Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.38-3 (2017) 396-399

Ambarzumyan Type Theorems for a Class of Sturm-Liouville Problem

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Received: 03.08.2017; Accepted: 14.09.2017 http://dx.doi.org/10.17776/csj.340393

Abstract: In this paper, we prove Ambarzumyan type theorems for an impulsive Sturm–Liouville problem with eigenparameter in the boundary conditions.

Keywords: Ambarzumyan theorem, Sturm-Liouville equation, Inverse problem.

Bir Sınıf Sturm-Liouville Problemi için Ambarzumyan Tipi Teoremler

Özet: Bu makalede, sınır koşulları parametreye bağlı, bir geçiş koşullu Sturm–Liouville problemi için Ambarzumyan tipi teoremler ispatlanmaktadır.

Anahtar Kelimeler: Ambarzumyan teoremi, Sturm-Liouville denklemi, Ters problem.

INTRODUCTION

Inverse spectral problems consist in recovering the coefficients of an operator from their spectral characteristics. The first study which started inverse spectral theory for Sturm-Liouville operator was investigated by Ambarzumyan [1] in 1929. He proved that if $q(x)$ is continuous function on $(0,1)$ and the eigenvalues of the problem

$$\begin{cases} -y'' + q(x)y = \lambda y, & x \in (0,1) \\ y'(0) = y'(1) = 0 \end{cases}$$

are given as $\lambda_n = n^2\pi^2, \ n \geq 0$, then $q(x) \equiv 0$.

We refer to some Ambarzumyan type theorems for the Sturm-Liouville and Dirac operators in [2]-[11].

Particularly, in [2], an extension of Ambarzumyan's theorem is given for Sturm-Liouville problem with general boundary conditions. In [3], the classical Ambarzumyan's theorem is proven for the regular Sturm-Liouville problem with the eigenvalue parameter in the boundary conditions. In [4], some particular generalizations of the classical Ambarzumyan theorem are proven for the regular Sturm-Liouville problem with the discontinuity conditions.

The aim of this paper is to prove two Ambarzumyan type theorems for the impulsive Sturm-Liouville problem with the eigenvalue parameter in one boundary condition.

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1. Preliminaries:

We consider the boundary value problem $L = L\left(q, \frac{a}{b}, \alpha_1, \alpha_2\right)$ generated by the regular Sturm-Liouville equation

$$-y'' + q(x)y = \lambda y, x \in (0,1) \qquad (1.1)$$

subject to the boundary conditions

$$y'(0) = 0 \qquad (1.2)$$

$$a(\lambda)y(1) + b(\lambda)y'(1) = 0 \qquad (1.3)$$

and the discontinuity conditions

$$\begin{cases} y\left(\frac{1}{2}+0\right) = \alpha_1 y\left(\frac{1}{2}-0\right) \\ y'\left(\frac{1}{2}+0\right) = \alpha_2 y'\left(\frac{1}{2}-0\right) \end{cases}, \qquad (1.4)$$

where λ is the spectral parameter; $q(x)$ is a continuous function on $(0,1)$; $\alpha_1, \alpha_2 \in R - \{1\}$ and for $a_k, b_k \in R,\ a_m \neq 0,\ b_m = 1, m \in Z^+$

$$a(\lambda) = \sum_{k=1}^{m} a_k \lambda^k, \quad b(\lambda) = \sum_{k=0}^{m} b_k \lambda^k . \qquad (1.5)$$

Let us denote a solution of (1.1) by $\varphi(x,\lambda)$ satisfying the initial conditions

$$\varphi(0,\lambda) = 1\ ,\ \varphi'(0,\lambda) = 0 \qquad (1.6)$$

and the discontinuity conditions (1.4).

The following asymptotics are given in [12]:

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}x + \omega(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\exp|\tau|x\right), \quad x < \frac{1}{2} \qquad (1.7)$$

$$\begin{aligned}\varphi(x,\lambda) &= \alpha^+ \cos\sqrt{\lambda}x + \alpha^- \cos\sqrt{\lambda}(1-x) + \alpha^+\omega(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \\ &- \alpha^-\left(\omega\left(\frac{1}{2}\right) - \omega(x)\right)\frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\exp|\tau|x\right), \quad x > \frac{1}{2}\end{aligned} \qquad (1.8)$$

and

$$\varphi'(x,\lambda) = -\sqrt{\lambda}\sin\sqrt{\lambda}x + \omega(x)\cos\sqrt{\lambda}x + o(\exp|\tau|x), \quad x < \frac{1}{2} \qquad (1.9)$$

$$\begin{aligned}\varphi'(x,\lambda) &= -\sqrt{\lambda}\alpha^+ \sin\sqrt{\lambda}x + \sqrt{\lambda}\alpha^- \sin\sqrt{\lambda}(1-x) + \alpha^+\omega(x)\cos\sqrt{\lambda}x \\ &- \alpha^-\left(\omega\left(\frac{1}{2}\right) - \omega(x)\right)\cos\sqrt{\lambda}(1-x) + o(\exp|\tau|x), \quad x > \frac{1}{2}\end{aligned} \qquad (1.10)$$

where $\omega(x) = \frac{1}{2}\int_0^x q(t)dt, \quad \alpha^{\mp} = \frac{1}{2}(\alpha_1 \mp \alpha_2),$ $\tau = Im\sqrt{\lambda}$.

The function

$$\Delta(\lambda) := a(\lambda)\varphi(1,\lambda) + b(\lambda)\varphi'(1,\lambda) \qquad (1.11)$$

is entire on λ and the roots of $\Delta(\lambda) = 0$ are coincide with eigenvalues of the problem L.

From (1.8), (1.10) and (1.11), we have

$$\begin{aligned}\Delta(\lambda) = -\alpha^+\lambda^m \Big\{&\sqrt{\lambda}\sin\sqrt{\lambda} - (\omega(1) + a_m)\cos\sqrt{\lambda} \\ &+ \frac{\alpha^-}{\alpha^+}\left(\omega\left(\frac{1}{2}\right) - \omega(1) - a_m\right) + o(\exp|\tau|)\Big\}\end{aligned} \qquad (1.12)$$

Let $\sigma(L)=\{\lambda_n\}_{n\geq 0}$ be the set of the eigenvalues of L. The numbers λ_n satisfy the following asymptotic formula for $n\to\infty$:

$$\lambda_n=(n-m)\pi+\frac{1}{(n-m)\pi}\left\{\omega(1)+a_m+(-1)^{n-m}\frac{\alpha^-}{\alpha^+}\left(\omega\left(\frac{1}{2}\right)-\omega(1)-a_m\right)\right\}+o\left(\frac{1}{n}\right). \quad (1.13)$$

2. Main Results:

We consider the problem $L_0=L\left(0,\frac{a}{b},\alpha_1,\alpha_2\right)$ together with L. It is obvious that eigenvalues of the problem L_0 satisfy the following asymptotic relation for $n\to\infty$

$$\lambda_n^0=(n-m)\pi+\frac{1}{(n-m)\pi}\left\{a_m-(-1)^{n-m}\frac{\alpha^-}{\alpha^+}a_m\right\}+o\left(\frac{1}{n}\right). \quad (2.1)$$

Lemma 1 *If* $\lambda_n=\lambda_n^0$ *for sufficiently large* n, *then* $\int_0^1 q(x)dx=0$.

Proof. If $\lambda_n=\lambda_n^0$ as $n\to\infty$, then

$$(n-m)\pi+\frac{1}{(n-m)\pi}\left\{\omega(1)+a_m+(-1)^{n-m}\frac{\alpha^-}{\alpha^+}\left(\omega\left(\frac{1}{2}\right)-\omega(1)-a_m\right)\right\}+o\left(\frac{1}{n}\right)$$
$$=(n-m)\pi+\frac{1}{(n-m)\pi}\left\{a_m-(-1)^{n-m}\frac{\alpha^-}{\alpha^+}a_m\right\}+o\left(\frac{1}{n}\right)$$

and so

$$\omega(1)+(-1)^{n-m}\frac{\alpha^-}{\alpha^+}\left(\omega\left(\frac{1}{2}\right)-\omega(1)-a_m\right)=o(1)$$

for sufficiently large n. Therefore, we get

$$\begin{cases}\omega(1)+\dfrac{\alpha^-}{\alpha^+}\left(\omega\left(\dfrac{1}{2}\right)-\omega(1)-a_m\right)=0\\ \omega(1)-\dfrac{\alpha^-}{\alpha^+}\left(\omega\left(\dfrac{1}{2}\right)-\omega(1)-a_m\right)=0\end{cases}.$$

Thus $\omega(1)=0$ i.e. $\int_0^1 q(x)dx=0$.

Theorem 1 *If*

$$\left\{(n-m)\pi+\frac{a_m}{(n-m)\pi}+o\left(\frac{1}{n}\right):n>n_0\right\}\cup\{0\}\subset\sigma(L)$$

for some $n_0\in\mathbb{N}$, *then* $q(x)\equiv 0$ *a.e. on* $(0,1)$.

Proof. From Lemma 1, it is obtained that $\int_0^1 q(x)dx\equiv 0$. On the other hand, since $0\in\sigma(L)$, we get $q(x)\equiv 0$ *a.e. on* $(0,1)$ from the classical Ambarzumyan theorem.

Theorem 2 *If* λ_s *is an eigenvalue of the problem* L *such that* $b(\lambda_s)\neq 0$ *and* $\int_0^1 q(x)dx-\lambda_s+\frac{a(\lambda_s)}{b(\lambda_s)}=0$, *then* $q(x)\equiv\lambda_s$, *a.e. on* $(0,1)$ *and* $a(\lambda_s)=0$.

Proof. Let $y_s(x)$ be the eigenfunction corresponding to λ_s. Then we can write for $x\in(0,1)$

$$\begin{cases}-y_s''(x)+q(x)y_s(x)=\lambda_s y_s(x)\\ y_s'(0)=0\\ a(\lambda_s)y_s(1)+b(\lambda_s)y_s'(1)=0.\end{cases} \quad (2.2)$$

It is clear that $y_s(0)\neq 0$ and $y_s(1)\neq 0$. Otherwise, since $b(\lambda_s)\neq 0$, $y_s'(0)=0$ or

$y_s'(1)=0$. In both cases, $y_s(x)\equiv 0$ by the uniqueness of the solution of an initial value problem.

The function $y_s(x)$ has finitely many isolated nodes on $(0,1)$ and $y_s(x_i)=0$ yields $y_s''(x_i)=0$ but $y_s'(x_i)\neq 0$. Then the function $\frac{y_s''(x)}{y_s(x)}$ is bounded in the neighborhood of each x_i.

From (2.2) and the relation $\frac{y_s''(x)}{y_s(x)}=\left(\frac{y_s'(x)}{y_s(x)}\right)'+\left(\frac{y_s'(x)}{y_s(x)}\right)^2$, we get $\left(\frac{y_s'(x)}{y_s(x)}\right)'=q(x)-\lambda_s-\left(\frac{y_s'(x)}{y_s(x)}\right)^2$.

By integrating of both sides from 0 to 1, the following equality is obtained

$$\int_0^1\left(\frac{y_s'(x)}{y_s(x)}\right)^2 dx=\frac{a(\lambda_s)}{b(\lambda_s)}+\int_0^1 q(x)dx-\lambda_s=0.$$

Thus $y_s'(x)\equiv 0$ and so $y_s(x)\equiv constant$. Hence, it is concluded from (2.2) that $q(x)=\lambda_s$ a.e. on $(0,1)$ and $a(\lambda_s)=0$.

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