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On the Solvability of Iterative Systems of Fractional-Order Differential Equations with Parameterized Integral Boundary Conditions

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Abstract

The aim of this paper is to determine the eigenvalue intervals of $\mu_{\mathtt{k}}$, $1 \leq \mathtt{k} \leq \mathtt{n}$ for which an iterative systems of a class of fractional-order differential equations with parameterized integral boundary conditions (BCs) has at least one positive solution by means of standard fixed point theorem of cone type. To the best of our knowledge, this will be the first time that we attempt to reach such findings for the topic at hand in the literature. The obtained results in the paper are illustrated with an example for their feasibility.

1. Introduction

There is a strong impetus for the study of nonlinear fractional systems, and significant research efforts have been made undertaken lately for these systems with the aim of implementing findings on the existence of positive solutions in related fields. At this point, differential calculus expanded its scope to include the dynamics of the complex real world, and new theories began to be put into effect and assessed on real data [1]. A variety of materials and processes with characteristics of heredity and memory can be accurately described by the nonlocal nature of fractional calculus [2, 3]. There are numerous applications in a variety of scientific disciplines, including biomathematics [4], viscoelasticity [5], non-Newtonian fluid mechanics [6], and characterization of anomalous diffusion [7].
Progressively, distinctive scientific advances and tools are created specifically for fractional differential equations (FDEqs). Due to this, a significant amount of scientists concentrate on boundary value problems (BVPs) for FDEqs involving various derivatives, such as Riemann–Liouville or Caputo, as well as some novel derivatives, including conformable fractional derivatives [8]. The literature on FDEqs of the conformable type is not enriched yet. The conformable fractional derivative was first proposed in 2014. The conformable derivative can be utilized for modeling many physical problems as DEqs with conformable fractional derivatives are easier to solve numerically in comparison to those with Riemann–Liouville or Caputo fractional derivatives. A new concept, known as the conformable fractional derivative, has recently [9, 10] been defined. Indeed, several researchers have previously applied conformable fractional derivatives to a wide range of domains, and numerous replicating methodologies have been established, see [11]. In different industries, such as telecommunication equipment, synthetic chemicals, automobiles, and pharmaceuticals, BVPs are frequently used. In these processes, positive solutions seem to be beneficial. In these contexts, the existence of positive solutions is often advantageous. For instance, in [12], the authors established the existence of multiple positive solutions for a coupled system of Riemann–Liouville FBVPs by means of an Avery generalization of the Leggett–Williams FPT. Subsequently, in [13], the same authors determined the eigenvalue intervals of the parameters leading to a positive solution for an iterative system of nonlinear Sturm–Liouville FBVPs by utilizing the Guo–Krasnosel'skii FPT on a cone. Additionally, in [14], the authors examined p-Laplacian fractional higher-order BVPs, establishing criteria for determining parameter values ensuring at least one positive solution. Furthermore, they derived sufficient conditions for the existence of an even number of positive solutions for FBVPs using an Avery–Henderson functional FPT. Moreover, in [15], the authors established the existence of at least three positive solutions to a system of FBVPs by employing a five-functionals FPT. Lastly, in [16], the authors investigated the eigenvalue intervals of parameters guaranteeing at least one positive solution for an iterative system of four-point FBVPs under suitable conditions.

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Recently, Zhou et al. [17] the existence, uniqueness, and multiplicity of findings associated with positive solutions to various types of conformable FBVPs. By using conventional fixed point theorems (FPTs) in conjunction with the theory of the cosine family of linear operators, Bouaouid [18] showed the existence and continuous dependence of mild solutions for a class of conformable FDEqs with nonlocal initial conditions. In their study of conformable stochastic functional DEqs of the neutral type, Xiao et al. [19] examined the existence and stability outcomes. A mild solution to a conformable FBVP was introduced by Jaiswal et al. [20] and the existence, uniqueness of solutions to the considered problem employing the contraction principle have been proven.

Conformable FDEqs with integral BCs provide a more flexible framework for modeling complex systems that exhibit non-local or memory-dependent behavior. Many real-world processes, such as heat conduction in non-homogeneous materials or transport phenomena in porous media, can be better described using fractional calculus. Gokdogan et al. demonstrated the uniqueness of solutions for sequential linear conformable FDEqs in [21]. Khuddush et al. [22] obtained the existence of positive solutions for an iterative system of conformable fractional dynamic BVPs on time scales by an application of FPT on a Banach space. Zhong and Wang in [23], where they studied the existence of positive solutions to the FBVP

$$\begin{aligned}&\mathtt{D}^{\mathtt{q}}\mathtt{u}(\mathtt{z})+\mathfrak{f}\big(\mathtt{z},\mathtt{u}(\mathtt{z})\big)=0, \quad \mathtt{z}\in(0,1),\\ &\mathtt{u}(0)=0,\ \ \mathtt{u}(1)=\lambda\int_0^1\mathtt{u}(\mathtt{z})d\mathtt{z},\end{aligned}$$

where $\mathtt{q}\in(1,2]$, λ is a constant and $\mathtt{D}^{\mathtt{q}}$ is the conformable derivative. By utilizing the solution-tube approach and Schauder's FPT, Bendouma et al. [24] investigated the existence of solutions to systems of conformable FDEqs concerning periodic conditions.

In [25], Haddouchi used the Kernel characteristics along with the FPT in a cone to investigate the existence of positive solutions to conformable FBVPs

$$\begin{aligned}&\mathtt{D}^{\mathtt{q}}\mathtt{u}(\mathtt{z})+\mathfrak{f}\big(\mathtt{z},\mathtt{u}(\mathtt{z})\big)=0, \quad \mathtt{z}\in(0,1),\\ &\mathtt{u}(0)=0,\ \ \mathtt{u}(1)=\lambda\int_0^\eta\mathtt{u}(\mathtt{z})d\mathtt{z},\end{aligned}$$

where $\mathtt{q}\in(1,2]$, $\eta\in(0,1]$, λ is a constant and $\mathtt{D}^{\mathtt{q}}$ is the conformable derivative.

Through the use of various FPTs found in the literature, numerous authors have explored the existence of positive solutions to a variety BVPs for ordinary, FDEqs during the past few years. Motivated and inspired by above highly decorated topics, by employing the Guo–Krasnosel'skii FPT of cone compression and expansion of norm kind (see [26, 27]) to the considered problem. More explicitly, we construct the Kernel for the associated linear FBVP, and estimate the bounds of this Kernel in more detail since they are essential for finding suitable fixed points for the newly indicated operator on a cone in a Banach space. Furthermore, it was explained how to utilize the fixed point technique and the bootstrapping argument to establish the existence of positive solutions to the iterative system. To the best of our knowledge, in this work, we attempt for the first time to determine the eigenvalue intervals of parameters that have positive solutions for the following iterative systems of conformable FDEqs

$$\left.\begin{aligned}\mathtt{D}^{\mathtt{q}}\mathtt{u}_{\mathtt{k}}(\mathtt{z})+\mu_{\mathtt{k}}\mathtt{p}_{\mathtt{k}}(\mathtt{z})\mathfrak{g}_{\mathtt{k}}\big(\mathtt{u}_{\mathtt{k}+1}(\mathtt{z})\big)=0\\ \mathtt{u}_{\mathtt{n}+1}(\mathtt{z})=\mathtt{u}_1(\mathtt{z}), \quad \mathtt{z}\in(0,1),\end{aligned}\right\} \tag{1.1}$$

with parameterized integral BCs

$$\left.\begin{aligned}\mathtt{u}_{\mathtt{k}}(0)=0, \quad \mathtt{u}_{\mathtt{k}}(1)=\vartheta\int_0^\xi\mathtt{u}_{\mathtt{k}}(\mathtt{z})d\mathtt{z},\\ \text{for } 1\le\mathtt{k}\le\mathtt{n},\end{aligned}\right\} \tag{1.2}$$

where $\mathtt{q}\in(1,2]$, $\xi\in(0,1]$, $\vartheta\in\mathbb{R}^+$ is constant and $\mathtt{D}^{\mathtt{q}}$ is the conformable fractional derivative. Iterative FDEqs have a variety of applications, which makes studying them preferable to non-iterative DEqs. For instance, IFDEqs are the most suitable for studying problems associated with infectious models and the kinetics of particles that are charged with delayed contact and can't be employed to study such problems via ordinary non-iterative DEqs. Iterative DEqs model dynamic systems where a variable's rate of change depends not only on its current value but also on its past values. These equations capture the influence of a system's history on its current state, often in a nonlinear fashion. They find applications across various fields, including modeling object motion, fluid dynamics, disease spread, chemical reactions, population growth, control systems, electrical circuits, and economic systems. The equation (1.1) relates a diffusion phenomena with source or reaction term. For example, in thermal conduction, it can be understood as a one dimensional heat conduction equation modeling steady states of a heating rod of length $\mathtt{c}$ with the controller at $\mathtt{r}=\mathtt{c}$, while the left end is held at 0°C and $\mathtt{h}$ is function of source distribution temperature over time delays in thermal conduction [28, 29]. The main advantage of studying IFDEqs over non-iterative DEqs exist in its various applications. For example, the problems related to infectious models and the motion of charge particles with retarded interaction are best described using IFDEqs and cannot be studied by general non DEqs.

We provide varied conditions for the functions $\mathfrak{g}_1,\mathfrak{g}_2,\cdots,\mathfrak{g}_{\mathtt{n}}$ and the intervals of $\mu_1,\mu_2,\cdots,\mu_{\mathtt{n}}$ ensuring that positive solutions to the iterative system of FBVP (1.1)–(1.2). A positive solution of the problem (1.1)–(1.2), we mean $\big(\mathtt{u}_1(\mathtt{z}),\mathtt{u}_2(\mathtt{z}),\cdots,\mathtt{u}_{\mathtt{n}}(\mathtt{z})\big)\in\big(\mathtt{C}^2[0,1]\big)^{\mathtt{n}}$ satisfying (1.1) and (1.2) with $\mathtt{u}_{\mathtt{k}}(\mathtt{z})>0, \mathtt{k}=1,2,\cdots,\mathtt{n}, \forall\, \mathtt{z}\in(0,1]$.

Throughout the article, we propose the following hypotheses:

(H_1) $\Delta=2-\vartheta\xi^2>0$.

(H_2) $\mathtt{p}_{\mathtt{k}}:[0,1]\to\mathbb{R}^+$ is continuous and $\mathtt{p}_{\mathtt{k}}$ does not vanish identically on any closed subinterval of $[0,1]$, for $\mathtt{k}=\overline{1,\mathtt{n}}$.

(H_3) $\mathfrak{g}_{\mathtt{k}}:\mathbb{R}^+\to\mathbb{R}^+$ is continuous, for $\mathtt{k}=\overline{1,\mathtt{n}}$.

(H_4) each of $\mathfrak{g}_{\mathtt{k}0}=\lim\limits_{\mathtt{x}\to0^+}\dfrac{\mathfrak{g}_{\mathtt{k}}(\mathtt{x})}{\mathtt{x}}$ and $\mathfrak{g}_{\mathtt{k}\infty}=\lim\limits_{\mathtt{x}\to\infty}\dfrac{\mathfrak{g}_{\mathtt{k}}(\mathtt{x})}{\mathtt{x}}$, for $1\le\mathtt{k}\le\mathtt{n}$, exists as positive real numbers.

The paper is arranged as follows: The preliminary results presented in Sect. 2 serve as foundations for the subsequent sections that follow. This covers the solution to the corresponding linear problem, an investigation of the characteristics of Kernels, and other pertinent information. The key existence theorems for the problem (1.1)–(1.2) are the focus of Sect. 3. In Sect. 4, an example is coined in support of validity of the findings concerning the earlier sections.

2. Preliminaries, Kernel and Bounds

In order to move on to the key results in the subsequent sections, the necessary results are provided here.

Definition 2.1. *[8] The conformable derivative of $\mathtt{h}:[0,\infty)\to\mathbb{R}$ is defined as*

$$\mathtt{D}_0^{\zeta}\mathtt{h}(\mathbf{r})=\lim_{\varepsilon\to 0}\left[\frac{\mathtt{h}(\mathbf{r}+\varepsilon\mathbf{r}^{1-\zeta})-\mathtt{h}(\mathbf{r})}{\varepsilon}\right],\quad \mathbf{r}>0,\quad \zeta\in(0,1],$$

and

$$\mathtt{D}_0^{\zeta}\mathtt{h}(0)=\lim_{\mathbf{r}\to 0^+}\mathtt{D}_0^{\zeta}\mathtt{h}(\mathbf{r}).$$

If $\mathtt{h}$ is differentiable then $\mathtt{D}_0^{\zeta}\mathtt{h}(\mathbf{r})=\mathbf{r}^{1-\zeta}\mathtt{h}'(\mathbf{r})$.

Definition 2.2. *[8] The conformable fractional integral of a function of order ζ is defined for $\mathtt{h}:[0,\infty)\to\mathbb{R}$ as*

$$\mathtt{I}_0^{\zeta}\mathtt{h}(\mathbf{r})=\int_0^{\mathbf{r}}\mathfrak{s}^{\zeta-1}\mathtt{h}(\mathfrak{s})d\mathfrak{s},\quad \mathfrak{s}>0,\ \zeta\in(0,1].$$

Lemma 2.3. *[30] Let $\zeta\in(0,1]$ and $\mathtt{h}:(0,\infty)\to\mathbb{R}$ be differentiable. Then*

$$\mathtt{I}_0^{\zeta}\mathtt{D}_0^{\zeta}\mathtt{h}(\mathbf{r})=\mathtt{h}(\mathbf{r})-\mathtt{h}(0),\quad \forall\,\mathbf{r}>0.$$

Lemma 2.4. *Suppose $(\mathtt{H}_1)$ holds, let $\mathtt{h}(\mathtt{z})\in\mathtt{C}([0,1],\mathbb{R})$. Then $\mathtt{u}_1(\mathtt{z})\in\mathtt{C}([0,1],\mathbb{R})$ is a solution of the FBVP*

$$^{\mathtt{H}}\mathtt{D}_{1^+}^{\mathtt{q}}\mathtt{u}_1(\mathtt{z})+\mathtt{h}(\mathtt{z})=0,\quad \mathtt{z}\in(0,1),\tag{2.1}$$

$$\mathtt{u}_1(0)=0,\quad \mathtt{u}_1(1)=\vartheta\int_0^{\xi}\mathtt{u}_1(\mathtt{z})d\mathtt{z},\tag{2.2}$$

has a unique solution

$$\mathtt{u}_1(\mathtt{z})=\int_0^1\aleph(\mathtt{z},\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y},$$

where

$$\aleph(\mathtt{z},\mathtt{y})=\aleph_1(\mathtt{z},\mathtt{y})+\frac{\vartheta\mathtt{z}}{\Delta}\aleph_2(\xi,\mathtt{y}),\tag{2.3}$$

$$\aleph_1(\mathtt{z},\mathtt{y})=\begin{cases}(1-\mathtt{z})\mathtt{y}^{\mathtt{q}-1}, & 0\le\mathtt{y}\le\mathtt{z}\le 1,\\ \mathtt{z}(1-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}, & 0\le\mathtt{z}\le\mathtt{y}\le 1,\end{cases}$$
$$\aleph_2(\mathtt{z},\mathtt{y})=\begin{cases}(2\mathtt{z}-\mathtt{z}^2-\mathtt{y})\mathtt{y}^{\mathtt{q}-1}, & \mathtt{y}\le\mathtt{z},\\ \mathtt{z}^2(1-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}, & \mathtt{z}\le\mathtt{y}.\end{cases}$$

Proof. Let $\mathtt{u}_1(\mathtt{z})\in\mathtt{C}^2[0,1]$ be a solution of FBVP (2.1)-(2.2) and is uniquely expressed as

$$\mathtt{u}_1(\mathtt{z})=\sum_{\mathtt{k}=1}^{2}\mathtt{c}_{\mathtt{k}}\mathtt{z}^{2-\mathtt{k}}-\int_1^{\mathtt{z}}(\mathtt{z}-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}\mathtt{h}(\mathtt{y})d\mathtt{y}.$$

By the condition (2.2), we get $\mathtt{c}_2=0$ and $\mathtt{c}_1=\mathtt{I}^{\mathtt{q}}\mathtt{h}(1)+\mathtt{u}_1(1)$. Hence the unique solution of FBVP (2.1)-(2.2) is

$$\begin{aligned}\mathtt{u}_1(\mathtt{z})&=\left\{\begin{array}{l}\displaystyle\int_0^{\mathtt{z}}(1-\mathtt{z})\mathtt{y}^{\mathtt{q}-1}\mathtt{h}(\mathtt{y})d\mathtt{y}+\int_{\mathtt{z}}^1\mathtt{z}(1-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}\mathtt{h}(\mathtt{y})d\mathtt{y}+\\ \displaystyle\frac{\vartheta\mathtt{z}}{\Delta}\int_0^{\xi}\mathtt{y}^{\mathtt{q}-2}\left[\xi^2(1-\mathtt{y})-(\xi-\mathtt{y})^2\right]\mathtt{h}(\mathtt{y})d\mathtt{y}+\\ \displaystyle\frac{\vartheta\mathtt{z}}{\Delta}\int_{\xi}^1\xi^2(1-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}\mathtt{h}(\mathtt{y})d\mathtt{y}\end{array}\right.\\
&=\left\{\begin{array}{l}\displaystyle\int_0^1\aleph_1(\mathtt{z},\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y}+\frac{\vartheta\mathtt{z}}{\Delta}\int_0^{\xi}\mathtt{y}^{\mathtt{q}-1}(2\xi-\xi^2-\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y}+\\ \displaystyle\frac{\vartheta\mathtt{z}}{\Delta}\int_{\xi}^1\xi^2(1-\mathtt{y})\mathtt{y}^{\mathtt{q}-2}\mathtt{h}(\mathtt{y})d\mathtt{y}\end{array}\right.\\
&=\int_0^1\aleph_1(\mathtt{z},\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y}+\frac{\vartheta\mathtt{z}}{\Delta}\int_0^1\aleph_2(\xi,\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y}\\
&=\int_0^1\aleph(\mathtt{z},\mathtt{y})\mathtt{h}(\mathtt{y})d\mathtt{y},\end{aligned}$$

where $\aleph(\mathtt{z},\mathtt{y})$ is given in (2.3). The proof is completed. □

Lemma 2.5. *The Kernel* $\aleph(z,y)$ *given in* (2.3) *is nonnegative, for all* $z,y \in [0,1]$.

Proof. The Kernel $\aleph(z,y)$ is given in (2.3). Let $0 \le z \le y \le 1$. Then:

$$\aleph_1(z,y) = z(1-y)y^{q-2} \ge 0.$$

Let $0 \le y \le z \le 1$. Then:

$$\aleph_1(z,y) = (1-z)y^{q-1} \ge 0.$$

On the other hand, let $0 \le \xi \le y \le 1$. Then:

$$\aleph_2(\xi,y) = \xi^2(1-y)y^{q-2} \ge 0.$$

Let $0 \le y \le \xi \le 1$. Then:

$$\aleph_2(\xi,y) = (2\xi - \xi^2 - y)y^{q-1} \ge 0.$$

Hence $\aleph(z,y) \ge 0$. □

Lemma 2.6. *Let* $\sigma \in \left(0,\frac{1}{2}\right)$. *The Kernel* $\aleph_1(z,y)$ *has the properties:*

(1) $\aleph_1(z,y) \le \aleph_1(y,y)$, $\forall\, z,y \in (0,1]$,

(2) $\aleph_1(z,y) \ge z(1-z)\aleph_1(y,y), \forall\, z,y \in (0,1]$,

(3) $\aleph_1(z,y) \ge \sigma^2\aleph_1(y,y), \forall\, z \in [\sigma, 1-\sigma], y \in (0,1]$.

Proof. We prove (1). Let $0 \le z \le y \le 1$. Then:

$$\begin{aligned}\aleph_1(z,y) &= z(1-y)y^{q-2}\\ &\le (1-y)y^{q-1}\\ &= \aleph_1(y,y).\end{aligned}$$

Let $0 \le y \le z \le 1$. Then:

$$\begin{aligned}\aleph_1(z,y) &= (1-z)y^{q-1}\\ &\le (1-y)y^{q-1}\\ &= \aleph_1(y,y).\end{aligned}$$

Hence the inequality (1). We establish the inequality (2). Let $0 \le z \le y \le 1$. Then:

$$\begin{aligned}\aleph_1(z,y) &= z(1-y)y^{q-2}\\ &\ge (1-z)y^{q-1}\\ &\ge z(1-z)\aleph_1(y,y).\end{aligned}$$

Let $0 \le y \le z \le 1$. Then:

$$\begin{aligned}\aleph_1(z,y) &= (1-z)y^{q-1}\\ &\ge (1-z)(1-y)y^{q-1}\\ &\ge z(1-z)\aleph_1(y,y).\end{aligned}$$

Hence the inequality (2). On the other hand, if $\sigma \in \left(0,\dfrac{1}{2}\right)$, then $\aleph_1(z,y)$ satisfies

$$\aleph_1(z,y) \ge \sigma^2\aleph_1(y,y), \forall\, z \in [\sigma, 1-\sigma], y \in (0,1].$$

□

Lemma 2.7. *Let* $\sigma \in \left(0,\frac{1}{2}\right)$. *The Kernels* $\aleph_1(z,y)$ *and* $\aleph_2(z,y)$ *have the properties:*

(1) $\aleph_2(z,y) \le \aleph_1(y,y)$, $\forall\, z,y \in (0,1]$,

(2) $\aleph_2(z,y) \ge \theta(z)\aleph_1(y,y), \forall\, z,y \in (0,1]$,

$$\text{where } \theta(z) = \min\left\{z^2, z(1-z)\right\} = \begin{cases} z^2, & z \le \frac{1}{2},\\ z(1-z), & z > \frac{1}{2},\end{cases}$$

(3) $\aleph_2(z,y) \ge \sigma^2\aleph_1(y,y), \forall\, z \in [\sigma, 1-\sigma], y \in (0,1]$.

Proof. Let $0 \le z \le y \le 1$. Then:

$$\begin{aligned}\aleph_2(z,y) &= z^2(1-y)y^{q-2}\\ &\le z(1-y)y^{q-2}\\ &\le (1-y)y^{q-1}\\ &= \aleph_1(y,y).\end{aligned}$$

Let $0 \le y \le z \le 1$. Then:

$$\begin{aligned}\aleph_2(z,y) &= (2z-z^2-y)y^{q-1}\\ &\le \left[(1-y)-(1-z)^2\right]y^{q-1}\\ &\le (1-y)y^{q-1}\\ &= \aleph_1(y,y).\end{aligned}$$

Hence the inequality (1). Let $0 \le z \le y \le 1$. Then:

$$\begin{aligned}\aleph_2(z,y) &= z^2(1-y)y^{q-2}\\ &\ge z^2 y(1-y)y^{q-2}\\ &= z^2(1-y)y^{q-1}\\ &= z^2\aleph_1(y,y).\end{aligned}$$

Let $0 \le y \le z \le 1$. Then:

$$\begin{aligned}\aleph_2(z,y) &= (2z-z^2-y)y^{q-1}\\ &= \left[z(1-z)+(z-y)\right]y^{q-1}\\ &\ge z(1-z)y^{q-1}\\ &\ge z(1-z)(1-y)y^{q-1}\\ &\ge z(1-z)\aleph_1(y,y).\end{aligned}$$

Therefore $\aleph_2(z,y) \ge \theta(z)\aleph_1(y,y), \forall\, z,y \in (0,1]$, where

$$\theta(z) = \min\left\{z^2, z(1-z)\right\} = \begin{cases} z^2, & z \le \frac{1}{2},\\ z(1-z), & z > \frac{1}{2}.\end{cases}$$

Hence the inequality (2). On the other hand, if $\sigma \in \left(0, \frac{1}{2}\right)$, then it follows immediately from (2):

$$\aleph_2(z,y) \ge \sigma^2\aleph_1(y,y), \forall\, z \in [\sigma, 1-\sigma], y \in (0,1].$$

□

3. Existence of Positive Solutions

An n-tuple $\Big(u_1(z), u_2(z), \cdots, u_n(z)\Big)$ is a solution of the FBVP (1.1)-(1.2) if and only if $u_k(z) \in C^2[0,1],\ k = 1,2,\cdots,n$ satisfies:

$$u_1(z) = \begin{cases} \mu_1 \int_0^1 \left[\aleph_1(z,y_1) + \frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\right] p_1(y_1)\\ g_1\Bigg(\mu_2 \int_0^1 \left[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right] p_2(y_2)\cdots\\ g_{n-1}\Bigg(\mu_n \int_0^1 \left[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\\ p_n(y_n) g_n\Big(u_1(y_n)\Big) dy_n\Bigg)\cdots dy_2\Bigg) dy_1, \end{cases}$$

and

$$\begin{cases} u_2(z) = \mu_2 \int_0^1 \left[\aleph_1(z,y) + \frac{\vartheta z}{\Delta}\aleph_2(\xi,y)\right] p_2(y) g_2\big(u_3(y)\big) dy,\\ u_3(z) = \mu_3 \int_0^1 \left[\aleph_1(z,y) + \frac{\vartheta z}{\Delta}\aleph_2(\xi,y)\right] p_3(y) g_3\big(u_4(y)\big) dy,\\ \cdots\\ u_n(z) = \mu_n \int_0^1 \left[\aleph_1(z,y) + \frac{\vartheta z}{\Delta}\aleph_2(\xi,y)\right] p_n(y) g_n\big(u_{n+1}(y)\big) dy, \end{cases}$$

where $u_{n+1}(z) = u_1(z), \quad 0 < z < 1$. By a positive solution of the FBVP (1.1)-(1.2), we mean $\big(u_1(z), u_2(z), \cdots, u_n(z)\big) \in \Big(C^2[0,1]\Big)^n$ which satisfying the FDEq (1.1) and BCs (1.2) with $u_k(z) > 0, k = \overline{1,n} \, \forall \, z \in [0,1]$.
Let $B = \big\{x : x \in C[0,1]\big\}$ be the Banach space endowed with the norm

$$\|x\| = \max_{z\in[0,1]} |x(z)|$$

and $P \subset B$ be a cone defined as

$$P = \Bigg\{ x \in B \ : \ x(z) \geq 0 \text{ on } [0,1] \text{ and } \min_{z\in\big[\sigma,1-\sigma\big]} x(z) \geq \sigma^2\|x\| \Bigg\},$$

where $\sigma \in \big(0, \frac{1}{2}\big)$. Construct an integral operator $T : P \to B$, for $u_1 \in P$, as

$$Tu_1(z) = \begin{cases} \mu_1 \displaystyle\int_0^1 \Big[\aleph_1(z,y_1) + \frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\Big] p_1(y_1) g_1 \Bigg(\mu_2 \int_0^1 \Big[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\Big] \\ p_2(y_2)\cdots g_{n-1}\Bigg(\mu_n \displaystyle\int_0^1 \Big[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\Big] p_n(y_n) g_n\Big(u_1(y_n)\Big) dy_n\Bigg)\cdots dy_2\Bigg) dy_1. \end{cases}$$

Notice from (H_1) and Lemma 2.5 that, for $u_1 \in P$, $Tu_1(z) \geq 0$ on $[0,1]$. In addition, we have

$$Tu_1(z) \leq \begin{cases} \mu_1 \displaystyle\int_0^1 \Big[\aleph_1(y_1,y_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_1)\Big] p_1(y_1) \\ g_1\Bigg(\mu_2 \displaystyle\int_0^1 \Big[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\Big] p_2(y_2)\cdots \\ g_{n-1}\Bigg(\mu_n \displaystyle\int_0^1 \Big[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\Big] \\ p_n(y_n) g_n\Big(u_1(y_n)\Big) dy_n\Bigg)\cdots dy_2\Bigg) dy_1 \end{cases}$$

so that

$$\left.\begin{aligned} \|Tu_1\| \leq \mu_1 &\int_0^1 \Big[\aleph_1(y_1,y_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_1)\Big] p_1(y_1) \\ g_1&\Bigg(\mu_2 \int_0^1 \Big[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\Big] \\ p_2(y_2)\cdots g_{n-1}&\Bigg(\mu_n \int_0^1 \Big[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta} \\ \aleph_2(\xi,y_n)\Big]&p_n(y_n) g_n\Big(u_1(y_n)\Big) dy_n\Bigg)\cdots dy_2\Bigg) dy_1. \end{aligned}\right\} \tag{3.1}$$

If $u_1 \in P$, from Lemmas 2.6, 2.7 and (3.1), we deduce that

$$\begin{aligned} \min_{z\in\big[\sigma,1-\sigma\big]} Tu_1(z) &= \begin{cases} \displaystyle\min_{z\in\big[\sigma,1-\sigma\big]} \mu_1 \int_0^1 \Big[\aleph_1(z,y_1) + \frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\Big] \\ p_1(y_1) g_1\Bigg(\mu_2 \displaystyle\int_0^1 \Big[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\Big] \\ p_2(y_2)\cdots g_{n-1}\Bigg(\mu_n \displaystyle\int_0^1 \Big[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta} \\ \aleph_2(\xi,y_n)\Big] p_n(y_n) g_n\Big(u_1(y_n)\Big) dy_n\Bigg)\cdots dy_2\Bigg) dy_1 \end{cases} \\ &\geq \begin{cases} \mu_1\sigma^2 \displaystyle\int_0^1 \Big[\aleph_1(y_1,y_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_1)\Big] \\ p_1(y_1) g_1\Bigg(\mu_2 \displaystyle\int_0^1 \Big[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\Big] \\ p_2(y_2)\cdots g_{n-1}\Bigg(\mu_n \displaystyle\int_0^1 \Big[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta} \\ \aleph_2(\xi,y_n)\Big] p_n(y_n) g_n\Big(u_1(y_n)\Big) dy_n\Bigg)\cdots dy_2\Bigg) dy_1 \end{cases} \\ &\geq \sigma^2\|Tu_1\|. \end{aligned}$$

Therefore $\displaystyle\min_{z\in\big[\sigma,1-\sigma\big]} Tu_1(z) \geq \sigma^2\|Tu_1\|$. Hence $Tu_1 \in P$ and so $T : P \to P$. An application of the Arzela–Ascoli Theorem indicates that the operator T remains completely continuous.

3.1. Notations

We introduce:

$$\sigma_1=\max\left\{\begin{array}{c}\left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_1(y)dy g_{1\infty}\right]^{-1},\\ \left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_2(y)dy g_{2\infty}\right]^{-1},\\ \cdots\\ \left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_n(y)dy g_{n\infty}\right]^{-1}\end{array}\right\},$$

$$\sigma_2=\min\left\{\begin{array}{c}\left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_1(y)dy g_{10}\right]^{-1},\\ \left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_2(y)dy g_{20}\right]^{-1},\\ \cdots\\ \left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_n(y)dy g_{n0}\right]^{-1}\end{array}\right\}.$$

Theorem 3.1. *Suppose* (H_1)-(H_4) *hold. Then for each* $\mu_k, k=\overline{1,n}$ *satisfying*

$$\sigma_1<\mu_k<\sigma_2,\quad k=\overline{1,n}, \tag{3.2}$$

there exists an n*-tuple* $(u_1,u_2,\cdots,u_n)$ *satisfying the FBVP* (1.1)-(1.2) *s.t.* $u_k(z)>0$, $k=\overline{1,n}$ *on* $(0,1)$.

Proof. Let μ_k, $k=\overline{1,n}$ be found as in (3.2). Now let $\varepsilon>0$ be chosen s.t.

$$\max\left\{\begin{array}{c}\left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_1(y)dy(g_{1\infty}-\varepsilon)\right]^{-1},\\ \left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_2(y)dy(g_{2\infty}-\varepsilon)\right]^{-1},\\ \cdots\\ \left[\sigma^2\int_\sigma^{1-\sigma}\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_n(y)dy(g_{n\infty}-\varepsilon)\right]^{-1}\end{array}\right\}\le\min\left\{\begin{array}{c}\mu_1,\\ \mu_2,\\ \cdot\\ \cdot\\ \cdot\\ \mu_n\end{array}\right\}$$

and

$$\max\left\{\begin{array}{c}\mu_1,\\ \mu_2,\\ \cdot\\ \cdot\\ \cdot\\ \mu_n\end{array}\right\}\le\min\left\{\begin{array}{c}\left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_1(y)dy(g_{10}+\varepsilon)\right]^{-1},\\ \left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_2(y)dy(g_{20}+\varepsilon)\right]^{-1},\\ \cdots\\ \left[\int_0^1\left[\aleph_1(y,y)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right]p_n(y)dy(g_{n0}+\varepsilon)\right]^{-1}\end{array}\right\}.$$

Furthermore, according to g_{k0}, $k=\overline{1,n}$, there exists an $N_1>0$ s.t., for each $1\le k\le n$, $g_k(x)\le(g_{k0}+\varepsilon)x$, $\quad 1<x\le N_1$. Let $u_1\in P$ with $\|u_1\|=N_1$. By Lemmas 2.6, 2.7 and the choice of ε, for $0\le y_{n-1}\le 1$,

$$\begin{aligned}\mu_n\int_0^1&\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)g_n(u_1(y_n))dy_n\\ &\le\mu_n\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)(g_{n0}+\varepsilon)u_1(y_n)dy_n\\ &\le\mu_n\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)dy_n(g_{n0}+\varepsilon)\|u_1\|\\ &\le\|u_1\|=N_1.\end{aligned}$$

It follows from Lemmas 2.6, 2.7 in the same way, for $0 \leq y_{n-2} \leq 1$,

$$\left.\begin{array}{l}\mu_{n-1}\int_0^1\left[\aleph_1(y_{n-2},y_{n-1})+\frac{\vartheta y_{n-2}}{\Delta}\aleph_2(\xi,y_{n-1})\right]p_{n-1}(y_{n-1})\\ \qquad g_{n-1}\left(\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\right.\\ \qquad\qquad\left.p_n(y_n)g_n(u_1(y_n))dy_n\right)dy_{n-1}\end{array}\right\}$$
$$\leq\left\{\begin{array}{l}\mu_{n-1}\int_0^1\left[\aleph_1(y,y_{n-1})+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_{n-1})\right]\\ p_{n-1}(y_{n-1})dy_{n-1}\left(g_{\overline{n-1}0}+\varepsilon\right)\|u_1\|\end{array}\right.$$
$$\leq\|u_1\|=N_1.$$

Proceeding with the bootstrapping assertion, for $0 \leq z \leq 1$,

$$\left.\begin{array}{l}\mu_1\int_0^1\left[\aleph_1(z,y_1)+\frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\right]p_1(y_1)\\ \quad g_1\left(\mu_2\int_0^1\left[\aleph_1(y_1,y_2)+\frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right]p_2(y_2)\cdots\right.\\ \qquad g_{n-1}\left(\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\right.\\ \qquad\qquad\left.\left.p_n(y_n)g_n(u_1(y_n))dy_n\right)\cdots dy_2\right)dy_1\end{array}\right\}\leq N_1,$$

so that, for $0 \leq z \leq 1$, $Tu_1(z) \leq N_1$. Hence $\|Tu_1\| \leq N_1 = \|u_1\|$. If we set $E_1 = \{x \in B : \|x\| < N_1\}$, then

$$\|Tu_1\| \leq \|u_1\|,\ \text{for } u_1 \in P \cap \partial E_1. \tag{3.3}$$

Additionally, according to $g_{k\infty}$, $k = \overline{1,n}$, there exists $\overline{N}_2 > 0$ s.t., for each $1 \leq k \leq n$, $g_k(x) \geq (g_{k\infty} - \varepsilon)x,\ x \geq \overline{N}_2$. Choose $N_2 = \max\left\{2N_1, \frac{\overline{N}_2}{\sigma^2}\right\}$. Let $u_1 \in P$ and $\|u_1\| = N_2$. Then

$$\min_{z\in[\sigma,1-\sigma]} u_1(z) \geq \sigma^2\|u_1\| \geq \overline{N}_2.$$

Based on Lemmas 2.6, 2.7 and choice of ε, for $0 \leq y_{n-1} \leq 1$, we have:

$$\begin{aligned}&\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)g_n(u_1(y_n))dy_n\\ &\geq\sigma^2\mu_n\int_\sigma^{1-\sigma}\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)(g_{n\infty}-\varepsilon)u_1(y_n)dy_n\\ &\geq\sigma^2\mu_n\int_\sigma^{1-\sigma}\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)dy_n(g_{n\infty}-\varepsilon)\|u_1\|\\ &\geq\|u_1\|=N_2.\end{aligned}$$

It stems in the same way from Lemmas 2.6, 2.7 and choice of ε, for $0 \leq y_{n-2} \leq 1$:

$$\left.\begin{array}{l}\mu_{n-1}\int_0^1\left[\aleph_1(y_{n-2},y_{n-1})+\frac{\vartheta y_{n-2}}{\Delta}\aleph_2(\xi,y_{n-1})\right]p_{n-1}(y_{n-1})\\ \qquad g_{n-1}\left(\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\right.\\ \qquad\qquad\left.p_n(y_n)g_n(u_1(y_n))dy_n\right)dy_{n-1}\end{array}\right\}$$
$$\geq\left\{\begin{array}{l}\sigma^2\mu_{n-1}\int_\sigma^{1-\sigma}\left[\aleph_1(y,y_{n-1})+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_{n-1})\right]\\ p_{n-1}(y_{n-1})dy_{n-1}\left(g_{\overline{n-1}\infty}-\varepsilon\right)\|u_1\|\end{array}\right.$$
$$\geq\|u_1\|=N_2.$$

By bootstrapping argument, we discover:

$$\left.\begin{array}{l}\mu_1\int_0^1\left[\aleph_1(z,y_1)+\frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\right]p_1(y_1)\\ \quad g_1\left(\mu_2\int_0^1\left[\aleph_1(y_1,y_2)+\frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right]p_2(y_2)\cdots\right.\\ \qquad g_{n-1}\left(\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\right.\\ \qquad\qquad\left.\left.p_n(y_n)g_n(u_1(y_n))dy_n\right)\cdots dy_2\right)dy_1\end{array}\right\}\geq N_2,$$

so that $\mathtt{Tu}_1(\mathtt{z}) \geq \mathtt{N}_2 = \|\mathtt{u}_1\|$. Hence $\|\mathtt{Tu}_1\| \geq \|\mathtt{u}_1\|$. So if we set $\mathtt{E}_2 = \left\{\mathtt{x} \in \mathtt{B} : \|\mathtt{x}\| < \mathtt{N}_2\right\}$, then

$$\|\mathtt{Tu}_1\| \geq \|\mathtt{u}_1\|, \text{ for } \mathtt{u}_1 \in \mathtt{P} \cap \partial \mathtt{E}_2. \tag{3.4}$$

By utilizing (3.3), (3.4) and Guo–Krasnosel'skii FPT (see [26,27]), we discover that $\mathtt{T}$ has a fixed point $\mathtt{u}_1 \in \mathtt{P} \cap (\overline{\mathtt{E}}_2 \backslash \mathtt{E}_1)$. Setting $\mathtt{u}_1 = \mathtt{u}_{\mathtt{n}+1}$ yields a positive solution $\left(\mathtt{u}_1, \mathtt{u}_2, \cdots, \mathtt{u}_\mathtt{n}\right)$ of the FBVP (1.1)–(1.2) iteratively indicated by:

$$\begin{aligned}\mathtt{u}_\mathtt{k}(\mathtt{z}) = \mu_\mathtt{k} \int_0^1 \left[\aleph_1(\mathtt{z},\mathtt{y}) + \frac{\vartheta \mathtt{z}}{\Delta} \aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_\mathtt{k}(\mathtt{y}) \mathfrak{g}_\mathtt{k}\left(\mathtt{u}_{\mathtt{k}+1}(\mathtt{y})\right) d\mathtt{y},\\ \mathtt{k} = \mathtt{n}, \mathtt{n}-1, \cdots, 1.\end{aligned}$$

□

3.2. Notations

We introduce:

$$\sigma_3 = \max\left\{\begin{array}{c} \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_1(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{10}\right]^{-1}, \\ \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_2(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{20}\right]^{-1}, \\ \cdots \\ \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_\mathtt{n}(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{\mathtt{n}0}\right]^{-1} \end{array}\right\} \text{ and}$$

$$\sigma_4 = \min\left\{\begin{array}{c} \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_1(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{1\infty}\right]^{-1}, \\ \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_2(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{2\infty}\right]^{-1}, \\ \cdots \\ \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_\mathtt{n}(\mathtt{y}) d\mathtt{y} \mathfrak{g}_{\mathtt{n}\infty}\right]^{-1} \end{array}\right\}.$$

Theorem 3.2. *Suppose* (H_1)-(H_4) *hold, then for each* $\mu_\mathtt{k}, \mathtt{k} = \overline{1,\mathtt{n}}$ *satisfying*

$$\sigma_3 < \mu_\mathtt{k} < \sigma_4, \ \mathtt{k} = \overline{1,\mathtt{n}}, \tag{3.5}$$

there exists an $\mathtt{n}$*-tuple* $\left(\mathtt{u}_1, \mathtt{u}_2, \cdots, \mathtt{u}_\mathtt{n}\right)$ *satisfying the FBVP* (1.1)-(1.2) *s.t.* $\mathtt{u}_\mathtt{k}(\mathtt{z}) > 0$, $\mathtt{k} = \overline{1,\mathtt{n}}$ *on* $(0,1)$.

Proof. Let $\mu_\mathtt{k}$, $\mathtt{k} = \overline{1,\mathtt{n}}$ be provided as in (3.5). Now let $\varepsilon > 0$ be chosen s.t.

$$\max\left\{\begin{array}{c} \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_1(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{10}-\varepsilon)\right]^{-1}, \\ \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_2(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{20}-\varepsilon)\right]^{-1}, \\ \cdots \\ \left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_\mathtt{n}(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{\mathtt{n}0}-\varepsilon)\right]^{-1} \end{array}\right\} \leq \min\left\{\begin{array}{c} \mu_1, \\ \mu_2, \\ \cdot \\ \cdot \\ \cdot \\ \mu_\mathtt{n} \end{array}\right\}$$

and

$$\max\left\{\begin{array}{c} \mu_1, \\ \mu_2, \\ \cdot \\ \cdot \\ \cdot \\ \mu_\mathtt{n} \end{array}\right\} \leq \min\left\{\begin{array}{c} \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_1(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{1\infty}+\varepsilon)\right]^{-1}, \\ \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_2(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{2\infty}+\varepsilon)\right]^{-1}, \\ \cdots \\ \left[\int_0^1 \left[\aleph_1(\mathtt{y},\mathtt{y}) + \frac{\vartheta}{\Delta}\aleph_2(\xi,\mathtt{y})\right] \mathtt{p}_\mathtt{n}(\mathtt{y}) d\mathtt{y} (\mathfrak{g}_{\mathtt{n}\infty}+\varepsilon)\right]^{-1} \end{array}\right\}.$$

Based on the rules of $\mathfrak{g}_{\mathtt{k}0}$, $1 \leq \mathtt{k} \leq \mathtt{n}$ there exists $\overline{\mathtt{N}}_3 > 0$ s.t., for each $1 \leq \mathtt{k} \leq \mathtt{n}$,

$$\mathfrak{g}_\mathtt{k}(\mathtt{x}) \geq \left(\mathfrak{g}_{\mathtt{k}0} - \varepsilon\right)\mathtt{x}, \ 1 < \mathtt{x} \leq \overline{\mathtt{N}}_3.$$

According to the definitions of $\mathfrak{g}_{k0}$, it follows that $\mathfrak{g}_{k0}(1)=0,\ 1\le k\le n$ and so there exist $1<\Theta_n<\Theta_{n-1}<\cdots<\Theta_2<\overline{N}_3$ s.t.

$$\left.\begin{aligned} &\mu_k\mathfrak{g}_k(z)\le\frac{\Theta_{k-1}}{\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_k(y)dy},\ z\in[1,\Theta_k],\\ &\qquad\qquad\qquad\qquad\qquad\qquad 3\le k\le n,\ \text{and}\\ &\mu_2\mathfrak{g}_2(z)\le\frac{\overline{N}_3}{\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_2(y)dy},\ z\in[1,\Theta_2]. \end{aligned}\right\}$$

Let $u_1\in P$ with $\|u_1\|=\Theta_n$. Then:

$$\begin{aligned} &\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)\mathfrak{g}_n\big(u_1(y_n)\big)dy_n\\ &\quad\le\mu_n\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)\mathfrak{g}_n\big(u_1(y_n)\big)dy_n\\ &\quad\le\frac{\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)\Theta_{n-1}dy_n}{\int_0^1\left[\aleph_1(y,y_n)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_n)\right]p_n(y_n)dy_n}\\ &\quad\le\Theta_{n-1}. \end{aligned}$$

Utilizing this bootstrapping technique, it implies that

$$\left.\begin{aligned} &\mu_2\int_0^1\left[\aleph_1(y_1,y_2)+\frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right]p_2(y_2)\\ &\quad\mathfrak{g}_2\Bigg(\mu_3\int_0^1\left[\aleph_1(y_2,y_3)+\frac{\vartheta y_2}{\Delta}\aleph_2(\xi,y_3)\right]p_3(y_3)\cdots\\ &\qquad\mathfrak{g}_{n-1}\Bigg(\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\\ &\qquad\qquad p_n(y_n)\mathfrak{g}_n\big(u_1(y_n)\big)dy_n\Bigg)\cdots dy_2\Bigg)dy_1 \end{aligned}\right\}\le\overline{N}_3.$$

Then

$$\begin{aligned} Tu_1(z)=&\left\{\begin{aligned} &\mu_1\int_0^1\left[\aleph_1(z,y_1)+\frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\right]p_1(y_1)\\ &\quad\mathfrak{g}_1\Bigg(\mu_2\int_0^1\left[\aleph_1(y_1,y_2)+\frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right]p_2(y_2)\cdots\\ &\qquad\mathfrak{g}_{n-1}\Bigg(\mu_n\int_0^1\left[\aleph_1(y_{n-1},y_n)+\frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right]\\ &\qquad\qquad p_n(y_n)\mathfrak{g}_n\big(u_1(y_n)\big)dy_n\Bigg)\cdots dy_2\Bigg)dy_1 \end{aligned}\right.\\ &\ge\sigma^2\mu_1\int_\sigma^{1-\sigma}\left[\aleph_1(y,y_1)+\frac{\vartheta}{\Delta}\aleph_2(\xi,y_1)\right]p_1(y_1)\big(\mathfrak{g}_{10}-\varepsilon\big)\|u_1\|dy_1\\ &\ge\|u_1\|. \end{aligned}$$

So $\|Tu_1\|\ge\|u_1\|$. If we set $E_1=\Big\{x\in B:\|x\|<\Theta_n\Big\}$, then

$$\|Tu_1\|\ge\|u_1\|,\ \text{for}\ u_1\in P\cap\partial E_1. \tag{3.6}$$

It follows that $\mathfrak{g}_k$, $1\le k\le n$ is unbounded at ∞. Since each $\mathfrak{g}_{k\infty}$ is considered to be a positive real number. For each $1\le k\le n$, set

$$\mathfrak{g}_k^*(x)=\sup_{y\in[1,x]}\mathfrak{g}_k(y).$$

Based on the definition of $\mathfrak{g}_{k\infty}$, $1\le k\le n$, there exists $\overline{N}_4$ s.t., for each $1\le k\le n$,

$$\mathfrak{g}_k^*(x)\le\big(\mathfrak{g}_{k\infty}+\varepsilon\big)x,\ x\ge\overline{N}_4.$$

It follows that there exists $N_4=\max\Big\{2\overline{N}_3,\overline{N}_4\Big\}$ s.t., for each $1\le k\le n$,

$$\mathfrak{g}_k^*(x)\le\mathfrak{g}_k^*\big(N_4\big),\ 1<x\le N_4.$$

Choose $u_1 \in P$ with $\|u_1\| = N_4$. Then, by using bootstrapping argument, we have:

$$
\begin{aligned}
Tu_1(z) &= \left\{
\begin{array}{l}
\mu_1 \int_0^1 \left[\aleph_1(z,y_1) + \frac{\vartheta z}{\Delta}\aleph_2(\xi,y_1)\right] p_1(y_1) \\
\quad g_1\Bigg(\mu_2 \int_0^1 \left[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right] p_2(y_2)\cdots \\
\qquad g_{n-1}\Bigg(\mu_n \int_0^1 \left[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right] \\
\qquad\qquad p_n(y_n) g_n(u_1(y_n))dy_n\Bigg)\cdots dy_2\Bigg)dy_1
\end{array}\right. \\
&\le \left\{
\begin{array}{l}
\mu_1 \int_0^1 \left[\aleph_1(y,y_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_1)\right] p_1(y_1) \\
\quad g_1^*\Bigg(\mu_2 \int_0^1 \left[\aleph_1(y_1,y_2) + \frac{\vartheta y_1}{\Delta}\aleph_2(\xi,y_2)\right] p_2(y_2)\cdots \\
\qquad g_{n-1}\Bigg(\mu_n \int_0^1 \left[\aleph_1(y_{n-1},y_n) + \frac{\vartheta y_{n-1}}{\Delta}\aleph_2(\xi,y_n)\right] \\
\qquad\qquad p_n(y_n) g_n(u_1(y_n))dy_n\Bigg)\cdots dy_2\Bigg)dy_1
\end{array}\right. \\
&\le \mu_1 \int_0^1 \left[\aleph_1(y,y_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_1)\right] p_1(y_1) g_1^*(N_4) dy_1 \\
&\le \mu_1 \int_0^1 \left[\aleph_1(y,y_1) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y_1)\right] p_1(y_1) dy_1 (g_{1\infty} + \varepsilon) N_4 \\
&\le N_4 = \|u_1\|.
\end{aligned}
$$

Thus $\|Tu_1\| \le \|u_1\|$. So, if we let $E_2 = \Big\{x \in B : \|x\| < N_4\Big\}$, then

$$
\|Tu_1\| \le \|u_1\|, \text{ for } u_1 \in P \cap \partial E_2. \tag{3.7}
$$

By utilizing (3.6), (3.7) and Guo–Krasnosel'skii FPT (see [26, 27]), we get that T has a fixed point $u_1 \in P \cap (\overline{E}_2 \backslash E_1)$, which in turn with $u_1 = u_{n+1}$ yields an n-tuple $(u_1, u_2, \cdots, u_n)$ satisfying the FBVP (1.1)-(1.2) for the chosen values of μ_k, $k = \overline{1,n}$. □

4. Application

Let $n = 2, p_1(z) = z+1, p_2(z) = z+2,\ \xi = \frac{1}{2}, \nu = 4, \sigma = \frac{1}{4},\ g_1(u) = u\left(1 - \frac{19}{20}e^{-u}\right),\ \ g_2(u) = u - \frac{39}{40}\sin u$. Then $\Delta = 1,\ g_{10} = \frac{1}{20},\ g_{20} = \frac{1}{40},\ g_{1\infty} = g_{2\infty} = 1$.

$$
\begin{aligned}
\sigma_1 &= \max_{1\le i\le 2}\left\{\left[\sigma^2 \int_\sigma^{1-\sigma} \left[\aleph_1(y,y) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right] p_i(y) dy g_{i\infty}\right]^{-1}\right\} \\
&= \max\{22.67613805, 13.41689359\} \\
&= 22.67613805.
\end{aligned}
$$

$$
\begin{aligned}
\sigma_2 &= \min_{1\le i\le 2}\left\{\left[\int_0^1 \left[\aleph_1(y,y) + \frac{\vartheta}{\Delta}\aleph_2(\xi,y)\right] p_i(y) dy g_{i0}\right]^{-1}\right\} \\
&= \min\{28.34517257, 33.54223396\} \\
&= 28.34517257.
\end{aligned}
$$

Theorem 3.1's requirements are all met. Therefore by Theorem 3.1 the following BVP

$$
\left.
\begin{array}{r}
D^{1.5}u_1(z) + \mu_1(z+1)u_2(z)\left(1 - \frac{19}{20}e^{u_2(z)}\right) = 0 \quad z \in (0,1), \\
D^{1.5}u_2(z) + \mu_2(z+2)\left(u_1(z) - \frac{39}{40}\sin u_1(z)\right) = 0, \quad z \in (0,1),
\end{array}
\right\}
$$

$$
u_k(0) = 0, \quad u_k(1) = 4\int_0^{1/2} u_k(z)dz, \quad \text{for } k = 1,2,
$$

has a positive solution if $22.67613805 < \mu_k < 28.34517257$ for $k = 1,2$.

5. Conclusion

In conclusion, this paper effectively fulfills its objective of identifying the eigenvalue intervals of $\mu_{\mathtt{k}},\ 1 \leq \mathtt{k} \leq \mathtt{n}$, for which an iterative system of a class of fractional-order DEqs with parameterized integral BCs possesses at least one positive solution. This is accomplished through the utilization of the standard fixed-point theorem of cone type. The significance of this work lies in its novelty; the authors assert that it represents the inaugural endeavor in the literature to derive such insights for this specific domain.

6. Comparison

In comparison to existing approaches, our study explores the eigenvalue intervals of $\mu_{\mathtt{k}},\ 1 \leq \mathtt{k} \leq \mathtt{n}$ for a class of FDEqs with parameterized integral BCs. By employing standard FPT of cone and combining an incomplete $\aleph$-function with a broad category of polynomials, the researchers devised generalized fractional calculus formulations [31]. Additionally, they utilized the natural transform method along with graph-based approaches to represent solutions for the M-Sturm-Liouville problem [32]. Moreover, the MDLTM was applied to provide analytic solutions for the fractional pseudo hyperbolic telegraph equation [33]. Notably, the existence and uniqueness of the model underlying the Caputo-Fabrizio-fractal-fractional derivative were demonstrated using FPTs [34]. Furthermore, the fundamental properties of a new integral transformation were elucidated, and its application to elementary functions was discussed in [35].

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