevant to our study [20]. We refer to the introduction of that paper for a broader literature review on Euclidean Schwarzschild space.

This paper is structured as follows. In Section 2, we give the basic geometric structure of Euclidean

Schwarzschild space, show that it is locally conformally Kähler and obtain its Kähler counterpart. In Section 3, we solve Lorentz equation to find analytical expressions for magnetic curves which are compatible with the almost complex structure of the Kähler space and also calculate the energy of these magnetic curves. Finally, we summarize the results and give potential research problems in Section 4.

Geometry of Euclidean Schwarzschild Space and Its Conformal Transformation

Euclidean Schwarzschild space is a complete solution to the Euclidean Einstein equations with zero cosmological constant (Λ) and it is characterized by the non-trivial topology of $\mathbb{R}^2\times S^2$ [21]. Metric for Euclidean Schwarzschild (ES) space is given by

$$\left(1-\frac{2m}{r}\right)\mathrm{d}\tau^2+\left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2) \tag{2}$$

It was shown that ES manifold admits a complex Hermitian structure [22]. Hermitian 2-form ω of a complex manifold (M^{2n},*g, J*) is an element of Λ^2 (bundle of 2-forms on *M*) for which self-dual (Ω_+) and anti-self-dual (Ω_-) bases can be defined as

$$\Omega_\pm = \{\boldsymbol{e}^1\wedge\boldsymbol{e}^2 \pm \boldsymbol{e}^3\wedge\boldsymbol{e}^4,\ \boldsymbol{e}^1\wedge\boldsymbol{e}^3 \pm \boldsymbol{e}^4\wedge\boldsymbol{e}^2,\ \boldsymbol{e}^1\wedge\boldsymbol{e}^4 \pm \boldsymbol{e}^2\wedge\boldsymbol{e}^3\} \tag{3}$$

where $\boldsymbol{e}^i$'s are orthonormal basis one-forms. We use the following orthonormal frame and the corresponding dual co-frame on ES space to describe its geometric structure .

$$\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_4\}=\{(1-2m/r)^{-1/2}\,\partial_\tau,(1-2m/r)^{1/2}\partial_r,(r\sin\theta)^{-1}\,\partial_\phi,r^{-1}\,\partial_\theta\} \tag{4}$$

$$\{\boldsymbol{e}^1,\boldsymbol{e}^2,\boldsymbol{e}^3,\boldsymbol{e}^4\}=\{(1-2m/r)^{1/2}\mathrm{d}\tau,(1-2m/r)^{-1/2}\mathrm{d}r,r\sin\theta\,\mathrm{d}\phi,r\,\mathrm{d}\theta\} \tag{5}$$

where $\partial_x\equiv\partial/\partial x$. The fundamental 2-form ω for a Hermitian manifold is given by

$$\omega(X,Y)=g(JX,Y) \qquad X, Y \in T_pM$$

We choose the orientation ($\boldsymbol{e}^1\wedge\boldsymbol{e}^2\wedge\boldsymbol{e}^3\wedge\boldsymbol{e}^4$) and the fundamental 2-form $\omega=\boldsymbol{e}^1\wedge\boldsymbol{e}^2+\boldsymbol{e}^3\wedge\boldsymbol{e}^4$ so that ω becomes self-dual. Almost complex structure that will yield the 2-form ω with respect to the chosen orientation will be:

$$\boldsymbol{J}\,\boldsymbol{e}^1=\boldsymbol{e}^2\ ,\ \boldsymbol{J}\,\boldsymbol{e}^2=-\boldsymbol{e}^1\ ,\ \boldsymbol{J}\,\boldsymbol{e}^3=\boldsymbol{e}^4\ ,\ \boldsymbol{J}\,\boldsymbol{e}^4=-\boldsymbol{e}^3 \tag{6}$$

Fundamental 2-form ω and coefficients of almost complex structure *J* can be written in coordinates as the following.

$$\omega=\mathrm{d}\tau\wedge\mathrm{d}r+r^2\sin\theta\,\mathrm{d}\phi\wedge\mathrm{d}\theta \tag{7}$$

$$J=\begin{pmatrix}0 & \frac{r-2m}{r} & 0 & 0\\ \frac{r}{2m-r} & 0 & 0 & 0\\ 0 & 0 & 0 & \sin\theta\\ 0 & 0 & -(\sin\theta)^{-1} & 0\end{pmatrix}$$

It is easy to see that the ω given above and other ω's formed by elements of $\Omega_\pm$ are not closed ($d\omega\neq 0$). Hence, ES manifold is not Kähler in its current form. A Hermitian manifold (M^{2n},*g, J*) is called locally conformally Kähler (l.c.K) if and only if there exists a globally defined closed 1-form (called Lee form) ξ on M^{2n} so that

$$d\omega=\xi\wedge\omega \tag{8}$$

Furthermore, it was shown that if ξ is exact ($\xi=\mathrm{d}f$) then there exists a Kähler metric $\hat{g}$ obtained by applying a conformal factor to the original metric g [23]:

$$\hat{g}=e^{-f}g \tag{9}$$

We find that the 1-form $\xi=2\mathrm{d}r/r$ is satisfying (8), hence conclude that ES manifold is locally conformally Kähler. The corresponding Kähler metric is obtained as:

$$\hat{g}=\frac{1}{r^2}\left(1-\frac{2m}{r}\right)\mathrm{d}\tau^2+\frac{1}{r^2}\left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2+\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2 \tag{10}$$

Magnetic curves will be constructed by using this Kähler metric associated to the space which we call conformally Euclidean Schwarzschild space. It will be convenient to use the orthonormal frame and the dual co-frame given in (4)-(5) scaled by r and 1/r, respectively. We work with the same orientation and fundamental 2-form given in basis 1-forms as

$$\hat{\omega}=\hat{\boldsymbol{e}}^1\wedge\hat{\boldsymbol{e}}^2+\hat{\boldsymbol{e}}^3\wedge\hat{\boldsymbol{e}}^4$$

$$= \frac{1}{r^2} \mathrm{d}\tau \wedge \mathrm{dr} + \sin\theta\, \mathrm{d}\phi \wedge \mathrm{d}\theta \tag{11}$$

This 2-form $\widehat{\omega}$ is closed as expected. Therefore, we obtained a Kähler manifold ($\widehat{\mathrm{M}}^{2n}$, $\hat{g}$, J) with the orthonormal frame and the corresponding dual co-frame given as

$$\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3, \hat{\boldsymbol{e}}_4\} = \{\, r(1 - 2\,m/r)^{-1/2}\, \partial_\tau\,, r(1 - 2\,m/r)^{1/2} \partial_r, (\sin\theta)^{-1}\, \partial_\phi, \partial_\theta\} \tag{12}$$

$$\{\hat{\boldsymbol{e}}^1, \hat{\boldsymbol{e}}^2, \hat{\boldsymbol{e}}^3, \hat{\boldsymbol{e}}^4\} = \{r^{-1}(1 - 2\,m/r)^{1/2} \mathrm{d}\tau, r^{-1}(1 - 2\,m/r)^{-1/2} \mathrm{dr}, \sin\theta\, \mathrm{d}\phi, \mathrm{d}\theta\} \tag{13}$$

Following covariant derivatives are obtained by using the orthonormal frame given above

$$\nabla_{\hat{e}_1} \hat{\boldsymbol{e}}_1 = r^{-1}(1 - 2\,m/r)^{-1/2}(r - 3m)\hat{\boldsymbol{e}}_2\,, \nabla_{\hat{e}_1} \hat{\boldsymbol{e}}_2 = r^{-1}(1 - 2\,m/r)^{-1/2}(3m - r)\hat{\boldsymbol{e}}_1$$

$$\nabla_{\hat{e}_3} \hat{\boldsymbol{e}}_3 = -\cot\theta\, \hat{\boldsymbol{e}}_4 \quad, \nabla_{\hat{e}_3} \hat{\boldsymbol{e}}_4 = \cot\theta\, \hat{\boldsymbol{e}}_3 \tag{14}$$

consisting of only non-zero ($\nabla_{\hat{e}_i} \hat{\boldsymbol{e}}_j \neq 0$) derivatives.

Magnetic Curves in Conformally Euclidean Schwarzschild Space

Let us consider a smooth curve γ(s) parameterized by its arclength in conformally ES space.

$$\gamma(s) = \big(\tau(s), r(s), \phi(s), \theta(s)\big)$$

We will be searching for solutions to the Lorentz equation $\nabla_{\dot\gamma}\dot\gamma = qJ(\dot\gamma)$, hence $\gamma(s)$ will be a magnetic curve. Note that we consider charged particles with unit mass (m_p=1). It is known that a trajectory for a Kähler magnetic field is characterized by the motion of a charged particle with unit mass, maintaining a constant speed under the influence of this magnetic field [12]. The unit tangent vector field can be written as

$$\dot\gamma(s) = \dot\tau(s)\partial_\tau + \dot r(s)\partial_r + \dot\phi(s)\partial_\phi + \dot\theta(s)\partial_\theta \tag{15}$$

$$= \frac{\dot\tau(s)\sqrt{1-2m/r(s)}}{r(s)}\, \hat{\boldsymbol{e}}_1 + \frac{\dot r(s)}{r(s)\sqrt{1-2m/r(s)}}\, \hat{\boldsymbol{e}}_2 + \dot\phi(s)\sin(\theta(s))\, \hat{\boldsymbol{e}}_3 + \dot\theta(s)\hat{\boldsymbol{e}}_4$$

Now we can apply J on $\dot\gamma$ to compute the right-hand side of Lorentz equation

$$J(\dot\gamma) = -\frac{\dot r(s)}{r(s)\sqrt{1-2m/r(s)}}\, \hat{\boldsymbol{e}}_1 + \frac{\dot\tau(s)\sqrt{1-2m/r(s)}}{r(s)}\, \hat{\boldsymbol{e}}_2 - \dot\theta(s)\hat{\boldsymbol{e}}_3 + \dot\phi(s)\sin(\theta(s))\, \hat{\boldsymbol{e}}_4 \tag{16}$$

Then the left-hand side of the Lorentz equation can be computed by using the covariant derivatives given in (14).

$$\nabla_{\dot\gamma}\dot\gamma = \frac{(2(3m-r(s))\dot r(s)\dot\tau(s)+r(s)(r(s)-2m)\ddot\tau(s))}{r(s)^{5/2}\,(r(s)-2m)^{1/2}}\hat{\boldsymbol{e}}_1 + \frac{(-12m^3\dot\tau(s)^2+16m^2r(s)\dot\tau(s)^2+mr(s)^2(\dot r(s)^2-7\dot\tau(s)^2)+r(s)^3(-2m\ddot r(s)-\dot r(s)^2+\dot\tau(s)^2)+r(s)^4\ddot r(s))}{r(s)^{7/2}(r(s)-2m)^{3/2}}\hat{\boldsymbol{e}}_2 + (2\cos(\theta(s))\dot\theta(s)\dot\phi(s) + \sin(\theta(s))\ddot\phi(s))\hat{\boldsymbol{e}}_3 + (\ddot\theta(s) - \sin(\theta(s))\cos(\theta(s))\dot\phi(s)^2)\hat{\boldsymbol{e}}_4 \tag{17}$$

The following system of ordinary differential equations are obtained by combining equations (16) and (17) which basically form the Lorentz equation for conformally ES space.

$$\frac{\dot r(qr^2+2(3m-r)\dot\tau)+r(r-2m)\ddot\tau}{r^{5/2}(r-2m)^{1/2}} = 0 \tag{18}$$

$$\frac{-12m^3\dot\tau^2+16m^2r\,\dot\tau^2-qr^2(r-2m)^2\dot\tau+mr^2(\dot r^2-7\dot\tau^2)+r^3(\dot\tau^2-\dot r^2-2m\ddot r)+r^4\ddot r}{r^{7/2}(r-2m)^{3/2}} = 0 \tag{19}$$

$$\dot\theta(q + 2\cos\theta\, \dot\phi) + \sin\theta\, \ddot\phi = 0 \tag{20}$$

$$\ddot\theta - \sin\theta\, \dot\phi(q + \cos\theta\, \dot\phi) = 0 \tag{21}$$

Let's rewrite equation (18) as

$$\dot r(q\, r^2 + 4(2m - r)\dot\tau) + \partial_s((r^2 - 2mr)\dot\tau) \tag{22}$$

Define a function $f(s)$

$$f(s) = (r(s)^2 - 2m\, r(s))\dot\tau(s) \tag{23}$$

Using $f(s)$ in equation (22) gives

$$qr(s)^2\dot r(s) - \frac{4f(s)}{r(s)}\dot r(s) + \dot f(s) = 0 \tag{24}$$

Multiplying both sides of equation (24) by $r(s)^{-4}$

$$qr(s)^{-2}\dot{r}(s) - 4f(s)r(s)^{-5}\dot{r}(s) + \dot{f}(s)r(s)^{-4} = 0$$

which can be written as an exact differential

$$\partial_s(f(s)r(s)^{-4} - qr(s)^{-1}) = 0$$

$$f(s)r(s)^{-4} - qr(s)^{-1} = a \Rightarrow f(s) = qr(s)^3 + a\,r(s)^4 \tag{25}$$

Then insert this in (23)

$$\dot{\tau}(s) = \frac{f(s)}{r(s)^2 - 2m\,r(s)} = \frac{qr(s)^2 + a\,r(s)^3}{r(s) - 2m} \tag{26}$$

Finally integrating (26) we get

$$\tau(s) = b + \int \frac{r(s)^2(q + a\,r(s))}{r(s) - 2m} ds \tag{27}$$

where a and b are arbitrary constants. Inserting this in equation (19) yields:

$$a^2r^5 + ar^4(q - 3a\,m) - 4a\,mqr^3 + r^2(\ddot{r} - mq^2) + m\dot{r}^2 - r(2m\,\ddot{r} + \dot{r}^2) = 0 \tag{28}$$

We couldn't find analytical solutions to equation (28), so we set $a = 0$ in equation (27) which then gives the following ODE

$$r^2(\ddot{r} - mq^2) + m\dot{r}^2 - r(2m\,\ddot{r} + \dot{r}^2) = 0 \tag{29}$$

Note that some numerical solutions can be found for equation (28) with a as non-zero constant. But in this study we are interested in exact solutions, so we will concentrate our focus on analytical solutions. It is possible to find some analytical solutions with $a \neq 0$, such as r=constant and $\tau(s) = cs + d$. However, those are relatively trivial solutions and as will be shown in the proceeding steps they are already included in our more general solution.

Solution to the ODE obtained in (29) is found as

$$r(s) = \frac{m(q^2 + {c_1}^2)(1 - \cosh(c_1(s + c_2)))}{{c_1}^2} \tag{30}$$

which leads to $\tau(s)$ by using equation (27) and setting b=c_3

$$\tau(s) = -4m \arctan\left(\frac{q\tanh(\frac{c_1}{2}(s + c_2))}{c_1}\right) - \frac{mq(q^2 + {c_1}^2)\sinh(c_1(s + c_2))}{{c_1}^3} + mq(3 + \frac{q^2}{{c_1}^2})(s + c_2) + c_3 \tag{31}$$

We apply a variable change $\theta(s) \to \arccos(X(s))$ in order to solve equations (20) and (21) which transform to

$$-\frac{\dot{X}\,(q + 2X\,\dot{\phi}) + (X^2 - 1)\ddot{\phi}}{\sqrt{1 - X^2}} = 0 \tag{32}$$

$$\frac{-\dot{\phi}(X^2 - 1)^2(q + X\,\dot{\phi}) + (X^2 - 1)\ddot{X} - X\,\dot{X}^2}{(1 - X^2)^{3/2}} = 0 \tag{33}$$

Equation (32) can be rewritten as

$$\partial_s(q\,X + \dot{\phi}(X^2 - 1)) = 0 \tag{34}$$

Integrating both sides yields

$$q\,X + \dot{\phi}(X^2 - 1) = c$$

$$\Rightarrow q\,X + c_5 = \dot{\phi}(1 - X^2) \Rightarrow \dot{\phi}(s) = \frac{q\,X(s) + c_5}{1 - X(s)^2} \tag{35}$$

Integrating (35) again gives

$$\phi(s) = c_4 + \int \frac{qX(s) + c_5}{1 - X(s)^2} ds \tag{36}$$

Inserting (36) in (33) yields

$$\frac{X(q^2 + X(qc_5 - \ddot{X}) + \dot{X}^2 + {c_5}^2) + q\,c_5 + \ddot{X}}{(1 - X^2)^{3/2}} = 0 \tag{37}$$

Solution to (37) is obtained as

$$X(s) = \frac{e^{-c_6(s + c_7)}({c_5}^2 + {c_6}^2)(q^2 + {c_6}^2) + e^{c_6(s + c_7)} + 2qc_5}{2{c_6}^2} \tag{38}$$

Inserting $X(s)$ and taking the integral in (36) gives

$$\phi(s) = \arctan\left(\frac{c_6(q + c_5)}{qc_5 + e^{c_6(s + c_7)} - {c_6}^2}\right) + \arctan\left(\frac{c_6(q - c_5)}{qc_5 + e^{c_6(s + c_7)} + {c_6}^2}\right) + c_4 \tag{39}$$

Remember that θ was transformed to the variable X by $\theta(s) \to \arccos(X(s))$. Hence,

$$\theta(s) = \arccos\left(\frac{e^{-c_6(s+c_7)}(c_5{}^2+c_6{}^2)(q^2+c_6{}^2)+e^{c_6(s+c_7)}+2qc_5}{2c_6{}^2}\right) \tag{40}$$

This concludes the solution of Lorentz equation for the magnetic curve γ(s). Note that we fixed only one initial condition in equation (27) and obtained a general solution for the Lorentz equation with seven arbitrary constants out of eight. However, one should be careful when choosing the constants in equation (40) since the domain of inverse cosine function is [-1, 1].

We give a plot of γ(s) for q=m=1 and all other constants were chosen by respecting the domains of the given functions, particularly fixing $\theta = 0$.

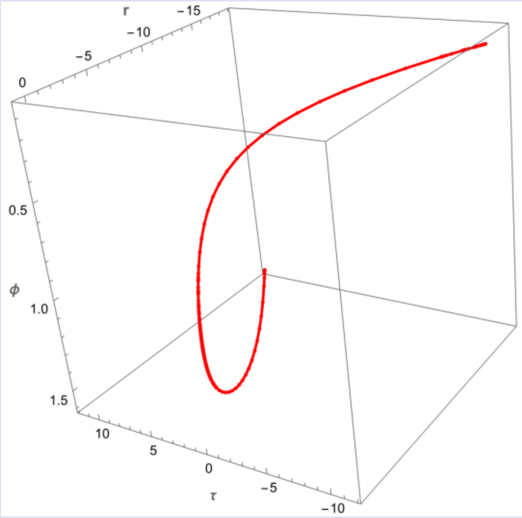

Figure 1. Magnetic curve γ(s) for q=m=1, $\theta = 0$ and $s \in [-3,3]$..

It is known that if magnetic field B on M has a global vector potential A, i.e. B=dA, then the energy functional of a smooth curve γ : [a, b] → M can be defined by [12]

$$E_A(\gamma) = \int_a^b \left[\frac{1}{2}\,\|\dot{\gamma}\|^2 + A(\dot{\gamma})\right] ds \tag{41}$$

We have given Kähler magnetic field B on ES space in equation (11) as a 2-form which can be written as dA

$$\begin{aligned} \widehat{\omega} &= \frac{1}{r^2}\mathrm{d}\tau \wedge \mathrm{d}r + \sin\theta\, \mathrm{d}\phi \wedge \mathrm{d}\theta \\ &= d\left(\frac{\mathrm{d}\tau}{r} + \cos\theta\, \mathrm{d}\phi\right) \;\Rightarrow\; A = \frac{\mathrm{d}\tau}{r} + \cos\theta\, \mathrm{d}\phi \end{aligned} \tag{42}$$

Since there exists a global vector potential A for the magnetic field we can calculate the energy for an interval of magnetic field curve γ(s). We first compute the integrand of the integral given in equation (41) by using the metric $\hat{g}$ and $\dot{\gamma}(s)$ given in equation (15) as follows

$$\begin{aligned} \|\dot{\gamma}\|^2 &= \frac{\dot{r}(s)^2}{r(s)(r(s)-2m)} + \frac{(r(s)-2m)\dot{\tau}(s)^2}{r(s)^3} + \dot{\theta}(s)^2 + \sin^2(\theta(s))\dot{\phi}(s)^2 \\ A(\dot{\gamma}) &= \frac{\dot{\tau}(s)}{r(s)} + \cos(\theta(s))\dot{\phi}(s) \end{aligned} \tag{43}$$

which yield the following after inserting the parameterized coordinates $\tau(s), r(s), \phi(s), \theta(s)$ given in equations (30), (31), (39) and (40)

$$\|\dot{\gamma}\|^2 = c_1{}^2 - c_6{}^2 \tag{44}$$

$$A(\dot{\gamma}) = c_6{}^2 e^{c_6(s+c_7)}\left(\frac{q-c_5}{(c_5{}^2+c_6{}^2)(q^2+c_6{}^2)+2(c_5q+c_6{}^2)e^{c_6(s+c_7)}+e^{2c_6(s+c_7)}} - \frac{q+c_5}{(c_5{}^2+c_6{}^2)(q^2+c_6{}^2)-2(c_6{}^2-c_5q)e^{c_6(s+c_7)}+e^{2c_6(s+c_7)}}\right) - \frac{2c_1{}^2q}{(q^2+c_1{}^2)\cosh(c_1(s+c_2))-q^2+c_1{}^2}$$

Finally, we take the integral given in (41) and obtain the energy of the magnetic curve associated with the potential A for an interval of $s \in [a,b]$ as

$$E_A(\gamma) = \frac{1}{2}({c_1}^2 - {c_6}^2)(b-a) - \arctan\left(\frac{c_6(q+c_5)}{e^{c_6(a+c_7)}+c_5q-{c_6}^2}\right) + \arctan\left(\frac{c_6(q-c_5)}{e^{c_6(a+c_7)}+c_5q+{c_6}^2}\right) + 2\arctan\left(\frac{q\tanh(\frac{1}{2}c_1(a+c_2))}{c_1}\right) + \arctan\left(\frac{c_6(q+c_5)}{e^{c_6(b+c_7)}+c_5q-{c_6}^2}\right) - \arctan\left(\frac{c_6(q-c_5)}{e^{c_6(b+c_7)}+c_5q+{c_6}^2}\right) - 2\arctan\left(\frac{q\tanh(\frac{1}{2}c_1(b+c_2))}{c_1}\right) \quad (45)$$

It is clear that the energy computed in (45) will always be finite for a finite interval [a,b] since the range of arctan function is (-π/2, π/2). Hence, it is possible to conclude that the proposed magnetic field and the associated magnetic curve solution is physically meaningful.

Conclusions and Outlook

In this study, we showed that Euclidean Schwarzschild space is locally conformally Kähler and transformed it into a Kähler space by applying a conformal factor. Then we solved Lorentz equation to find analytical expressions for magnetic curve which is compatible with the almost complex structure of the newly found Kähler space. Finally, we found a global vector potential A whose exterior derivative gives the magnetic field and calculated the energy of the magnetic curves for this vector potential. We think that the following research problems might be interesting for future study.

- What are the J-trajectories of Euclidean Schwarzschild space?
- J-trajectories of $\mathbb{R}\times S^3$ was already studied in literature. $\mathbb{R}\times S^3$ is also known to be a locally conformally Kähler space. So, can we apply the same procedure that we did in this study and find the Kähler magnetic curves for the Kähler counterpart of $\mathbb{R}\times S^3$?
- More generally, let (M, g, J) be a locally conformally Kähler space and (M', g', J') be its corresponding Kähler space. Can we find a general relationship between J-trajectory γ on M and Kähler magnetic curve γ′ on M'?
- Among the magnetic curves found in literature, which ones can be considered physical, and how can we classify them?

Although there are some studies on Kähler magnetic curves, there are few in the literature that address the explicit construction of these curves, particularly in relation to physics. We expect that this study, along with the proposed problems outlined above, will offer insights for future studies on four-dimensional Kähler magnetic curves.

Acknowledgments

The authors declare that there is no conflict of interest.

Conflicts of interest

There are no conflicts of interest in this work.

References

[1] Hinterleitner I., Mikes J., On F-planar mappings of spaces with affine connections, *Note Mat.*, 27(1) (2007) 111–118.

[2] Cabrerizo J.L., Fernandez M., Gomez J., On the existence of almost contact structure and the contact magnetic field, *Acta. Math. Hungar.*, 125 (2009) 191-199.

[3] Cabrerizo J.L., Fernandez M., Gomez J., The contact magnetic flow in 3D Sasakian manifolds, *J. Phys. A: Math. Theor.*, 42(19) (2009) 195-201.

[4] Munteanu M. I., Nistor A.I., The classification of Killing magnetic curves in $S^2\times R$, *J. Geom. Phys.*, 62(2) (2012) 170-182.

[5] Druţă-Romaniuc S.L., Inoguchi J., Munteanu M.I., Nistor A.I., Magnetic curves in Sasakian manifolds, *J. Nonlinear Math. Phys.*, 22(3) (2015) 428–447.

[6] Druţă-Romaniuc S.L., Inoguchi J., Munteanu M.I., Nistor A.I., Magnetic curves in cosymplectic manifolds, *Rep. Math. Phys.*, 78 (1) (2016) 33-48.

[7] Erjavec Z., Inoguchi, J., Magnetic curves in Sol_3, *J. Nonlinear Math. Phys.*, 25(2) (2018) 198-210.

[8] Inoguchi, J., Munteanu, M.I., Magnetic curves in the real special linear group, *Adv. Theor. Math. Phys.*, 23 (8) (2019) 2161-2205.

[9] Kelekçi Ö., Dündar F.S., Ayar G., Classification of Killing magnetic curves in $\mathbb{H}^3$, *Int. J. Geom. Methods Mod. Phys.*, 20 (14) (2023) 2450006.

[10] Adachi T., Kähler magnetic fields on a complex projective space, *Proc. Japan Acad.*, 70 (1994) 12-13.

[11] Adachi T., Kähler Magnetic flow for a manifold of constant holomorphic sectional curvature, *Tokyo J. Math.*, 18 (1995) 473-483.

[12] Adachi T., Kähler magnetic fields on Kähler manifolds of negative curvature, *Differential Geom. Appl.*, 29 (2011) S2-S8.

[13] Kalinin, D.A., Trajectories of charged particles in Kähler magnetic fields, *Rep. Math. Phys.*, 39(3) (1997) 299-309.

[14] Ateş O., Munteanu M. I., Periodic J-trajectories on $R\times S^3$, *J. Geom. Phys.*, 133 (2018) 141-152.

[15] Inoguchi J., Lee J., J-trajectories in Vaisman manifolds, *Differential Geom. Appl.*, 82(101882) (2022) 1-21.

[16] Inoguchi J., J-trajectories in Locally Conformal Kahler Manifolds with Parallel Anti Lee Field, *International Electronic Journal of Geometry,* 13(2) (2020) 30-44.

[17] Jleli M., Munteanu M.I., Magnetic curves on flat para-Kähler manifolds, *Turkish Journal of Mathematics*, 39(6) (2015) 963-969.

[18] Erjavec Z., Inoguchi J., Magnetic curves in $\mathbb{H}^3\times\mathbb{R}$, *J. Korean Math. Soc.*, 58(6) (2021) 1501–1511.

[19] Yau S.T., Einstein manifolds with zero Ricci curvature, Surveys in differential geometry: essays on Einstein manifolds, Int. Press, Boston, MA, (1999) 1-14.

[20] Battista E., Esposito G., Geodesic motion in Euclidean Schwarzschild geometry, *Eur. Phys. J. C,* 82(1088) (2022) 1-13.

[21] Etesi G., Hausel T., Geometric interpretation of Schwarzschild instantons, *J. Geom. Phys.*, 37 (1–2) (2001) 126-136.

[22] Rajan D., Complex spacetimes and the Newman–Janis trick, Master thesis, Victoria University of Wellington, School of Mathematics and Statistics, 2015.

[23] Dragomir S., Ornea L., Locally Conformal Kähler Geometry,1st ed. Birkhäuser Boston, (1998) 1–5.