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Modified Ridge Estimator for Poisson Regression

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ABSTRACT

Poisson regression is a statistical model used to model the relationship between a count-valued-dependent variable and one or more independent variables. A frequently encountered problem when modeling such relationships is multicollinearity, which occurs when the independent variables are highly correlated with each other. Multicollinearity can affect the maximum likelihood (ML) estimates of unknown model parameters, making them unstable and inaccurate. In this study, we propose a modified ridge parameter estimator to combat multicollinearity in Poisson regression. We conducted extensive simulations to evaluate the performance of our proposed estimator using the mean squared error (MSE). We also apply our estimator to real data. The results show that our proposed estimator outperforms the ML estimator in both simulations and real data applications.

Keywords: Poisson regression, Multicollinearity, Ridge estimator, Monte Carlo simulations, Maximum likelihood estimation.

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Introduction

Poisson regression is a statistical technique used to model the relationship between a count-valued response variable and one or more independent variables [1]. It is commonly used in a variety of fields, such as epidemiology, marketing, and finance, to model the incidence of diseases, the number of customers who visit a store, and the number of financial transactions that occur in a given period of time.

One of the key assumptions of Poisson regression is that the independent variables are not correlated. However, in practice, it is common for independent variables to be correlated, which is known as multicollinearity [2]. Multicollinearity can lead to several problems, including unstable coefficient estimates, inflated standard errors, and decreased statistical power [3].

Ridge regression is a regularization technique that can be used to reduce the effects of multicollinearity in Poisson regression models [4]. Ridge regression works by adding a small penalty term to the ML function, which shrinks the coefficient estimates towards zero [5]. This shrinkage can help improve the stability of the coefficient estimates and reduce the impact of multicollinearity in the model.

A number of different ridge estimators have been proposed for Poisson regression models [6-14] However, the performance of these estimators can vary depending on the specific characteristics of the data.

The purpose of this study is to evaluate the performance of existing ridge estimators and propose new ridge estimators that are more effective in combating multicollinearity in Poisson regression models. Using Monte Carlo simulations, we compare the performance of the different estimators in terms of their MSEs.

Materials and Methods

Poisson Regression

A random variable Y is said to follow a Poisson probability distribution if the vector of $Y = [y_1, y_2, y_3, \dots, y_n]^T$ are count numbers, and the $y_i's$ are independent and identically distributed probability mass function with:

$$f(Y = y) = \frac{e^{-\mu}\mu^{y}}{y!}, \qquad \mu > 0 \tag{1}$$

where μ is the mean occurrence of the event y for a specified interval of time and e is the base of the natural logarithm.

The mean and variance of the Poisson distribution are equal:

$$E(Y) = E(Var) = \mu \tag{2}$$

Poisson regression is a statistical model used to model count data. Instead of modeling the linear relationship between the dependent variable and the independent variable(s), Poisson regression models the linear relationship between the log of the expected value of the dependent variable and the independent variable(s) [15] . This ensures that the expected value of the dependent variable is always positive.

The Poisson model can be written as follows:

$$\ln(\mu_i) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} = x_i\beta \tag{3}$$

where x_i is the i^{th} row of the design matrix X.

The ML method is used to estimate the Poisson regression model coefficients. The likelihood function is given by:

$$L(\beta) = \prod_{i=1}^{n} f(Y = y) \tag{4}$$

Maximizing the likelihood is the same as maximizing the log-likelihood.

$$l(\beta) = \log[\sum_{i=1}^{n} f(Y = y)] \tag{5}$$

From Equation (1), the likelihood is:

$$L(\beta) = \prod_{i=1}^{n} \frac{e^{-\mu_i}\mu_i^{y_i}}{y_i!} \tag{6}$$

The log likelihood is

$$l(\beta) = \sum_{i=1}^{n}[-\mu_i + y_i \log(\mu_i) - \log(y_i)] \tag{7}$$

substituting $\mu_i = \exp(x_i\beta)$ from equation (3) with equation (7):

$$l(\beta) = \sum_{i=1}^{n}[-\exp(x_i\beta) + y_i(x_i\beta) - \log(y_i)] \tag{8}$$

The ML estimates for the coefficients are obtained by taking the first derivative of the log-likelihood function and equating it to zero. This results in the following equations:

$$\frac{\partial l(\beta)}{\partial(\beta)} = \sum_{i=1}^{n}[y_i - \exp(x_i\beta)]\, x_i = 0 \tag{9}$$

These equations are called score equations and can be solved numerically using iterative methods such as iteratively reweighted least squares (IRLS).

The IRLS algorithm produces the following Poisson Maximum Likelihood Estimator (PMLE) for β:

$$\hat{\beta}_{PMLE} = (X^tWX)^{-1}X^tWz \tag{10}$$

where $z = \log(\mu_i) + \frac{y_i - \mu_i}{\mu_i}$ and W is a diagonal matrix with weights $w_{ii} = \exp(x_i\beta)$.

The PMLE is asymptotically normally distributed with a covariance matrix equal to the negative inverse of the Hessian matrix:

$$I_Y(\beta) = -E\left[\sum_{i=1}^{n} -\exp(x_i\beta)x_i x_i^t\right] = (X^tWX)^{-1} \tag{11}$$

The MSE of the PMLE is given by:

$$MSE(\beta_{MLE}) = E\left[\left(\hat{\beta}_{PMLE} - \beta\right)^t\left(\hat{\beta}_{PMLE} - \beta\right)\right] = trace(X^tWX)^{-1} = \sum_{i=1}^{p+1}\frac{1}{\lambda_i} \tag{12}$$

where λ_i is the i^{th} eigenvalue of the matrix $X^t\,WX$.

Poisson Ridge Regression

When the independent variables in a Poisson regression model are highly correlated, the $X^t\,WX$ matrix becomes ill-conditioned, meaning that its determinant is close to zero. This can lead to an inflated variance and instability of the ML estimator.

To address this issue, [4] introduced the concept of ridge regression. Ridge regression involves adding a small positive term, known as the ridge parameter, to the diagonal of the $X^t\,X$ matrix in linear regression.

[16] proposed the Poisson Ridge Regression Estimator (PRRE) to address the problem of multicollinearity in the Poisson regression model. The PRRE is defined as follows:

$$\hat{\beta}_{PRRE} = (X^tWX + \mathrm{k}I)^{-1}X^tWX\,\hat{\beta}_{PMLE} \tag{13}$$

where $k > 0$ and I represents the $p \times p$ identity matrix. When the parameter k is set to 0, the PRRE simplifies the ML estimator. The PRRE helps alleviate the multicollinearity issue and improves the stability of the estimator. The MSE of the PRRE can be calculated as follows:

$$MSE(\beta_{PRRE}) = E\left[\left(\hat{\beta}_{PRRE} - \beta\right)^t\left(\hat{\beta}_{PRRE} - \beta\right)\right] = \sum_{j=1}^{p+1}\frac{\lambda_j}{\left(\lambda_j + k\right)^2} + k^2\sum_{j=1}^{p+1}\frac{\alpha_j^2}{\left(\lambda_j + k\right)^2} \tag{14}$$

where α_j represents $U\hat{\beta}_{PMLE}$, where U is a matrix, whose columns are the eigenvectors of the X^tWX matrix. The values λ_j correspond to the eigenvalues associated with the eigenvectors of the X^tWX matrix.

For $k > 0$, the MSE of the PRRE is consistently lower than that of the Poisson regression ML estimator. This indicates that the PRRE provides improved accuracy and precision compared with the ML estimator in the Poisson regression model.

Some Existing Ridge Estimators

In this section, we have conducted a systematic review of 30 existing ridge estimators. Inspired by the works of [9,16,17]. We have proposed modified versions of some of these estimators. Table 1 summarizes the existing ridge estimators we examined.

Table 1. Summary of Existing Ridge Estimators

No.	Estimator	Reference	No.	Estimator	Reference
1	$k=0$	MLE [14]	17	$k_{GK}=k_{HK}+\frac{2}{(\lambda_{max}+\lambda_{min})'}$	[18]
2	$k_{HK1}=\frac{1}{\hat{a}_{max}^2}$	[4]	18	$k_{D1}=\frac{2p}{\sum_{i=1}^{p}\lambda_{max}\,\hat{a}_i^2}$	[19]
3	$k_{HK2}=\frac{1}{\sum_{i=1}^{p}\hat{a}_{max}^2}$	[4]	19	$k_{D2}=Median\left(\frac{2}{\lambda_{max}\,\hat{a}_i^2}\right)$	[19]
4	$k_{HKB}=\frac{p}{\sum_{i=1}^{p}\hat{a}_{max}^2}$	[4]	20	$k_{D3}=\frac{2}{\lambda_{max}\left(\prod_{i=1}^{p}\hat{a}_i^2\right)^{\frac{1}{p}}}$	[19]
5	$k_{LW1}=\frac{1}{\lambda_i\,\hat{a}_i^2}$	[20]	21	$k_{D4}=\frac{2}{\lambda_{max}p}\sum_{i=1}^{p}\frac{1}{\hat{a}_i^2}$	[19]
6	$k_{LW2}=\frac{p}{\sum_{i=1}^{p}\lambda_i\,\hat{a}_i^2}$	[20]	22	$k_{Y4}=Max\left(\sqrt{\frac{1}{\lambda_i\,\hat{a}_i^2}}\right)$	[17]
7	$k_{HSL}=\frac{\sum_{i=1}^{p}(\lambda_i\,\hat{a}_i)^2}{\left(\sum_{i=1}^{p}\lambda_i\,\hat{a}_i^2\right)^2}$	[21]	23	$k_{Y6}=Max\left(\sqrt{\lambda_i\,\hat{a}_i^2}\right)$	[17]
8	$k_{AM}=\frac{1}{p}\sum_{i=1}^{p}\frac{1}{\hat{a}_i^2}$	[6]	24	$k_{Y9}=\frac{p}{\sum_{i=1}^{p}\sqrt{\frac{1}{\lambda_i\,\hat{a}_i^2}}}$	[17]
9	$k_{GM}=\frac{1}{\left(\prod_{i=1}^{p}\hat{a}_i^2\right)^{\frac{1}{p}}}$	[6]	25	$k_{AS1}=\frac{1}{\hat{a}_{max}}+\frac{1}{\lambda_{max}}$	[22]
10	$k_{KS}=\frac{\lambda_{max}}{(n-p)+\lambda_{max}\,\hat{a}_{max}^2}$	[7]	26	$k_{AS2}=Max\left(\frac{1}{\hat{a}_i}+\frac{1}{\lambda_i}\right)$	[22]
11	$k_{A1}=\frac{1}{p}\sum_{i=1}^{p}\left(\frac{\lambda_{max}}{(n-p)+\lambda_{max}\,\hat{a}_{max}^2}\right)$	[8]	27	$k_{AS3}=\frac{1}{Min\left(\frac{1}{\hat{a}_i}+\frac{1}{\lambda_i}\right)}$	[22]
12	$k_{A2}=Max\left(\frac{\lambda_i}{(n-p)+\lambda_i\,\hat{a}_i^2}\right)$	[8]	28	$k_{AY1}=\frac{\sqrt{5}p}{\lambda_{max}\sum_{i=1}^{p}\hat{a}_i^2}$	[23]
13	$k_{A3}=Median\left(\frac{\lambda_i}{(n-p)+\lambda_i\,\hat{a}_i^2}\right)$	[8]	29	$k_{AY2}=\frac{p}{\sqrt{\lambda_{max}}\sum_{i=1}^{p}\hat{a}_i^2}$	[23]
14	$k_{MK4}=\left(\prod_{i=1}^{p}\sqrt{\hat{a}_i^2}\right)^{\frac{1}{p}}$	[9]	30	$k_{AY3}=\frac{2p}{\sum_{i=1}^{p}\left(\lambda_i^{\frac{1}{4}}\right)\sum_{i=1}^{p}\hat{a}_i^2}$	[23]
15	$k_{MK5}=\left(\prod_{i=1}^{p}\frac{1}{\sqrt{\hat{a}_i^2}}\right)^{\frac{1}{p}}$	[9]	31	$k_{AY4}=\frac{2p}{\sqrt{\sum_{i=1}^{p}\lambda_i\sum_{i=1}^{p}\hat{a}_i^2}}$	[23]
16	$k_{MK6}=\text{Median}\left(\sqrt{\hat{a}_i^2}\right)$	[9]			

Proposed Ridge Estimators

By incorporating insights from these previous works, we have devised novel modifications to the existing estimators, aiming to enhance their performance. Based on work of Asar and Genç (2017) we applied square root transformation and the absolute value of $\hat{\alpha}$ and proposed the following ridge estimators:

$k_{SK1} = \text{Max}\left(\frac{1}{\sqrt{abs(\hat{\alpha}_i)}}\right)$	$k_{SK2} = \text{Median}\left(\frac{1}{\sqrt{abs(\hat{\alpha}_i)}}\right)$
$k_{SK3} = \frac{1}{p}\sum_{i=1}^{p}\frac{1}{\sqrt{abs(\hat{\alpha}_i)}}$	$k_{SK4} = \left(\prod_{i=1}^{p}\frac{1}{\sqrt{abs(\hat{\alpha}_i)}}\right)^{\frac{1}{p}}$
$k_{SK5} = \frac{p}{\sum_{i=1}^{p}\left(\frac{1}{\sqrt{abs(\hat{\alpha}_i)}}\right)}$	$k_{SK7} = \sqrt{\sum_{i=1}^{p}\hat{\alpha}_i^2\sum_{i=1}^{p}\lambda_i}$

Simulation Design

To investigate the performance of Poisson regression's MSE under multicollinearity, we generated the dependent variable, Y_i, from a Poisson distribution with parameter μ_i. Here, μ_i is equal to $\exp(x_i\beta)$, where x_i represents the i^{th} row of the design matrix X, $\beta = (\beta_0, \beta_1, \ldots, \beta_p)'$ is the vector of coefficients, and i ranges from 1 to n.

To control the correlation between the independent variables, we generated them using the following equation:

$$x_{ij} = (1 - \rho^2)^{\frac{1}{2}} z_{ij} + \rho\, z_{i,p} \tag{15}$$

where i ranges from 1 to n, and j ranges from 1 to p. In this equation, z_{ij} represents independent pseudo-normal random variables, and ρ^2 represents the degree of correlation between any two random variables. In order to observe how the severity of correlation affects the performance of Poisson MSE, we considered four levels of correlation: $\rho^2 = 0.90, 0.95, 0.99$, and 0.999.

Apart from considering the degree of correlation, we also examined the effects of the Poisson intercept β_0, the number of independent variables (p) and the number of observations (n) on the performance of the Poisson MSE. Specifically, we chose different values for the intercept β_0, including $-1, 0$, and 1. As the intercept decreases, the average value of μ_i decreases as well. Consequently, this decrease in average value results in a higher frequency of zero values for the dependent variable. This can lead to convergence problems in the iteratively reweighted least squares (IRLS) algorithm. The slope coefficients $(\beta_0, \beta_1, \cdots, \beta_p)'$ were selected in a way that ensures the sum of their squares $\sum_{j=0}^{p}\beta_j^2 = 1$.

To analyze the performance of the Poisson MSE, we generated two models, one with 5 independent variables and another with 8 independent variables. Additionally, we varied the sample size, with n set to 50,100, and 200 for each model.

For conducting these simulations, we utilized MATLAB as the computational tool. The MSE was calculated using the formula:

$$MSE(\hat{\beta}) = \frac{1}{5000}\sum_{j=1}^{5000}(\hat{\beta} - \beta)'(\hat{\beta} - \beta) \tag{16}$$

Results

The results of the simulation study are displayed in Tables 2-7. Four factors that affect the performance of the Poisson MSE estimator—namely, sample size, degree of correlation, the intercept of the Poisson model, and the number of independent variables—were considered and examined through Monte Carlo simulation. The findings are summarized as follows:

The effect of the intercept on the MSE is positive. In other words, as the intercept increases from -1 to 1, the MSE of all estimators generally decreases.

When considering the effect of the degree of correlation, the results show that as the degree of correlation increases, the MSE also increases. However, this pattern does not hold true for all estimators. For example, in cases where the sample size is small ($n = 50$) or the number of independent variables is large, the MSE of the estimators k_{AM}, k_{GM}, k_{A4}, k_{Y4}, k_{A6}, k_{AS1}, k_{SK1}, and k_{SK6} decrease as the degree of correlation increases, especially if the intercept value is kept at 0 or 1. The MSE of PMLE is the worst performing estimator, while k_{SK6} is the best performing estimator when the intercept is 1 regardless of other factors, k_{SK1} is the best performing estimator when the intercept is -1, and k_{Y6} is the best performing estimator when the intercept is 0.

The effect of sample size on the MSE is desirable. As the sample size increases, the MSE also decreases. For example, when the degree of correlation is 0.999 and all other factors are kept constant, the MSE decreases significantly, especially for PMLE. On the other hand, as the number of independent variables increases, the MSE also increases. The PMLE estimator has the worst performance in this case.

When considering the estimator with the highest performance across all instances, it is evident that estimator k_{SK6} outperforms all other estimators. Additionally, when analyzing the estimators that secure the top three positions in terms of MSE performance, estimators k_{SK1} and k_{Y6} had consistently ranked second as shown in Figure. 1.

Real Data Application

In this section we apply a real dataset to examine the effectiveness of our proposed estimators. We used the aircraft damage dataset introduced by [15]. This dataset has been used in previous studies by [11], [13] and others. The dataset contains information about 30 strike missions flown by two types of aircraft: the McDonnell Douglas A-4 Skyhawk and the Grumman A-6 Intruder. The explanatory variables in the dataset are defined as follows: x_1 serves as a binary variable indicating the aircraft type, where A-4 is coded as 0 and A-6 is coded as 1. Additionally, x_2 and x_3 represent the bomb load in tons and the total months of aircrew experience, respectively. The response variable (y) is the number of locations where damage was inflicted on the aircraft. [11] claims that the data suffers from multicollinearity. The eigenvalues of the design matrix of X^tWX is $[4.289 \quad 789.849 \quad 283543.296]$. This is supported by the high condition number of the design matrix, which is 257.125. The condition number, CN, is a measure of multicollinearity, and is calculated as follows:

$$CN = \sqrt{\frac{\max(eigenvalue)}{\min(eigenvalue)}} \quad (17)$$

Where $\max(eigenvalue)$ is the largest eigenvalue of the design matrix X^tWX, and $\min(eigenvalue)$ is the smallest eigenvalue of the design matrix X^tWX. A condition number greater than 30 is generally considered to indicate the presence of multicollinearity [11], [13].

The performance of the estimators was assessed using MSE, which was computed using Equation 14. The estimated coefficients are presented in Table 8. The results of the application showed that our proposed ridge estimator k_{SK1} has the smallest MSE value, which shows its superiority in real applications. The worst performing estimator is the Poisson maximum likelihood estimator (PMLE).

Table 2: Estimated MSE when $n = 50$ and $p = 4$

β_0	-1				0				1			
	ρ				ρ				ρ			
	0.900	0.950	0.990	0.999	0.900	0.950	0.990	0.999	0.900	0.950	0.990	0.999
Estimator	MSE				MSE				MSE			
$PMLE$	1.061	2.125	11.978	125.769	0.380	0.790	4.219	46.588	0.134	0.291	1.500	16.104
k_{HK1}	0.577	0.935	3.964	38.241	0.152	0.233	0.749	8.017	0.128	0.134	0.201	1.091
k_{HK2}	0.825	1.504	7.280	71.456	0.310	0.585	2.611	27.180	0.126	0.259	1.087	9.501
k_{HKB}	0.450	0.692	2.740	25.452	0.179	0.286	1.002	10.108	0.103	0.183	0.511	3.520
k_{LW1}	1.055	2.103	11.309	87.446	0.380	0.789	4.203	44.720	0.134	0.291	1.500	16.057
k_{LW2}	1.004	1.943	10.306	103.921	0.371	0.757	3.818	39.231	0.133	0.286	1.421	13.988
k_{HSL}	1.061	2.125	11.970	125.132	0.380	0.790	4.219	46.585	0.134	0.291	1.500	16.104
k_{AM}	0.608	0.608	0.599	0.386	0.112	0.111	0.119	0.084	0.223	0.227	0.226	0.420
k_{GM}	0.663	0.893	0.698	0.159	0.360	0.665	1.475	0.484	0.131	0.273	0.855	0.518
k_{KS}	0.552	0.924	3.906	36.806	0.218	0.390	1.426	14.131	0.108	0.209	0.719	4.936
k_{A1}	0.851	1.490	6.398	28.575	0.237	0.450	1.741	6.474	0.133	0.228	0.928	7.937
k_{A2}	0.757	1.182	4.194	10.671	0.180	0.308	0.950	1.892	0.147	0.209	0.669	4.911
k_{A3}	1.002	1.930	8.932	58.914	0.340	0.682	3.249	24.876	0.131	0.282	1.376	11.537
k_{MK4}	0.570	0.734	0.598	0.120	0.333	0.568	1.020	0.338	0.127	0.255	0.692	0.468
k_{MK5}	0.369	0.459	0.662	0.641	0.162	0.231	0.388	0.470	0.096	0.160	0.359	0.659
k_{MK6}	0.531	0.672	0.571	0.138	0.328	0.550	0.905	0.283	0.125	0.244	0.617	0.441
k_{GK}	0.576	0.929	3.792	28.126	0.152	0.233	0.745	7.573	0.128	0.134	0.201	1.086
k_{D1}	1.057	2.113	11.862	124.186	0.380	0.788	4.205	46.389	0.134	0.290	1.499	16.080
k_{D2}	1.052	2.096	11.434	106.557	0.379	0.784	4.148	44.246	0.133	0.290	1.498	15.979
k_{D3}	1.049	2.089	11.477	112.982	0.378	0.784	4.148	44.611	0.133	0.290	1.496	15.987
k_{D4}	0.993	1.917	10.053	85.073	0.348	0.720	3.629	32.734	0.132	0.279	1.422	15.056
k_{Y4}	0.430	0.458	0.410	0.275	0.167	0.220	0.243	0.114	0.102	0.159	0.263	0.192
k_{Y6}	0.678	0.661	0.634	0.615	0.040	0.036	0.031	0.030	0.164	0.160	0.149	0.143
k_{Y9}	0.438	0.639	1.055	1.015	0.282	0.453	0.983	1.479	0.113	0.215	0.583	1.107
k_{AS1}	0.676	1.210	6.149	54.433	0.274	0.491	2.212	23.787	0.122	0.240	0.879	8.212
k_{AS2}	0.827	0.789	0.587	1.067	0.166	0.140	0.095	0.057	0.361	0.353	0.245	0.093
k_{AS3}	0.417	0.507	0.503	0.178	0.271	0.414	0.664	0.371	0.110	0.195	0.384	0.293
k_{AY1}	1.056	2.112	11.849	124.009	0.380	0.788	4.203	46.366	0.134	0.290	1.499	16.078
k_{AY2}	1.015	1.993	10.797	110.642	0.372	0.763	3.967	43.154	0.133	0.288	1.466	15.396
k_{AY3}	0.859	1.563	7.345	68.591	0.332	0.637	2.869	28.925	0.130	0.273	1.222	10.988
k_{AY4}	0.975	1.885	9.921	100.193	0.365	0.739	3.761	40.522	0.133	0.286	1.434	14.799
k_{SK1}	0.328	0.369	0.374	0.159	0.148	0.198	0.241	0.105	0.089	0.138	0.265	0.296
k_{SK2}	0.428	0.534	0.613	0.354	0.221	0.326	0.540	0.438	0.110	0.201	0.464	0.571
k_{SK3}	0.391	0.479	0.560	0.287	0.201	0.288	0.428	0.283	0.104	0.179	0.395	0.512
k_{SK4}	0.409	0.509	0.610	0.341	0.217	0.315	0.491	0.380	0.108	0.191	0.430	0.568
k_{SK5}	0.425	0.538	0.668	0.415	0.229	0.339	0.554	0.498	0.111	0.199	0.460	0.631
k_{SK6}	0.429	0.446	0.595	0.971	0.058	0.050	0.027	0.033	0.062	0.060	0.050	0.088

Table 3: Estimated MSE when $n = 50$ and $p = 8$

β_0	-1				0				1			
	ρ				ρ				ρ			
	0.9	0.95	0.99	0.999	0.9	0.95	0.99	0.999	0.9	0.95	0.99	0.999
Estimator	MSE				MSE				MSE			
$PMLE$	2.274	4.463	25.599	256.677	0.776	1.547	8.507	91.043	0.276	0.571	2.967	32.504
k_{HK1}	0.994	1.653	8.725	80.008	0.240	0.388	1.563	15.921	0.127	0.160	0.315	2.569
k_{HK2}	1.728	3.152	16.606	160.467	0.613	1.138	5.612	57.365	0.253	0.491	2.145	20.659
k_{HKB}	0.624	0.976	4.488	41.347	0.243	0.386	1.564	15.230	0.161	0.252	0.714	5.571
k_{LW1}	2.271	4.444	25.029	225.854	0.776	1.547	8.494	89.959	0.276	0.571	2.966	32.465
k_{LW2}	2.049	3.814	20.384	198.289	0.731	1.409	7.105	70.567	0.270	0.549	2.637	25.968
k_{HSL}	2.274	4.463	25.597	252.078	0.776	1.547	8.507	91.043	0.276	0.571	2.967	32.504
k_{AM}	0.762	0.689	0.616	0.842	0.093	0.072	0.066	0.094	0.274	0.265	0.226	0.288
k_{GM}	1.519	2.091	2.314	0.362	0.747	1.391	3.600	1.722	0.271	0.542	1.959	2.188
k_{KS}	1.032	1.758	9.103	84.399	0.420	0.741	3.052	30.881	0.210	0.382	1.237	11.403
k_{A1}	0.994	1.602	5.249	31.757	0.180	0.291	1.188	5.374	0.196	0.297	0.794	5.701
k_{A2}	0.725	0.929	2.118	10.395	0.101	0.125	0.383	1.049	0.203	0.238	0.380	2.102
k_{A3}	1.831	3.332	13.630	87.751	0.505	0.938	4.478	35.021	0.254	0.492	2.117	16.700
k_{MK4}	1.165	1.489	1.642	0.423	0.674	1.134	2.216	1.222	0.258	0.491	1.425	1.568
k_{MK5}	0.454	0.569	0.948	1.613	0.185	0.251	0.517	1.021	0.152	0.223	0.433	0.992
k_{MK6}	1.066	1.331	1.440	0.447	0.669	1.115	2.057	1.021	0.256	0.482	1.323	1.351
k_{GK}	0.993	1.649	8.543	68.828	0.240	0.388	1.559	15.689	0.127	0.160	0.315	2.564
k_{D1}	2.271	4.454	25.529	255.479	0.775	1.546	8.499	90.950	0.276	0.571	2.966	32.491
k_{D2}	2.262	4.423	24.962	239.680	0.772	1.539	8.426	89.428	0.276	0.571	2.963	32.392
k_{D3}	2.258	4.410	24.992	243.203	0.772	1.539	8.423	89.578	0.276	0.571	2.962	32.383
k_{D4}	2.033	3.879	20.483	196.406	0.682	1.334	7.096	68.843	0.269	0.548	2.773	29.479
k_{Y4}	0.473	0.486	0.426	0.304	0.166	0.188	0.213	0.113	0.155	0.203	0.245	0.177
k_{Y6}	0.900	0.874	0.841	0.830	0.033	0.028	0.022	0.020	0.303	0.297	0.281	0.271
k_{Y9}	0.959	1.412	3.019	4.193	0.583	0.958	2.336	4.484	0.225	0.413	1.302	3.507
k_{AS1}	1.383	2.543	14.131	130.708	0.528	0.948	4.748	49.801	0.235	0.431	1.730	17.759
k_{AS2}	1.086	0.997	0.835	0.955	0.181	0.141	0.079	0.046	0.474	0.461	0.391	0.207
k_{AS3}	0.782	0.989	1.157	0.661	0.468	0.669	1.143	1.044	0.209	0.342	0.700	0.848
k_{AY1}	2.271	4.453	25.521	255.345	0.775	1.546	8.498	90.939	0.276	0.571	2.966	32.490
k_{AY2}	2.196	4.254	24.053	238.648	0.761	1.509	8.201	87.268	0.275	0.567	2.920	31.666
k_{AY3}	1.838	3.336	16.931	154.109	0.670	1.260	6.154	60.380	0.265	0.528	2.428	23.287
k_{AY4}	2.127	4.080	22.812	224.917	0.747	1.474	7.937	84.108	0.274	0.563	2.876	30.919
k_{SK1}	0.383	0.424	0.468	0.370	0.159	0.196	0.288	0.223	0.135	0.185	0.285	0.385
k_{SK2}	0.628	0.785	1.115	1.047	0.321	0.442	0.834	1.101	0.194	0.309	0.660	1.144
k_{SK3}	0.532	0.655	0.904	0.852	0.276	0.368	0.639	0.734	0.180	0.277	0.537	0.894
k_{SK4}	0.585	0.728	1.051	1.045	0.316	0.430	0.770	0.989	0.191	0.301	0.617	1.067
k_{SK5}	0.630	0.793	1.195	1.279	0.352	0.487	0.899	1.267	0.198	0.319	0.685	1.236
k_{SK6}	0.523	0.530	0.658	0.979	0.065	0.054	0.025	0.019	0.098	0.094	0.083	0.146

Table 4: Estimated MSE when $n = 100$ and $p = 4$

β_0	-1				0				1			
	ρ				ρ				ρ			
	0.900	0.950	0.990	0.999	0.900	0.950	0.990	0.999	0.900	0.950	0.990	0.999
Estimator	MSE				MSE				MSE			
$PMLE$	0.345	0.734	3.847	43.740	0.128	0.262	1.435	15.521	0.047	0.093	0.517	5.789
k_{HK1}	0.255	0.406	1.200	10.800	0.077	0.113	0.255	2.008	0.071	0.069	0.116	0.329
k_{HK2}	0.307	0.602	2.564	25.676	0.117	0.223	0.991	9.182	0.046	0.089	0.432	3.742
k_{HKB}	0.217	0.351	1.059	9.393	0.087	0.139	0.431	3.408	0.043	0.077	0.261	1.513
k_{LW1}	0.345	0.733	3.831	41.579	0.128	0.262	1.435	15.459	0.047	0.093	0.517	5.787
k_{LW2}	0.341	0.718	3.602	39.067	0.127	0.260	1.391	14.283	0.047	0.093	0.510	5.501
k_{HSL}	0.345	0.734	3.847	43.737	0.128	0.262	1.435	15.521	0.047	0.093	0.517	5.789
k_{AM}	0.483	0.510	0.506	0.494	0.081	0.074	0.085	0.075	0.193	0.202	0.158	0.349
k_{GM}	0.308	0.577	1.008	0.173	0.127	0.255	1.020	1.004	0.047	0.092	0.452	0.918
k_{KS}	0.252	0.449	1.482	12.809	0.096	0.184	0.628	4.889	0.045	0.083	0.333	2.146
k_{A1}	0.343	0.626	2.803	25.428	0.106	0.203	0.986	6.516	0.062	0.095	0.390	4.078
k_{A2}	0.353	0.567	2.212	15.840	0.094	0.166	0.706	2.948	0.080	0.105	0.310	3.120
k_{A3}	0.342	0.721	3.610	33.225	0.124	0.250	1.305	12.430	0.047	0.092	0.507	5.330
k_{MK4}	0.287	0.508	0.808	0.190	0.124	0.242	0.807	0.681	0.047	0.091	0.403	0.753
k_{MK5}	0.209	0.305	0.539	0.698	0.083	0.132	0.305	0.469	0.042	0.070	0.219	0.596
k_{MK6}	0.276	0.470	0.718	0.208	0.124	0.241	0.774	0.578	0.047	0.090	0.371	0.668
k_{GK}	0.255	0.406	1.196	10.444	0.077	0.113	0.255	1.999	0.071	0.069	0.116	0.329
k_{D1}	0.345	0.733	3.840	43.583	0.128	0.262	1.434	15.501	0.047	0.093	0.517	5.787
k_{D2}	0.345	0.733	3.827	42.508	0.128	0.262	1.431	15.363	0.047	0.093	0.516	5.783
k_{D3}	0.345	0.732	3.820	42.831	0.128	0.262	1.431	15.381	0.047	0.093	0.516	5.781
k_{D4}	0.344	0.705	3.551	39.257	0.125	0.253	1.367	14.115	0.049	0.093	0.503	5.601
k_{Y4}	0.253	0.325	0.364	0.204	0.091	0.142	0.268	0.179	0.045	0.074	0.190	0.263
k_{Y6}	0.510	0.496	0.476	0.456	0.025	0.023	0.019	0.018	0.120	0.117	0.108	0.101
k_{Y9}	0.224	0.399	0.859	1.125	0.114	0.207	0.613	1.166	0.044	0.083	0.330	0.950
k_{AS1}	0.271	0.506	2.071	22.417	0.113	0.205	0.811	8.067	0.046	0.088	0.378	3.095
k_{AS2}	0.719	0.733	0.535	0.431	0.129	0.106	0.069	0.035	0.330	0.343	0.228	0.124
k_{AS3}	0.238	0.358	0.495	0.217	0.115	0.207	0.548	0.558	0.045	0.082	0.273	0.416
k_{AY1}	0.345	0.733	3.839	43.565	0.128	0.262	1.434	15.498	0.047	0.093	0.517	5.786
k_{AY2}	0.341	0.719	3.674	40.680	0.127	0.260	1.398	14.842	0.047	0.093	0.513	5.670
k_{AY3}	0.319	0.639	2.786	27.174	0.122	0.240	1.129	10.547	0.047	0.092	0.473	4.478
k_{AY4}	0.337	0.706	3.522	38.244	0.127	0.257	1.364	14.264	0.047	0.093	0.510	5.559
k_{SK1}	0.196	0.260	0.360	0.212	0.082	0.127	0.243	0.160	0.041	0.065	0.175	0.353
k_{SK2}	0.239	0.378	0.615	0.445	0.102	0.173	0.430	0.527	0.044	0.080	0.292	0.640
k_{SK3}	0.223	0.335	0.537	0.400	0.098	0.163	0.375	0.393	0.043	0.076	0.249	0.559
k_{SK4}	0.231	0.356	0.583	0.458	0.102	0.172	0.411	0.479	0.044	0.079	0.270	0.610
k_{SK5}	0.236	0.372	0.626	0.531	0.105	0.180	0.445	0.575	0.045	0.081	0.286	0.661
k_{SK6}	0.248	0.251	0.310	0.624	0.036	0.035	0.019	0.015	0.035	0.035	0.028	0.036

Table 5: Estimated MSE when $n = 100$ and $p = 8$

β_0	-1				0				1			
	ρ				ρ				ρ			
	0.900	0.950	0.990	0.999	0.900	0.950	0.990	0.999	0.900	0.950	0.990	0.999
Estimator	MSE				MSE				MSE			
$PMLE$	0.469	0.933	4.964	54.532	0.171	0.342	1.858	19.833	0.063	0.127	0.679	7.111
k_{HK1}	0.251	0.365	1.234	11.702	0.080	0.107	0.229	1.957	0.052	0.074	0.108	0.301
k_{HK2}	0.419	0.779	3.550	35.446	0.156	0.295	1.358	12.931	0.062	0.122	0.576	4.884
k_{HKB}	0.243	0.363	1.151	9.776	0.097	0.148	0.460	3.674	0.054	0.093	0.280	1.487
k_{LW1}	0.469	0.933	4.961	54.202	0.171	0.342	1.858	19.821	0.063	0.127	0.679	7.111
k_{LW2}	0.460	0.901	4.503	46.559	0.169	0.337	1.763	17.375	0.063	0.127	0.664	6.509
k_{HSL}	0.469	0.933	4.964	54.532	0.171	0.342	1.858	19.833	0.063	0.127	0.679	7.111
k_{AM}	0.596	0.549	0.490	0.572	0.076	0.057	0.045	0.063	0.212	0.234	0.219	0.171
k_{GM}	0.445	0.827	2.473	1.318	0.171	0.339	1.630	3.597	0.063	0.127	0.649	2.819
k_{KS}	0.325	0.533	2.057	19.401	0.134	0.232	0.793	6.954	0.060	0.112	0.401	2.861
k_{A1}	0.400	0.645	2.190	17.721	0.082	0.148	0.635	4.492	0.073	0.115	0.330	2.521
k_{A2}	0.399	0.512	1.147	7.921	0.058	0.084	0.278	1.440	0.101	0.129	0.217	1.131
k_{A3}	0.457	0.886	4.277	36.985	0.149	0.287	1.409	13.859	0.063	0.124	0.595	5.845
k_{MK4}	0.413	0.720	1.764	1.125	0.168	0.327	1.303	2.238	0.063	0.125	0.586	1.922
k_{MK5}	0.243	0.338	0.637	1.287	0.084	0.127	0.313	0.782	0.053	0.087	0.222	0.672
k_{MK6}	0.402	0.684	1.564	1.023	0.168	0.327	1.291	1.926	0.063	0.124	0.582	1.660
k_{GK}	0.251	0.365	1.233	11.653	0.080	0.107	0.229	1.955	0.052	0.074	0.108	0.301
k_{D1}	0.469	0.933	4.962	54.500	0.171	0.342	1.858	19.829	0.063	0.127	0.679	7.111
k_{D2}	0.469	0.932	4.954	54.226	0.171	0.342	1.855	19.785	0.063	0.127	0.679	7.109
k_{D3}	0.469	0.932	4.952	54.239	0.171	0.342	1.856	19.783	0.063	0.127	0.679	7.108
k_{D4}	0.460	0.893	4.620	48.659	0.164	0.328	1.750	18.226	0.064	0.127	0.660	6.836
k_{Y4}	0.274	0.323	0.346	0.222	0.090	0.130	0.210	0.184	0.056	0.091	0.180	0.224
k_{Y6}	0.737	0.715	0.674	0.659	0.021	0.018	0.014	0.012	0.242	0.235	0.218	0.209
k_{Y9}	0.338	0.594	1.792	4.279	0.159	0.296	1.100	3.363	0.059	0.115	0.506	2.246
k_{AS1}	0.349	0.617	2.851	30.387	0.151	0.272	1.127	11.037	0.061	0.118	0.498	3.991
k_{AS2}	0.940	0.888	0.783	0.483	0.162	0.120	0.070	0.031	0.408	0.435	0.411	0.228
k_{AS3}	0.352	0.551	0.990	0.873	0.152	0.268	0.768	1.328	0.060	0.112	0.389	0.934
k_{AY1}	0.469	0.933	4.962	54.496	0.171	0.342	1.857	19.828	0.063	0.127	0.679	7.111
k_{AY2}	0.465	0.922	4.845	52.727	0.170	0.340	1.833	19.433	0.063	0.127	0.677	7.037
k_{AY3}	0.439	0.834	3.873	37.575	0.164	0.318	1.543	14.677	0.063	0.125	0.630	5.744
k_{AY4}	0.462	0.911	4.737	51.132	0.170	0.338	1.810	19.065	0.063	0.127	0.674	6.966
k_{SK1}	0.218	0.276	0.391	0.416	0.083	0.122	0.241	0.315	0.050	0.080	0.178	0.357
k_{SK2}	0.306	0.459	0.925	1.206	0.125	0.203	0.522	1.127	0.058	0.105	0.327	0.991
k_{SK3}	0.280	0.403	0.742	0.982	0.116	0.185	0.446	0.818	0.057	0.100	0.289	0.764
k_{SK4}	0.295	0.436	0.845	1.171	0.125	0.203	0.512	1.007	0.058	0.104	0.323	0.894
k_{SK5}	0.306	0.459	0.931	1.367	0.131	0.218	0.571	1.196	0.059	0.107	0.351	1.003
k_{SK6}	0.290	0.287	0.313	0.592	0.039	0.039	0.023	0.008	0.051	0.052	0.044	0.050

Table 6: Estimated MSE when $n = 200$ and $p = 4$

β_0	-1				0				1			
	ρ				ρ				ρ			
	0.900	0.950	0.990	0.999	0.900	0.950	0.990	0.999	0.900	0.950	0.990	0.999
Estimator	MSE				MSE				MSE			
$PMLE$	0.141	0.283	1.517	16.440	0.050	0.100	0.539	6.087	0.018	0.037	0.199	2.229
k_{HK1}	0.131	0.208	0.544	3.581	0.041	0.064	0.119	0.645	0.031	0.037	0.094	0.102
k_{HK2}	0.134	0.256	1.130	9.886	0.048	0.093	0.416	3.722	0.018	0.036	0.183	1.576
k_{HKB}	0.112	0.185	0.543	3.664	0.042	0.072	0.212	1.410	0.018	0.034	0.138	0.704
k_{LW1}	0.141	0.283	1.516	16.337	0.050	0.100	0.539	6.084	0.018	0.037	0.199	2.228
k_{LW2}	0.141	0.281	1.485	15.526	0.049	0.100	0.535	5.883	0.018	0.037	0.199	2.192
k_{HSL}	0.141	0.283	1.517	16.440	0.050	0.100	0.539	6.087	0.018	0.037	0.199	2.229
k_{AM}	0.382	0.455	0.379	0.531	0.060	0.058	0.059	0.070	0.162	0.187	0.133	0.225
k_{GM}	0.137	0.266	0.889	0.434	0.049	0.100	0.495	1.436	0.018	0.037	0.193	1.002
k_{KS}	0.124	0.224	0.767	5.030	0.044	0.086	0.306	2.044	0.018	0.036	0.151	1.020
k_{A1}	0.155	0.282	1.259	12.795	0.048	0.087	0.436	4.277	0.029	0.045	0.174	1.805
k_{A2}	0.175	0.288	1.067	10.368	0.047	0.079	0.359	2.828	0.042	0.058	0.154	1.489
k_{A3}	0.141	0.281	1.494	15.215	0.049	0.099	0.520	5.644	0.018	0.037	0.198	2.184
k_{MK4}	0.133	0.250	0.732	0.408	0.049	0.098	0.442	0.988	0.018	0.037	0.185	0.805
k_{MK5}	0.111	0.172	0.400	0.693	0.040	0.069	0.191	0.448	0.017	0.032	0.122	0.456
k_{MK6}	0.131	0.241	0.651	0.395	0.049	0.098	0.440	0.853	0.018	0.037	0.179	0.707
k_{GK}	0.131	0.208	0.544	3.567	0.041	0.064	0.119	0.645	0.031	0.037	0.094	0.102
k_{D1}	0.141	0.282	1.516	16.420	0.050	0.100	0.539	6.084	0.018	0.037	0.199	2.228
k_{D2}	0.141	0.282	1.515	16.353	0.050	0.100	0.539	6.074	0.018	0.037	0.199	2.228
k_{D3}	0.141	0.282	1.515	16.359	0.050	0.100	0.539	6.075	0.018	0.037	0.199	2.228
k_{D4}	0.144	0.284	1.457	15.603	0.049	0.099	0.525	5.886	0.019	0.037	0.198	2.189
k_{Y4}	0.131	0.198	0.310	0.218	0.043	0.074	0.191	0.255	0.019	0.034	0.117	0.276
k_{Y6}	0.364	0.355	0.334	0.318	0.016	0.015	0.012	0.011	0.080	0.078	0.071	0.067
k_{Y9}	0.112	0.208	0.609	1.079	0.047	0.091	0.348	0.993	0.018	0.035	0.162	0.716
k_{AS1}	0.127	0.235	0.904	8.471	0.047	0.091	0.357	3.146	0.018	0.036	0.173	1.251
k_{AS2}	0.619	0.702	0.500	0.205	0.104	0.095	0.063	0.028	0.294	0.331	0.222	0.143
k_{AS3}	0.122	0.204	0.441	0.307	0.048	0.092	0.336	0.683	0.018	0.035	0.152	0.446
k_{AY1}	0.141	0.282	1.516	16.418	0.050	0.100	0.539	6.084	0.018	0.037	0.199	2.228
k_{AY2}	0.141	0.281	1.487	15.756	0.049	0.100	0.534	5.936	0.018	0.037	0.199	2.208
k_{AY3}	0.137	0.267	1.254	11.229	0.049	0.097	0.472	4.508	0.018	0.037	0.193	1.896
k_{AY4}	0.140	0.279	1.459	15.159	0.049	0.100	0.528	5.798	0.018	0.037	0.198	2.188
k_{SK1}	0.107	0.155	0.296	0.296	0.040	0.068	0.175	0.223	0.017	0.031	0.105	0.318
k_{SK2}	0.121	0.207	0.509	0.561	0.045	0.083	0.263	0.593	0.018	0.035	0.152	0.565
k_{SK3}	0.116	0.189	0.436	0.513	0.044	0.080	0.244	0.463	0.018	0.034	0.137	0.484
k_{SK4}	0.118	0.198	0.472	0.568	0.045	0.083	0.262	0.533	0.018	0.035	0.145	0.526
k_{SK5}	0.120	0.204	0.502	0.631	0.046	0.085	0.278	0.606	0.018	0.035	0.151	0.562
k_{SK6}	0.143	0.142	0.153	0.325	0.021	0.023	0.015	0.007	0.019	0.020	0.017	0.015

Table 7: Estimated MSE when $n = 200$ and $p = 8$

β_0	-1				0				1			
	ρ				ρ				ρ			
	0.900	0.950	0.990	0.999	0.900	0.950	0.990	0.999	0.900	0.950	0.990	0.999
Estimator	MSE				MSE				MSE			
$PMLE$	0.134	0.264	1.411	15.475	0.049	0.099	0.520	5.705	0.018	0.036	0.193	2.071
k_{HK1}	0.099	0.149	0.345	2.321	0.032	0.042	0.062	0.378	0.024	0.047	0.091	0.070
k_{HK2}	0.129	0.245	1.128	10.513	0.048	0.094	0.429	3.916	0.018	0.036	0.181	1.587
k_{HKB}	0.102	0.162	0.466	3.151	0.039	0.066	0.192	1.182	0.018	0.033	0.124	0.598
k_{LW1}	0.134	0.264	1.411	15.469	0.049	0.099	0.520	5.704	0.018	0.036	0.193	2.071
k_{LW2}	0.134	0.262	1.374	14.292	0.049	0.099	0.514	5.446	0.018	0.036	0.192	2.022
k_{HSL}	0.134	0.264	1.411	15.475	0.049	0.099	0.520	5.705	0.018	0.036	0.193	2.071
k_{AM}	0.449	0.442	0.437	0.371	0.059	0.046	0.029	0.045	0.158	0.183	0.200	0.125
k_{GM}	0.133	0.258	1.226	2.731	0.049	0.099	0.511	3.235	0.018	0.036	0.192	1.667
k_{KS}	0.119	0.203	0.715	5.778	0.044	0.075	0.257	2.313	0.018	0.035	0.141	1.083
k_{A1}	0.152	0.244	0.869	8.038	0.036	0.064	0.284	2.661	0.029	0.044	0.139	1.072
k_{A2}	0.192	0.251	0.576	4.339	0.033	0.046	0.160	1.282	0.049	0.063	0.120	0.582
k_{A3}	0.134	0.261	1.329	13.883	0.047	0.094	0.468	4.855	0.018	0.036	0.184	1.906
k_{MK4}	0.131	0.248	1.018	1.960	0.049	0.098	0.481	2.160	0.018	0.036	0.187	1.304
k_{MK5}	0.104	0.163	0.366	1.019	0.035	0.058	0.166	0.551	0.017	0.031	0.103	0.411
k_{MK6}	0.130	0.245	0.983	1.658	0.049	0.098	0.482	2.057	0.018	0.036	0.188	1.237
k_{GK}	0.099	0.149	0.345	2.320	0.032	0.042	0.062	0.378	0.024	0.047	0.091	0.070
k_{D1}	0.134	0.264	1.410	15.473	0.049	0.099	0.520	5.704	0.018	0.036	0.193	2.071
k_{D2}	0.134	0.264	1.410	15.463	0.049	0.099	0.520	5.702	0.018	0.036	0.193	2.071
k_{D3}	0.134	0.264	1.410	15.461	0.049	0.099	0.520	5.702	0.018	0.036	0.193	2.071
k_{D4}	0.137	0.261	1.363	14.794	0.049	0.097	0.509	5.544	0.018	0.036	0.191	2.035
k_{Y4}	0.122	0.173	0.269	0.235	0.039	0.064	0.152	0.232	0.018	0.033	0.102	0.226
k_{Y6}	0.580	0.559	0.517	0.500	0.014	0.012	0.009	0.007	0.176	0.172	0.155	0.144
k_{Y9}	0.116	0.217	0.881	3.158	0.048	0.095	0.423	2.092	0.018	0.035	0.174	1.168
k_{AS1}	0.120	0.216	0.911	8.749	0.048	0.092	0.385	3.260	0.018	0.035	0.172	1.281
k_{AS2}	0.792	0.785	0.764	0.405	0.131	0.101	0.056	0.026	0.332	0.371	0.397	0.232
k_{AS3}	0.125	0.224	0.630	1.100	0.048	0.092	0.365	1.180	0.018	0.035	0.159	0.706
k_{AY1}	0.134	0.264	1.410	15.473	0.049	0.099	0.520	5.704	0.018	0.036	0.193	2.071
k_{AY2}	0.134	0.263	1.399	15.218	0.049	0.099	0.517	5.649	0.018	0.036	0.193	2.063
k_{AY3}	0.132	0.255	1.241	11.830	0.049	0.097	0.477	4.616	0.018	0.036	0.189	1.844
k_{AY4}	0.134	0.262	1.388	14.976	0.049	0.099	0.515	5.595	0.018	0.036	0.193	2.055
k_{SK1}	0.100	0.145	0.274	0.454	0.036	0.059	0.155	0.342	0.017	0.030	0.093	0.281
k_{SK2}	0.117	0.199	0.537	1.290	0.044	0.080	0.265	0.877	0.018	0.034	0.138	0.628
k_{SK3}	0.112	0.186	0.461	1.013	0.043	0.076	0.242	0.711	0.018	0.033	0.129	0.521
k_{SK4}	0.115	0.195	0.515	1.180	0.044	0.080	0.266	0.825	0.018	0.034	0.139	0.595
k_{SK5}	0.117	0.200	0.559	1.328	0.045	0.084	0.288	0.933	0.018	0.034	0.147	0.656
k_{SK6}	0.159	0.159	0.149	0.269	0.020	0.023	0.019	0.004	0.025	0.027	0.026	0.018

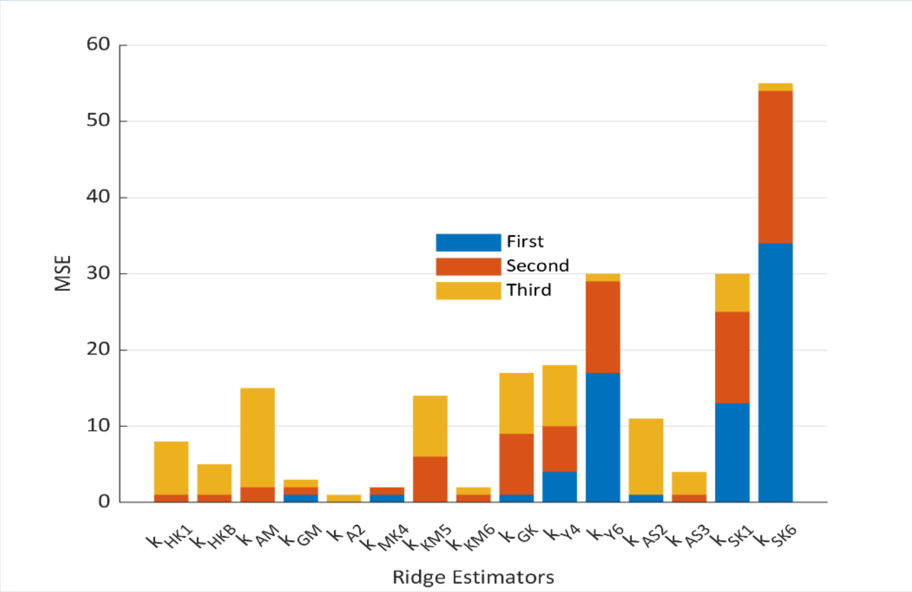

Figure 1: Estimators Achieving Top Three Positions in MSE Performance

Table 8: Poisson Regression Estimates for the aircraft damage dataset

Estimator	β_0	β_1	β_2	β_3	MSE
$PMLE$	-0.406	0.569	0.165	-0.014	1.029
k_{HK1}	-0.140	0.339	0.172	-0.015	0.254
k_{HK2}	-0.193	0.414	0.169	-0.015	0.304
k_{HKB}	-0.078	0.225	0.178	-0.016	0.272
k_{LW1}	-0.401	0.567	0.165	-0.014	1.003
k_{LW2}	-0.400	0.567	0.165	-0.014	0.999
k_{HSL}	-0.406	0.569	0.165	-0.014	1.029
k_{AM}	0.001	0.003	0.035	0.004	0.503
k_{GM}	-0.402	0.567	0.165	-0.014	1.006
k_{KS}	-0.168	0.380	0.171	-0.015	0.273
k_{A1}	0.000	0.030	0.163	-0.013	0.456
k_{A2}	0.003	0.013	0.112	-0.006	0.479
k_{A3}	-0.282	0.501	0.166	-0.014	0.508
k_{MK4}	-0.375	0.556	0.165	-0.014	0.874
k_{MK5}	-0.079	0.227	0.178	-0.016	0.271
k_{MK6}	-0.353	0.545	0.165	-0.014	0.772
k_{GK}	-0.140	0.339	0.172	-0.015	0.254
k_{D1}	-0.406	0.569	0.165	-0.014	1.029
k_{D2}	-0.406	0.569	0.165	-0.014	1.029
k_{D3}	-0.406	0.569	0.165	-0.014	1.028
k_{D4}	-0.399	0.566	0.165	-0.014	0.994
k_{Y4}	-0.151	0.355	0.172	-0.015	0.259
k_{Y6}	-0.103	0.273	0.176	-0.016	0.254
k_{Y9}	-0.261	0.483	0.166	-0.015	0.446
k_{AS1}	-0.168	0.380	0.171	-0.015	0.273
k_{AS2}	0.000	0.001	0.011	0.006	0.512
k_{AS3}	-0.333	0.534	0.165	-0.014	0.687
k_{AY1}	-0.406	0.569	0.165	-0.014	1.029
k_{AY2}	-0.402	0.567	0.165	-0.014	1.009
k_{AY3}	-0.311	0.521	0.165	-0.014	0.605
k_{AY4}	-0.399	0.566	0.165	-0.014	0.990
k_{SK1}	-0.058	0.180	0.180	-0.016	0.300
k_{SK2}	-0.187	0.406	0.169	-0.015	0.296
k_{SK3}	-0.125	0.314	0.174	-0.016	0.250
k_{SK4}	-0.160	0.369	0.171	-0.015	0.266
k_{SK5}	-0.184	0.401	0.170	-0.015	0.291
k_{SK6}	-0.003	0.039	0.171	-0.014	0.445

Conclusion

This study evaluated the performance of several existing ridge estimators and introduced new ridge estimators which effectively address multicollinearity in Poisson regression models. Through an extensive simulation study, it was found that the proposed estimator, k_{SK6}, consistently outperformed all other estimators in terms of minimizing the MSE. Furthermore, estimators k_{SK1} and k_{Y6} ranked consistently as the second-best performers based on MSE values. When applied to a real dataset, the proposed estimator k_{SK1} demonstrated the lowest MSE, indicating its superior performance in practical applications. In contrast, the Poisson maximum likelihood estimator emerged as the least effective, with the highest MSE, underscoring its limitations in dealing with multicollinearity.

Conflicts of interest

There are no conflicts of interest in this work.

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