Uniqueness Theorems for Sturm-Liouville Operator with Parameter Dependent Boundary Conditions and Finite Number Of Transmission Conditions

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Abstract: In this paper, we prove some uniqueness theorems for the solution of inverse spectral problems of Sturm–Liouville operators with boundary conditions depending linearly on the spectral parameter and with a finite number of transmission conditions.

Keywords: Inverse problem, Sturm-Liouville operator, Weyl function, Prüfer Angle.

Sonlu Sayıda Süreksizlik Koşullarına Sahip ve Sınır Koşulları Parametreye Bağlı Sturm-Liouville Problemi için Teklik Teoremleri

Özet: Bu makalede, sonlu sayıda süreksizlik koşullarına sahip ve sınır koşulları spektral parametreye lineer şekilde bağlı Sturm–Liouville operatörlerin ters spektral problemlerinin çözümü için bazı teklik teoremleri ispatlayacağız.

Anahtar Kelimeler: Ters problem, Sturm-Liouville operatör, Weyl fonksiyonu, Prüfer Açısı.

1. INTRODUCTION

First important results for inverse problem of a regular Sturm-Liouville operator were given by Ambarzumyan in 1929 [1] and Borg in 1945 [2]. In the following years, results which is obtained in these works have been generalized to various versions for Sturm-Liouville operator.

Inverse problems for Sturm–Liouville equations with boundary conditions linearly dependent on the spectral parameter were investigated in [3-14] Such problems often arise from physical problems, for example, vibration of a string, quantum mechanics and geophysics. In [7] and [12], an operator-theoretic formulation of the problems with the spectral parameter contained in only one of the boundary conditions has been given. Boundary conditions depend nonlinearly on the spectral parameter were also considered in [15-19].

Sturm-Liouville problems with transmission conditions at interior points arise in a variety of applications in applied sciences. For general background of these kind of problems, we refer (e.g.) to the monographs [21-27].

1. Preliminaries:

Consider a boundary value problem generated by the Sturm Liouville equation for $x \in I = \bigcup_{i=0}^{N}(d_i, d_{i+1})$

$$\ell y := -y'' + q(x)y = \lambda w(x)y \qquad (1.1)$$

subject to the boundary conditions

$$\lambda(y'(0) + h_0 y(0)) = h_1 y'(0) + h_2 y(0) \qquad (1.2)$$

$$\lambda(y'(1) + H_0 y(1)) = H_1 y'(1) + H_2 y(1) \qquad (1.3)$$

and a finite number of discontinuity conditions

$$\begin{cases} y(d_i + 0) = \alpha_i y(d_i - 0) \\ y'(d_i + 0) = \alpha_i^{-1} y'(d_i - 0) \\ \qquad -(\omega_i \lambda + \beta_i)y(d_i - 0) \end{cases} \qquad (1.4)$$

where $q(x)$ is real valued function in $L_2(0,1)$; β_i, h_j and H_j, $j = 0,1,2$, are real numbers; $\alpha_i \in \mathbb{R}^+$, $d_0 = 0$, $d_i \in (0,1)$, $d_{N+1} = 1$, $\rho_1 := h_2 - h_0 h_1 > 0$, $\rho_2 := H_0 H_1 - H_2 > 0$, $w(x) = 1/\sigma_k^2$, $d_k < x < d_{k+1}$, $\sigma_k \in \mathbb{R}$ for $k = 0, \dots, N$, $\sigma_0 = 1$ and λ is a spectral parameter. We denote the problem (1.1)-(1.4) by $L = L(q, \mathbf{h}, \mathbf{H}, \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_N)$ where $\mathbf{h} = (h_0, h_1, h_2)$, $\mathbf{H} = (H_0, H_1, H_2)$, $\mathbf{s}_i = (d_i, \alpha_i, \omega_i, \beta_i)$, $i = 1, \dots, N$.

Consider a Hilbert Space $H = L_2(0,1) \oplus \mathbb{C}^{N+2}$, equipped with the inner product

$$\langle Y, Z \rangle := \int_0^1 w^2(x) y(x) \overline{z(x)} dx$$
$$+ \sum_{i=1}^{2} \frac{1}{\rho_i} Y_i \overline{Z_i} + \sum_{i=1}^{N} \alpha_i \omega_i Y_{i+2} \overline{Z_{i+2}},$$

where $Y = (y(x)/w(x), Y_1, Y_2, \dots, Y_{N+2})^T$, $Z = (z(x)/w(x), Z_1, Z_2, \dots, Z_{N+2})^T \in H$.

Define an operator T with the domain

$$D(T) = \{Y \in H : y(x) \quad \text{and} \quad y'(x) \quad \text{are}$$
absolutely continuous in I, $\ell Y \in L_2(0,1)$, $y(d_i + 0) = \alpha_i y(d_i - 0)$, $Y_1 = y'(0) + h_0 y(0)$, $Y_2 = y_1'(1) + H_0 y(1)$, $Y_{i+2} = \omega_i y(d_i - 0)$, $i = 1, \dots, N\}$

such that,

$$T(Y)$$
$$:= \begin{pmatrix} -y''(x) + q(x)y(x) \\ h_1 y'(0) + h_2 y(0) \\ H_1 y'(1) + H_2 y(1) \\ -y'(d_1 + 0) + \alpha_1^{-1} y'(d_1 - 0) - \beta_1 y(d_1 - 0) \\ -y'(d_2 + 0) + \alpha_2^{-1} y'(d_2 - 0) - \beta_2 y(d_2 - 0) \\ \dots \\ -y'(d_N + 0) + \alpha_N^{-1} y'(d_N - 0) - \beta_N y(d_N - 0) \end{pmatrix}$$

$$(1.5)$$

It can be proven, using classical methods in the similar works (see for example [20]), that the operator T is symmetric in H; the eigenvalues problem for the operator T and the problem L coincide.

Let $\varphi(x, \lambda)$ be the solution of equation (1.1) satisfying the initial conditions $y(0) = h_1 - \lambda := a$, $y'(0) = \lambda h_0 - h_2 := b$ and transmission conditions (1.4). This solution is found in the interval $0 = d_0 < x < d_1$ by the method of the variation of parameters as

$$\varphi(x, \lambda) = a \cos \sqrt{\lambda} x + \frac{b}{\sqrt{\lambda}} \sin \sqrt{\lambda} x$$
$$+ \frac{1}{\sqrt{\lambda}} \int_0^x \sin \sqrt{\lambda}(x - t)q(t)\varphi(t, \lambda)dt.$$

In each interval $d_i < x < d_{i+1} = 1$ $(i = 1, 2, \dots, n)$ the solutions such as

$$\varphi(x,\lambda) = A_i\left(\sqrt{\lambda}\right)\cos\sqrt{\lambda}x + B_i\left(\sqrt{\lambda}\right)\sin\sqrt{\lambda}x$$

$$+ \frac{1}{\sqrt{\lambda}}\int_0^x \sin\sqrt{\lambda}(x-t)q(t)\varphi(t,\lambda)dt$$

are searched. For each interval if the coefficients $A_i\left(\sqrt{\lambda}\right)$ and $B_i\left(\sqrt{\lambda}\right)$ are found by using transmission conditions (1.4) and substitued in then the following asymptotics hold for the solution $\varphi(x,\lambda)$

$\varphi(x,\lambda)$

$$= \begin{cases} -\lambda\cos\sqrt{\lambda}\gamma(x) \\ +O\left(\sqrt{\lambda}\exp|\tau|\gamma(x)\right), \quad d_0 < x < d_1 \\ -\omega_1\sigma_1\lambda^{3/2}\cos\sqrt{\lambda}\gamma(d_1)\times \\ \times\sin\sqrt{\lambda}(\gamma(d_1)-\gamma(x)) \\ +O\left(\lambda\exp|\tau|\gamma(x)\right), \quad d_1 < x < d_2 \\ -\omega_1\omega_2\sigma_1\sigma_2\lambda^2\cos\sqrt{\lambda}\gamma(d_1)\times \\ \times\sin\sqrt{\lambda}(\gamma(d_1)-\gamma(d_2))\times \\ \times\sin\sqrt{\lambda}(\gamma(d_2)-\gamma(x)) \\ +O\left(\lambda^{3/2}\exp|\tau|\gamma(x)\right), \quad d_2 < x < d_3 \\ \vdots \\ -\left(\prod_{k=1}^n \omega_k\sigma_k\right)\lambda^{(n+2)/2}\cos\sqrt{\lambda}\gamma(d_1)\times \\ \times\sin\sqrt{\lambda}(\gamma(d_1)-\gamma(d_2))\times \\ \times\cdots\sin\sqrt{\lambda}(\gamma(d_{n-1})-\gamma(d_n))\times \\ \times\sin\sqrt{\lambda}(\gamma(d_n)-\gamma(x)) \\ +O\left(\lambda^{(n+1)/2}\exp|\tau|\gamma(x)\right), \quad d_n < x < d_{n+1} \end{cases}$$

$\varphi'(x,\lambda)$

$$= \begin{cases} \lambda^{3/2}\sin\sqrt{\lambda}\gamma(x) \\ +O\left(\lambda\exp|\tau|\gamma(x)\right), \quad d_0 < x < d_1 \\ \omega_1\lambda^2\cos\sqrt{\lambda}\gamma(d_1)\times \\ \times\cos\sqrt{\lambda}(\gamma(d_1)-\gamma(x)) \\ +O\left(\lambda^{3/2}\exp|\tau|\gamma(x)\right), \quad d_1 < x < d_2 \\ \omega_1\omega_2\sigma_1\lambda^{5/2}\cos\sqrt{\lambda}\gamma(d_1)\times \\ \times\sin\sqrt{\lambda}(\gamma(d_1)-\gamma(d_2))\times \\ \times\cos\sqrt{\lambda}(\gamma(d_2)-\gamma(x)) \\ +O\left(\lambda^2\exp|\tau|\gamma(x)\right), \quad d_2 < x < d_3 \\ \vdots \\ \left(\prod_{k=1}^n \omega_k\sigma_{k-1}\right)\lambda^{(n+3)/2}\cos\sqrt{\lambda}\gamma(d_1)\times \\ \times\sin\sqrt{\lambda}(\gamma(d_1)-\gamma(d_2))\times \\ \times\cdots\sin\sqrt{\lambda}(\gamma(d_{n-1})-\gamma(d_n))\times \\ \times\cos\sqrt{\lambda}(\gamma(d_n)-\gamma(x)) \\ +O\left(\lambda^{(n+2)/2}\exp|\tau|\gamma(x)\right), \quad d_n < x < d_{n+1} \end{cases}$$

Similarly, let $\psi(x,\lambda)$ be the solution of equation (1.1) satisfying the initial conditions $y(1) = H_1 - \lambda := a$, $y'(1) = \lambda H_0 - H_2 := b$ and transmission conditions (1.4). This solution is found in the interval $d_n < x < d_{n+1} = 1$ by the method of the variation of parameters as

$$\psi(x,\lambda) = \tilde{a}\cos\sqrt{\lambda}(1-x) - \frac{\tilde{b}}{\sqrt{\lambda}}\sin\sqrt{\lambda}x$$

$$+ \frac{1}{\sqrt{\lambda}}\int_x^1 \sin\sqrt{\lambda}(t-x)q(t)\varphi(t,\lambda)dt.$$

In each interval $d_{n-i} < x < d_{n-i+1}$ $(i = n, n-1, \ldots, 1)$ the solutions such as

$$\psi(x,\lambda) = \tilde{A}_i\left(\sqrt{\lambda}\right)\cos\sqrt{\lambda}x + \tilde{B}_i\left(\sqrt{\lambda}\right)\sin\sqrt{\lambda}x$$

$$+ \frac{1}{\sqrt{\lambda}}\int_0^x \sin\sqrt{\lambda}(x-t)q(t)\varphi(t,\lambda)dt$$

are searched. For each interval if the coefficients $\tilde{A}_i\left(\sqrt{\lambda}\right)$ and $\tilde{B}_i\left(\sqrt{\lambda}\right)$ are found by using transmission conditions (1.4) and written in the

their places then the following asymptotics hold for the solution $\psi(x,\lambda)$

$$\psi(x,\lambda)$$

$$
=\begin{cases}
-\lambda\cos\sqrt{\lambda}\,(\gamma(1)-\gamma(x)) \\
+O\big(\sqrt{\lambda}\exp|\tau|(\gamma(1)-\gamma(x))\big)\,,\quad d_n<x<d_{n+1} \\
-\omega_n\sigma_n\lambda^{3/2}\cos\sqrt{\lambda}\,(\gamma(1)-\gamma(d_n))\times \\
\times\sin\sqrt{\lambda}\,(\gamma(x)-\gamma(d_n)) \\
+O\big(\lambda\exp|\tau|(\gamma(1)-\gamma(x))\big)\,,\quad d_{n-1}<x<d_n \\
-\omega_n\omega_{n-1}\sigma_n\sigma_{n-1}\lambda^2\cos\sqrt{\lambda}\,(\gamma(1)-\gamma(d_n))\times \\
\times\sin\sqrt{\lambda}\,(\gamma(d_{n-1})-\gamma(d_n)) \\
\times\sin\sqrt{\lambda}\,(\gamma(x)-\gamma(d_{n-1})) \\
+O\big(\lambda^{3/2}\exp|\tau|(\gamma(1)-\gamma(x))\big),\ d_{n-2}<x<d_{n-1} \\
\vdots \\
-\left(\prod_{k=1}^{n}\omega_k\sigma_k\right)\lambda^{(n+2)/2}\cos\sqrt{\lambda}\,(\gamma(1)-\gamma(d_n))\times \\
\times\sin\sqrt{\lambda}\,(\gamma(d_{n-1})-\gamma(d_n))\times \\
\times\cdots\sin\sqrt{\lambda}\,(\gamma(d_1)-\gamma(d_2))\sin\sqrt{\lambda}\,(\gamma(x)-\gamma(d_1)) \\
+O\big(\lambda^{(n+1)/2}\exp|\tau|(\gamma(1)-\gamma(x))\big)\,,\ d_0<x<d_1
\end{cases}
$$

$$\psi'(x,\lambda)$$

$$
=\begin{cases}
-\lambda^{3/2}\sigma_n^{-1}\sin\sqrt{\lambda}\,(\gamma(1)-\gamma(x)) \\
+O\big(\lambda\exp|\tau|(\gamma(1)-\gamma(x))\big)\,,\ d_n<x<d_{n+1} \\
-\omega_n\sigma_n\sigma_{n-1}^{-1}\lambda^2\cos\sqrt{\lambda}\,(\gamma(1)-\gamma(d_n))\times \\
\times\cos\sqrt{\lambda}\,(\gamma(x)-\gamma(d_n)) \\
+O\big(\lambda^{3/2}\exp|\tau|(\gamma(1)-\gamma(x))\big)\,,\ d_{n-1}<x<d_n \\
-\omega_n\omega_{n-1}\sigma_n\sigma_{n-1}\sigma_{n-2}^{-1}\lambda^{5/2}\cos\sqrt{\lambda}\,(\gamma(1)-\gamma(d_n))\times \\
\times\sin\sqrt{\lambda}\,(\gamma(d_{n-1})-\gamma(d_n)) \\
\times\cos\sqrt{\lambda}\,(\gamma(x)-\gamma(d_{n-1})) \\
+O\big(\lambda^2\exp|\tau|(\gamma(1)-\gamma(x))\big)\,,\ d_{n-2}<x<d_{n-1} \\
\vdots \\
-\left(\prod_{k=1}^{n}\omega_k\sigma_k\right)\lambda^{(n+3)/2}\cos\sqrt{\lambda}\,(\gamma(1)-\gamma(d_n))\times \\
\times\sin\sqrt{\lambda}\,(\gamma(d_{n-1})-\gamma(d_n))\times \\
\times\cdots\sin\sqrt{\lambda}\,(\gamma(d_1)-\gamma(d_2))\times \\
\times\cos\sqrt{\lambda}\,(\gamma(x)-\gamma(d_1)) \\
+O\big(\lambda^{(n+2)/2}\exp|\tau|(\gamma(1)-\gamma(x))\big)\,,\ d_0<x<d_1
\end{cases}
$$

These solutions are entire functions of λ and satisfy the relation $\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n)$ for each eigenvalue λ_n, where

$$\beta_n=-\frac{\psi'(0,\lambda_n)+h_0\psi(0,\lambda_n)}{\rho_1}.$$

The characteristic function $\Delta(\lambda)$ and norming constants α_n of the problem L are defined as follows.

$$
\begin{aligned}
\Delta(\lambda) &= W[\varphi,\psi]\\
&= \lambda(\varphi'(1,\lambda)+H_0\varphi(1,\lambda))\\
&\quad -H_1\varphi'(1,\lambda)-H_2\varphi(1,\lambda)\\
&= -\lambda(\psi'(0,\lambda)+h_0\psi(0,\lambda))\\
&\quad +h_1\psi'(0,\lambda)+h_2\psi(0,\lambda)
\end{aligned}
\tag{1.6}
$$

$$
\begin{aligned}
\alpha_n &:= \int_0^1 w(x)\varphi^2(x,\lambda_n)\,dx\\
&\quad +\frac{1}{\rho_1}\big(\varphi'(0,\lambda_n)+h_0\varphi(0,\lambda_n)\big)^2\\
&\quad +\frac{1}{\rho_2}\big(\varphi'(1,\lambda_n)+H_0\varphi(1,\lambda_n)\big)^2\\
&\quad +\sum_{i=1}^{N}\alpha_i\omega_i\varphi^2(d_i-0,\lambda_n)
\end{aligned}
\tag{1.7}
$$

It is obvious that, $\Delta(\lambda)$ is an entire function in λ and the zeros namely $\{\lambda_n\}$ of $\Delta(\lambda)$ coincide with eigenvalues of the problem L. Now, from the asymptotics of solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, we can write

$$
\begin{aligned}
\Delta(\lambda) &= \lambda^{(N+5)/2}\left(\prod_{i=1}^{N}\omega_i\sigma_{i-1}\right)\times\\
&\quad \times\cos\sqrt{\lambda}\,\gamma(d_1)\times\\
&\quad \times\sin\sqrt{\lambda}\,(\gamma(d_1)-\gamma(d_2))\times\\
&\quad \times\ldots\sin\sqrt{\lambda}\,(\gamma(d_{N-1})-\gamma(d_N))\times\\
&\quad \times\cos\sqrt{\lambda}\,(\gamma(d_N)-\gamma(1))\\
&\quad +O\big(\lambda^{(N+4)/2}\exp|\tau|\gamma(1)\big)
\end{aligned}
\tag{1.8}
$$

where $\tau = Im\sqrt{\lambda}$, $\gamma(x) = \int\limits_0^x \sqrt{w(t)}dt$.

Lemma 1 *i- All eigenvalues of the problem L are real.*

ii- $\Delta'(\lambda_n) = \beta_n \alpha_n$, *so eigenvalues are simple zeros of* $\Delta(\lambda)$.

iii- Two eigenfunctions $\varphi(x, \lambda_1)$ and $\varphi(x, \lambda_2)$, corresponding to different eigenvalues λ_1 and λ_2, are orthogonal, i.e.,

$$\int\limits_0^1 \varphi(x, \lambda_1)\varphi(x, \lambda_2)dx$$
$$+\frac{1}{\rho_1}\big(\varphi'(0, \lambda_1) + h_0\varphi(0, \lambda_1)\big)\times$$
$$\times\big(\varphi'(0, \lambda_2) + h_0\varphi(0, \lambda_2)\big) \qquad (1.9)$$
$$+\frac{1}{\rho_2}\big(\varphi'(1, \lambda_1) + H_0\varphi(1, \lambda_1)\big)\times$$
$$\times\big(\varphi'(1, \lambda_2) + H_0\varphi(1, \lambda_2)\big)$$
$$+\sum_{i=1}^N \alpha_i \omega_i \varphi(d_i - 0, \lambda_1)\varphi(d_i - 0, \lambda_2) = 0.$$

Proof. Since the operator T is selfadjoint, all eigenvalues are real and two different eigenfunctions are orthogonal. This proves (i) and (iii). Let us show that the simplicity of the eigenvalues λ_n; write the following equations,

$$\psi''(x, \lambda) + q(x)\psi(x, \lambda) = \lambda w(x)\psi(x, \lambda),$$
$$\varphi''(x, \lambda_n) + q(x)\varphi(x, \lambda_n) = \lambda_n w(x)\varphi(x, \lambda_n).$$

If these equations are multiplied by $\varphi(x, \lambda_n)$ and $\psi(x, \lambda)$, respectively and subtracting them side by side and finally integrating over the interval $[0,1]$, the equality

$$\big[\varphi'(x, \lambda_n)\psi(x, \lambda) - \psi'(x, \lambda)\varphi(x, \lambda_n)\big]\Big|_0^{d_1 - 0}$$
$$+ \big[\varphi'(x, \lambda_n)\psi(x, \lambda) - \psi'(x, \lambda)\varphi(x, \lambda_n)\big]\Big|_{d_1 + 0}^{d_2 - 0}$$
$$+ \dots +$$
$$+ \big[\varphi'(x, \lambda_n)\psi(x, \lambda) - \psi'(x, \lambda)\varphi(x, \lambda_n)\big]\Big|_{d_N + 0}^{1}$$
$$= (\lambda - \lambda_n)\int\limits_0^1 w(x)\psi(x, \lambda)\varphi(x, \lambda_n)dx$$

is obtained. Add and subtract $\Delta(\lambda)$ in the left-hand side of the last equality and use initial conditions (1.5) to get

$$\Delta(\lambda) + (\lambda - \lambda_n)\big(\psi'(0, \lambda) + h_0\psi(0, \lambda)\big)$$
$$- (\lambda - \lambda_n)\big(\varphi'(1, \lambda_n) + H_0\varphi(1, \lambda_n)\big)$$
$$- (\lambda - \lambda_n)\sum_{i=1}^N \alpha_i \gamma_i \psi(d_i - 0, \lambda)\varphi(d_i - 0, \lambda_n)$$
$$= (\lambda - \lambda_n)\int\limits_0^1 w(x)\psi(x, \lambda)\varphi(x, \lambda_n)dx,$$

rewrite this equality as

$$\frac{\Delta(\lambda)}{\lambda - \lambda_n} = \int\limits_0^1 w(x)\psi(x, \lambda)\varphi(x, \lambda_n)dx$$
$$+ \big(\varphi'(1, \lambda_n) + H_0\varphi(1, \lambda_n)\big)$$
$$- \big(\psi'(0, \lambda) + h_0\psi(0, \lambda)\big)$$
$$+ \sum_{i=1}^N \alpha_i \gamma_i \psi(d_i - 0, \lambda)\varphi(d_i - 0, \lambda_n)$$

$$= \int\limits_0^1 w(x)\psi(x, \lambda)\varphi(x, \lambda_n)dx$$
$$- \frac{\big(\psi'(0, \lambda) + h_0\psi(0, \lambda)\big)\big(\varphi'(0, \lambda_n) + h_0\varphi(0, \lambda_n)\big)}{h_0 h_1 - h_2}$$
$$+ \frac{\big(\varphi'(1, \lambda_n) + H_0\varphi(1, \lambda_n)\big)\big(\psi'(1, \lambda) + H_0\psi(1, \lambda)\big)}{H_0 H_1 - H_2}$$
$$+ \sum_{i=1}^N \alpha_i \gamma_i \psi(d_i - 0, \lambda)\varphi(d_i - 0, \lambda_n)$$

$$= \int\limits_0^1 w(x)\psi(x, \lambda)\varphi(x, \lambda_n)dx$$
$$+ \frac{\big(\psi'(0, \lambda) + h_0\psi(0, \lambda)\big)\big(\varphi'(0, \lambda_n) + h_0\varphi(0, \lambda_n)\big)}{\rho_1}$$

$$+ \frac{\left(\varphi'(1,\lambda_n) + H_0\varphi(1,\lambda_n)\right)\left(\psi'(1,\lambda) + H_0\psi(1,\lambda)\right)}{\rho_2}$$

$$+ \sum_{i=1}^{N} \alpha_i \gamma_i \psi(d_i - 0, \lambda)\varphi(d_i - 0, \lambda_n)$$

As $\lambda \to \lambda_n$ and from the equalities $\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n)$ and (1.7)

$$\Delta'(\lambda_n) = \beta_n \alpha_n \qquad (1.10)$$

is obtained. Thus $\Delta'(\lambda_n) \neq 0$.

2. Main Results:

In this section, we prove three theorems, uniquely determined by i) Weyl function, ii) Prüfer angle and iii) eigenvalues and norming constants, for uniqueness of the solution of the inverse problem. We consider a boundary value problem $\tilde{L}$, together with L, of the same form but with different coefficients $\tilde{q}(x)$, $\tilde{\mathbf{h}}$, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{s}}_i$, $i = 1, \ldots, N$.

Let the function $\chi(x,\lambda)$ denotes the solution of (1.1) under the initial conditions $\chi(0,\lambda) = \rho_1^{-1}$, $\chi'(0,\lambda) = -\rho_1^{-1}h_0$ and the transmission conditions (1.4). It is clear that the solution $\psi(x,\lambda)$ satisfies the following relation

$$\psi(x,\lambda) = \Delta(\lambda)\chi(x,\lambda)$$
$$- \frac{\psi'(0,\lambda) + h_0\psi(0,\lambda)}{\rho_1}\varphi(x,\lambda). \qquad (2.1)$$

If we denote

$$m(\lambda) := -\frac{\psi'(0,\lambda) + h_0\psi(0,\lambda)}{\rho_1\Delta(\lambda)} \qquad (2.2)$$

then we have

$$\frac{\psi(x,\lambda)}{\Delta(\lambda)} = \chi(x,\lambda) + m(\lambda)\varphi(x,\lambda). \qquad (2.3)$$

The function $m(\lambda)$ is called the Weyl function of the boundary value problem (1.1)-(1.4).

Theorem 1 *If* $m(\lambda) = \tilde{m}(\lambda)$, *then* $L = \tilde{L}$, *i.e.,* $q(x) = \tilde{q}(x)$, *almost everywhere in* I; $\mathbf{h} = \tilde{\mathbf{h}}$, $\mathbf{H} = \tilde{\mathbf{H}}$ *and* $\mathbf{s}_i = \tilde{\mathbf{s}}_i$, $i = 1, \ldots, n$.

Proof. Let us define the functions $P_1(x,\lambda)$ and $P_2(x,\lambda)$ as follows,

$$P_1(x,\lambda) = \varphi(x,\lambda)\tilde{\Phi}'(x,\lambda) - \Phi(x,\lambda)\tilde{\varphi}'(x,\lambda) \qquad (2.4)$$

$$P_2(x,\lambda) = \Phi(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\tilde{\Phi}(x,\lambda) \qquad (2.5)$$

where $\Phi(x,\lambda) = \frac{\psi(x,\lambda)}{\Delta(\lambda)}$. If $m(\lambda) = \tilde{m}(\lambda)$ then from (2.3)-(2.5), $P_1(x,\lambda)$ and $P_2(x,\lambda)$ are entire functions in λ. Denote $G_\delta = \left\{\lambda \in \mathbb{C} : \lambda = k^2, |k - k_n| > \delta, n = 1, 2, \ldots\right\}$ and

$$\tilde{G}_\delta = \left\{\lambda \in \mathbb{C} : \lambda = k^2, |k - \tilde{k}_n| > \delta, n = 1, 2, \ldots\right\},$$

where δ is sufficiently small number, k_n and $\tilde{k}_n$ are square roots of the eigenvalues of the problem L and $\tilde{L}$, respectively. One can easily show that the asymptotics for $d_i < x < d_{i+1}, i = 0, \ldots, n$

$$\begin{cases} \Phi(x,\lambda) = O\left(\lambda^{-\frac{i+3}{2}}\exp\left(-|\tau|\gamma(x)\right)\right) \\ \Phi'(x,\lambda) = O\left(\lambda^{-\frac{i+2}{2}}\exp\left(-|\tau|\gamma(x)\right)\right) \end{cases} \qquad (2.6)$$

are valid for sufficiently large $|\lambda|$. Thus, the following inequalities are obtained from the asymptotics

$$|P_1(x,\lambda)| \leq C_\delta ,$$
$$|P_2(x,\lambda)| \leq C_\delta|\lambda|^{-1/2} , \quad \lambda \in G_\delta \cap \tilde{G}_\delta \qquad (2.7)$$

According to the last inequalities and Liouville's theorem, $P_1(x,\lambda) = A(x)$ and $P_2(x,\lambda) \equiv 0$, for $x \in [0,1] \setminus \{d_1, d_2, \tilde{d}_1, \tilde{d}_2,\ldots,\tilde{d}_{n-1}, \tilde{d}_n\}$. Use (2.4) and (2.5) again to take

$$\varphi(x,\lambda) = A(x)\tilde{\varphi}(x,\lambda),$$
$$\Phi(x,\lambda) = A(x)\tilde{\Phi}(x,\lambda). \tag{2.8}$$

Since $W[\Phi(x,\lambda),\varphi(x,\lambda)] \equiv 1$ and similarly $W[\tilde{\Phi}(x,\lambda),\tilde{\varphi}(x,\lambda)] \equiv 1$, then $A^2(x) = 1$ for x in I.

On the other hand, the asymptotic expressions

$$\varphi(x,\lambda) = \theta(\lambda)\exp\left(-i\sqrt{\lambda}\gamma(x)\right)(1+o(1)) \tag{2.9}$$

$$\tilde{\varphi}(x,\lambda) = \tilde{\theta}(\lambda)\exp\left(-i\sqrt{\lambda}\gamma(x)\right)(1+o(1)) \tag{2.10}$$

is valid for sufficiently large λ on the imaginary axis, where

$$\theta(\lambda) = \begin{cases} \dfrac{\lambda}{2}, & d_0 < x < d_1 \\ -\dfrac{1}{2}\left(\displaystyle\prod_{k=1}^{i} \omega_k \sigma_k\right)\lambda^{(i+2)/2} C_i, & d_i < x < d_{i+1} \end{cases} \tag{2.11}$$

$$\tilde{\theta}(\lambda) = \begin{cases} \dfrac{\lambda}{2}, & \tilde{d}_0 < x < \tilde{d} \\ -\dfrac{1}{2}\left(\displaystyle\prod_{k=1}^{i} \tilde{\omega}_k \sigma_k\right)\lambda^{(i+2)/2} \tilde{C}_i, & \tilde{d}_i < x < \tilde{d}_{i+1} \end{cases} \tag{2.12}$$

$$C_i = \cos\sqrt{\lambda}\gamma(d_1)\prod_{i=1}^{n}\sin\sqrt{\lambda}(\gamma(d_{i-1})-\gamma(d_i)),$$

$$\tilde{C}_i = \cos\sqrt{\lambda}\gamma(\tilde{d}_1)\prod_{i=1}^{n}\sin\sqrt{\lambda}(\gamma(\tilde{d}_{i-1})-\gamma(\tilde{d}_i)),$$

$i = 1,\ldots,n$, $d_0 = 0$, $d_{n+1} = 1$, $\tilde{d}_0 = 0$ and $\tilde{d}_{n+1} = 1$.

From (2.8)-(2.12), we can see that $d_i = \tilde{d}_i$ for $i = \overline{1,n}$. Moreover, if we use $\theta(\lambda) = A(x)\tilde{\theta}(\lambda)$ and $A^2(x) = 1$, we get $\omega_i = \tilde{\omega}_i$ for $i = 1,\ldots,n$ and $A(x) = 1$ from (2.11)-(2.12). Hence,

$$\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda) \quad \text{and} \quad \frac{\psi'(x,\lambda)}{\psi(x,\lambda)} \equiv \frac{\tilde{\psi}'(x,\lambda)}{\tilde{\psi}(x,\lambda)}$$

It can be obtained from (1.1)-(1.4) that $q(x) = \tilde{q}(x)$, a.e. in I; $\mathbf{s}_i = \tilde{\mathbf{s}}_i$, $i = 1,\ldots,n$ and $\mathbf{h} = \tilde{\mathbf{h}}$, $\mathbf{H} = \tilde{\mathbf{H}}$. Consequently $L = \tilde{L}$.

The function called Prüfer angle is defined by

$$P(\lambda)$$
$$:= \begin{cases} \cot^{-1}\dfrac{\psi'(0,\lambda)}{\psi(0,\lambda)} & \text{if } \psi(0,\lambda) \neq 0, \\ \tan^{-1}\dfrac{\psi(0,\lambda)}{\psi'(0,\lambda)} & \text{if } \psi'(0,\lambda) \neq 0 \end{cases} \tag{2.13}$$

Theorem 2 *If $P(\lambda) = \tilde{P}(\lambda)$ and $\mathbf{h} = \tilde{\mathbf{h}}$, $L = \tilde{L}$; i.e. the problem L is uniquely determined by $P(\lambda)$ and $U(y)$.*

Proof. It is obvious from (6), (11) and (22) that the equality

$$m(\lambda)\left[\lambda - \frac{h_1\cot P(\lambda) + h_2}{h_0 + \cot P(\lambda)}\right] = -\rho_1^{-1}$$

holds. Therefore, under the hypothesis of the theorem, we get $m(\lambda) = \tilde{m}(\lambda)$. This completes the proof.

Theorem 3 *The problem L is uniquely determined by $\{\lambda_n, \alpha_n\}_{n \geq 0}$*

Proof. The meromorphic function $m(\lambda)$ has simple poles at λ_n and its residues at these poles are

$$Res\{m(\lambda), \lambda_n\} = \frac{\psi'(0, \lambda_n) + h_0\psi(0, \lambda_n)}{\rho_1\Delta'(\lambda_n)}$$

$$= -\frac{\beta_n}{\Delta'(\lambda_n)} = -\frac{1}{\alpha_n}.$$

Denote $\Gamma_n = \{\lambda : \lambda = k^2, \quad |k| = \sqrt{\lambda_n} + \varepsilon\}$,

where ε is sufficiently small number. Consider the contour integral

$$F_n(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_n}\frac{m(\eta)}{(\eta - \lambda)}d\eta, \quad \lambda \in int\Gamma_n.$$

It can be calculated that $\lim_{n\to\infty}F_n(\lambda) = 0$ and from Residue theorem that

$$m(\lambda) = \sum_{n=0}^{\infty}\frac{1}{\alpha_n(\lambda_n - \lambda)}. \qquad (2.14)$$

Consequently, if $\lambda_n = \tilde{\lambda}_n$ and $\alpha_n = \tilde{\alpha}_n$ for all n, then from (2.14), $m(\lambda) = \tilde{m}(\lambda)$. Hence, Theorem 1 yields $L = \tilde{L}$ when $\{\lambda_n, \alpha_n\}_{n\geq 0} = \{\tilde{\lambda}_n, \tilde{\alpha}_n\}_{n\geq 0}$.

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