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Some Sums Involving Generalized Harmonic and r −Derangement Numbers

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ABSTRACT

In this paper, we derive some sums involving generalized harmonic and r −derangement numbers by using generating functions of these numbers and some combinatorial identities. The relationship between Daehee numbers and generalized harmonic numbers of rank r, $H(n,r,\alpha)$ is given. In addition, sums including Daehee numbers of order r, D_n^r, generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$, Cauchy numbers of order r, C_n^r and the stirling numbers of the first kind, $s(n,i)$ are also calculated.

Keywords: Generalized harmonic numbers, r − Derangement numbers, Generating functions, Special numbers.

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Introduction

Harmonic numbers are important in various branches of combinatorics and number theory. The harmonic numbers are defined by

$$H_0 = 0 \text{ and } H_n = \sum_{i=1}^{n} \frac{1}{i} \quad \text{for } n = 1, 2, \cdots.$$

Some harmonic numbers are $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \cdots$. Recently, some authors have generalized them[1, 2, 4, 8, 13, 19].

In [8], for any $\alpha \in \mathbb{R}^+$, the generalized harmonic numbers $H_n(\alpha)$ are defined by

$$H_0(\alpha) = 0 \text{ and } H_n(\alpha) = \sum_{i=1}^{n} \frac{1}{i\alpha^i} \quad \text{for } n = 1, 2, \cdots.$$

When $\alpha = 1$, $H_n(1) = H_n$ and the generating function of these numbers is

$$\sum_{n=1}^{\infty} H_n(\alpha)x^n = -\frac{\ln\left(1 - \frac{x}{\alpha}\right)}{1 - x}.$$

In [12], for the generalized harmonic numbers $H_n(\alpha)$, Ömür and Bilgin defined the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ by: For $r < 0$ or $n \le 0$, $H_n^r(\alpha) = 0$ and for $n \ge 1$,

$$H_n^r(\alpha) = \sum_{i=1}^{n} H_i^{r-1}(\alpha), \qquad r \ge 1,$$

where $H_n^0(\alpha) = \frac{1}{n\alpha^n}$.

When $\alpha = 1$, $H_n^r(1) = H_n^r$ are the hyperharmonic numbers of order r. The generating function of $H_n^r(\alpha)$ is

$$\sum_{n=1}^{\infty} H_n^r(\alpha)x^n = -\frac{\ln\left(1 - \frac{x}{\alpha}\right)}{(1 - x)^r}. \tag{1}$$

In [4, 19], the generalized harmonic numbers $H(n,r)$ of rank r are defined as for $n \ge 1, r \ge 0$,

$$H(n,r) = \sum_{1 \le n_0 + n_1 + \cdots + n_r \le n} \frac{1}{n_0 n_1 \cdots n_r}.$$

When $r = 0$, $H(n,0) = H_n$. The generating function of the generalized harmonic numbers $H(n,r)$ of rank r is

$$\sum_{n=1}^{\infty} H(n,r)x^n = \frac{(-\ln(1 - x))^{r+1}}{1 - x}.$$

In [7], inspiring from these definitions, $H(n,r,\alpha)$ are defined as for $n \ge 1, r \ge 0$,

$$H(n,r,\alpha) = \sum_{1 \le n_0 + n_1 + \cdots + n_r \le n} \frac{1}{n_0 n_1 \cdots n_r \alpha^{n_0 + n_1 + \cdots + n_r}}.$$

For $\alpha = 1$, $H(n,r,1) = H(n,r)$. The generating function of the generalized harmonic numbers of rank r, $H(n,r,\alpha)$ is given by

$$\sum_{n=0}^{\infty} H(n,r,\alpha)x^n = \frac{\left(-\ln\left(1 - \frac{x}{\alpha}\right)\right)^{r+1}}{1 - x}. \tag{2}$$

The Cauchy numbers of order r, C_n^r have an exponential generating function

$$\left(\frac{x}{\ln(1+x)}\right)^r = \sum_{n=0}^{\infty} C_n^r \frac{x^n}{n!}. \quad (3)$$

The Daehee numbers of order r, D_n^r have an exponential generating function

$$\left(\frac{\ln(1+x)}{x}\right)^r = \sum_{n=0}^{\infty} D_n^r \frac{x^n}{n!}. \quad (4)$$

When $r = 1$, $D_n^1 = D_n$ are called Daehee numbers.

In [22], the r −derangement numbers, $D_r(n)$ have an exponential generating function

$$\frac{x^r e^{-x}}{(1-x)^{r+1}} = \sum_{n=0}^{\infty} D_r(n) \frac{x^n}{n!}. \quad (5)$$

For $r = 0$, $D_0(n) = d_n$ are called derangement numbers.

The generalized geometric series are given by for positive integers a, b,

$$\frac{x^b}{(1-x)^{a+1}} = \sum_{n=b}^{\infty} \binom{n+a-b}{a} x^n. \quad (6)$$

The exponential generating function is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \quad (7)$$

The Stirling numbers of the first kind $s(n,i)$ are defined by

$$x^{\underline{n}} = \sum_{i=0}^{n} s(n,i) x^i,$$

where $x^{\underline{n}} = \begin{cases} x(x-1)\cdots(x-n+1) & \text{if } n \geq 1, \\ 0 & \text{if } n = 0. \end{cases}$

The Stirling numbers of the first kind $s(n,k)$ satisfy the recurrence relation

$$s(n+1,k) = s(n,k-1) - ns(n,k),$$

and the generating functions of these numbers are given by

$$\sum_{n=0}^{\infty} s(n,k) \frac{x^n}{n!} = \frac{(\ln(1+x))^k}{k!}, k \geq 0. \quad (8)$$

Recently, using generating functions, there are some works including generalized harmonic, r −derangement and special numbers by authors [5, 6, 7, 9, 10, 11, 14, 15, 16, 17, 18, 20, 23, 24, 25, 26, 27, 31]. At the same time, many studies have been carried out on the degenerate states of these numbers [28, 29, 30, 32, 33].

In [22], the authors gave many formulas for the r −derangement numbers. For example, for a positive integer r and $r \leq n$,

$$(r+1)!\, L(n,r+1) = \sum_{i=1}^{n} \binom{n}{i} i D_r(n-i),$$

where $L(n,k)$ are the Lah numbers.

In [18], Rim, et al. examined some identities relating the hyperharmonic and the Daehee numbers. For example, for any positive integer n,

$$n!\, H_n^r = (-1)^{n-1} \sum_{i=0}^{n-1} \binom{n}{i} (n-i)(-r)^{\underline{i-1}} D_i.$$

In [7], some sums including generalized harmonic numbers have been obtained by Duran et al. For example, for any positive integers n and r,

$$H(n,r,\alpha) = (-1)^{n-r} \sum_{i=0}^{n} \frac{(-1)^i s(n-i,r) r!}{\alpha^{n-i}(n-i)!} H_i(\alpha).$$

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$ be two infinite series. The Cauchy product of these series is given as follows:

$$F(x)G(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for all positive integer n.

In this paper, we derive some sums involving generalized harmonic and r −derangement numbers by using generating functions of these numbers and some combinatorial identities. The relationship between Daehee numbers and generalized harmonic numbers of rank r, $H(n,r,\alpha)$ is given. In addition, sums including Daehee numbers of order r, D_n^r, generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$, Cauchy numbers of order r, C_n^r and the stirling numbers of the first kind, $s(n,i)$ are also calculated.

Some Sums Involving Generalized Harmonic and r-Derangement Numbers

This section, we will give some sums including generalized harmonic and $\boldsymbol{r}$ −derangement numbers by using generating functions of these numbers and some combinatorial identities.

Theorem 1. Let n, r and m be positive integers. For $n \geq m(r+1)$, then

$$\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{j-i}r^{j-i}\binom{n-j+r-1}{n-j}\binom{j}{i}\frac{D_{m(r+1)}(i)}{j!}$$
$$=\sum_{l_1+l_2+\cdots+l_{r+1}=n}\frac{D_m(l_1)D_m(l_2)\cdots D_m(l_{r+1})}{l_1!\,l_2!\cdots l_{r+1}!}.$$

Proof. By (5), (6) and (7), we consider

$$\frac{x^{m(r+1)}e^{-x}}{(1-x)^{m(r+1)+1}}e^{-rx}\frac{1}{(1-x)^r}$$
$$=\sum_{n=0}^{\infty}\frac{D_{m(r+1)}(n)}{n!}x^n\sum_{n=0}^{\infty}(-1)^n\frac{r^n}{n!}x^n\sum_{n=0}^{\infty}\binom{n+r-1}{n}x^n$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}(-r)^{n-i}\frac{D_{m(r+1)}(i)}{i!\,(n-i)!}x^n\sum_{n=0}^{\infty}\binom{n+r-1}{n}x^n,$$

and from product of generating functions, equals

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}(-r)^{j-i}\binom{n-j+r-1}{n-j}\binom{j}{i}\frac{D_{m(r+1)}(i)}{j!}x^n,$$

and

$$\frac{x^{m(r+1)}e^{-x}}{(1-x)^{m(r+1)+1}}e^{-rx}\frac{1}{(1-x)^r}$$
$$=\left(\frac{x^me^{-x}}{(1-x)^{m+1}}\right)\left(\frac{x^me^{-x}}{(1-x)^{m+1}}\right)\cdots\left(\frac{x^me^{-x}}{(1-x)^{m+1}}\right)$$
$$=\sum_{l_1=0}^{\infty}\frac{D_m(l_1)}{l_1!}x^{l_1}\sum_{l_2=0}^{\infty}\frac{D_m(l_2)}{l_2!}x^{l_2}\cdots\sum_{l_{r+1}=0}^{\infty}\frac{D_m(l_{r+1})}{l_{r+1}!}x^{l_{r+1}}$$
$$=\sum_{n=0}^{\infty}\sum_{l_1+l_2+\cdots+l_{r+1}=n}\frac{D_m(l_1)D_m(l_2)\cdots D_m(l_{r+1})}{l_1!\,l_2!\cdots l_{r+1}!}x^n.$$

Hence, comparing the coefficients on both sides, we have the proof.

Theorem 2. For $n\geq r$, then

$$(-1)^{n-r}\frac{D_{n-r}^r}{\alpha^n(n-r)!}=H(n,r-1,\alpha)-H(n-1,r-1,\alpha),$$

and for $n>2r$,

$$\sum_{i=0}^{n}\binom{i-1}{r-1}H(n-i,r,\alpha)$$
$$=\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{n}{j}\binom{j}{i}\frac{(-1)^{i-r-1}s(i,r+1)(r+1)!}{\alpha^i n!}D_r(j-i).$$

Proof. From (4), we have

$$\sum_{n=r}^{\infty}(-1)^{n-r}\frac{D_{n-r}^r}{\alpha^n(n-r)!}x^n$$
$$=\frac{x^r}{\alpha^r}\sum_{n=0}^{\infty}(-1)^n\frac{D_n^r}{\alpha^n}\frac{x^n}{n!}=\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r.$$

By (2), we get

$$\sum_{n=r}^{\infty}(-1)^{n-r}\frac{D_{n-r}^r}{\alpha^n(n-r)!}x^n=\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{1-x}(1-x)$$
$$=\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{1-x}-x\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{1-x}$$
$$=\sum_{n=0}^{\infty}H(n,r-1,\alpha)x^n-\sum_{n=0}^{\infty}H(n,r-1,\alpha)x^{n+1}$$
$$=\sum_{n=1}^{\infty}\big(H(n,r-1,\alpha)-H(n-1,r-1,\alpha)\big)x^n.$$

Secondly, by product of exponential generating functions, we write

$$\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}\frac{x^re^{-x}}{(1-x)^{r+1}}e^x$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^{n-r-1}s(n,r+1)(r+1)!}{\alpha^n}\frac{x^n}{n!}$$
$$\times\sum_{n=0}^{\infty}D_r(n)\frac{x^n}{n!}\sum_{n=0}^{\infty}\frac{x^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^{i-r-1}s(i,r+1)D_r(n-i)(r+1)!}{\alpha^i}\frac{x^n}{n!}$$
$$\times\sum_{n=0}^{\infty}\frac{x^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{n}{j}\binom{j}{i}\frac{(-1)^{i-r-1}s(i,r+1)D_r(j-i)(r+1)!}{\alpha^i n!}x^n,$$

and by (2) and (6), we have

$$\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}\frac{x^r}{(1-x)^{r+1}}$$
$$=\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x}\frac{x^r}{(1-x)^r}$$
$$=\sum_{n=0}^{\infty}H(n,r,\alpha)x^n\sum_{n=r}^{\infty}\binom{n-1}{r-1}x^n$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{i-1}{r-1}H(n-i,r,\alpha)x^n.$$

Comparing the coefficients of x^n in the first and last series. So, we have the proof.

Theorem 3. Let m be positive integer. For $n>r$, then

$$\sum_{i=0}^{n}\frac{(-1)^i}{i!\,\alpha^{i-m}}C_i^mH(n+m-i,r+m,\alpha)=H(n,r,\alpha),$$

and

$$\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{j}{i}\frac{(-1)^i}{j!\,\alpha^{i-r}}C_i^r d_{j-i}H(n-j,r-1,\alpha)$$
$$=\sum_{i=0}^{n}(-1)^i\binom{r-1}{i}\frac{D_r(n-i)}{(n-i)!}.$$

Proof. From (2) and (3), we write

$$\sum_{n=0}^{\infty}H(n,r,\alpha)x^n=\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x}$$
$$=\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+m+1}}{1-x}\frac{(-x/\alpha)^m}{\left(\ln\left(1-\frac{x}{\alpha}\right)\right)^m}\frac{\alpha^m}{x^m}$$
$$=\sum_{n=0}^{\infty}H(n,r+m,\alpha)x^{n-m}\sum_{n=0}^{\infty}(-1)^n\alpha^{m-n}C_n^m\frac{x^n}{n!}$$
$$=\sum_{n=0}^{\infty}H(n+m,r+m,\alpha)x^n\sum_{n=0}^{\infty}(-1)^n\alpha^{m-n}C_n^m\frac{x^n}{n!},$$

and by product of exponential generating functions,

$$\sum_{n=0}^{\infty}H(n,r,\alpha)x^n$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(-1)^i}{i!\,\alpha^{i-m}}C_i^m H(n+m-i,r+m,\alpha)x^n,$$

as claimed.

Secondly, we will give the proof of the other sum. From Binomial theorem and (5), we have

$$(1-x)^{r-1}\frac{x^re^{-x}}{(1-x)^{r+1}}$$
$$=\sum_{n=0}^{r-1}(-1)^n\binom{r-1}{n}x^n\sum_{n=0}^{\infty}D_r(n)\frac{x^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^i\binom{r-1}{i}\frac{D_r(n-i)}{(n-i)!}x^n, \quad (9)$$

and by product of generating functions,

$$(1-x)^{r-1}\frac{x^re^{-x}}{(1-x)^{r+1}}$$
$$=\frac{e^{-x}}{1-x}\left(\frac{-\frac{x}{\alpha}}{\ln\left(1-\frac{x}{\alpha}\right)}\right)^r\alpha^r\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{1-x}$$
$$=\sum_{n=0}^{\infty}d_n\frac{x^n}{n!}\sum_{n=0}^{\infty}\frac{(-1)^nC_n^r}{\alpha^{n-r}}\frac{x^n}{n!}\sum_{n=0}^{\infty}H(n,r-1,\alpha)x^n$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{C_i^rd_{n-i}}{\alpha^{i-r}}\frac{x^n}{n!}\sum_{n=0}^{\infty}H(n,r-1,\alpha)x^n$$
$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^i\binom{j}{i}\frac{C_i^rd_{j-i}H(n-j,r-1,\alpha)}{\alpha^{i-r}j!}x^n. \quad (10)$$

By (9) and (10), comparing the coefficients on both sides, we get the desired result. So, the proof is complete.

Theorem 4. For $\boldsymbol{n \geq m}$ and $\boldsymbol{n > r \geq 1}$, then

$$\sum_{j=0}^{n}\sum_{i=0}^{j}H(i,r-1,\alpha)H_{j-i}^m(\alpha)\frac{d_{n-j}}{(n-j)!}$$
$$=\sum_{i=0}^{n}H(n-i+m,r,\alpha)\frac{D_m(i)}{i!},$$

and for $n \geq r \geq 1$,

$$D_r(n)$$
$$=n!\sum_{j=0}^{n-r}\sum_{i=0}^{j}\frac{(-1)^{i+j}D_{r-1}(i+r-1)C_{j-i}^rH(n-j,r-1,\alpha)}{(i+r-1)!\,(j-i)!\,\alpha^{j-i-r}}.$$

Proof. From product of exponential generating functions, we write

$$\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{(1-x)^{m+2}}e^{-x}$$
$$=\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x}\frac{x^me^{-x}}{(1-x)^{m+1}}x^{-m}$$
$$=\sum_{n=0}^{\infty}H(n,r,\alpha)\,x^{n-m}\sum_{n=0}^{\infty}\frac{D_m(n)}{n!}x^n$$
$$=\sum_{n=-m}^{\infty}H(n+m,r,\alpha)\,x^n\sum_{n=0}^{\infty}\frac{D_m(n)}{n!}x^n$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}H(i+m,r,\alpha)\frac{D_m(n-i)}{(n-i)\,!}x^n, \quad (11)$$

and

$$\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{(1-x)^{m+2}}e^{-x}$$
$$=\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r}}{1-x}\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^m}\frac{e^{-x}}{1-x}$$
$$=\sum_{n=0}^{\infty}H(n,r-1,\alpha)\,x^n\sum_{n=0}^{\infty}H_n^m(\alpha)\,x^n\sum_{n=0}^{\infty}\frac{d_n}{n!}x^n$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}H(i,r-1,\alpha)\,H_{n-i}^m(\alpha)x^n\sum_{n=0}^{\infty}\frac{d_n}{n!}x^n$$
$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}H(i,r-1,\alpha)H_{j-i}^m(\alpha)\frac{d_{n-j}}{(n-j)!}x^n. \quad (12)$$

From here, (11) and (12) yield the desired result.

Now, we will give the other sum. With the help of generating functions of (2), (3) and (5), we get

$$\sum_{n=0}^{\infty}\frac{D_r(n)}{n!}x^n=\frac{x^r e^{-x}}{(1-x)^{r+1}}$$

$$=x^{-r+1}\frac{x^{r-1}e^{-x}}{(1-x)^r}\left(\frac{-\frac{x}{\alpha}}{\ln\left(1-\frac{x}{\alpha}\right)}\right)^r\alpha^r\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{1-x}$$

$$=\sum_{n=r-1}^{\infty}\frac{D_{r-1}(n)x^{n-r+1}}{n!}\sum_{n=0}^{\infty}\frac{(-1)^n C_n^r}{\alpha^{n-r}}\frac{x^n}{n!}$$

$$\times\sum_{n=0}^{\infty}H(n,r-1,\alpha)x^n$$

$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^{n-i}\frac{D_{r-1}(i+r-1)C_{n-i}^r}{(i+r-1)!\,(n-i)!\,\alpha^{n-i-r}}x^n$$

$$\times\sum_{n=0}^{\infty}H(n,r-1,\alpha)x^n$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{j-i}\frac{D_{r-1}(i+r-1)C_{j-i}^r H(n-j,r-1,\alpha)}{(i+r-1)!\,(j-i)!\,\alpha^{j-i-r}}x^n.$$

Thus, the desired result is given.

Theorem 5. For $n\geq 2r+1$, then

$$\sum_{i=0}^{n}(-1)^{i+r+1}\binom{n}{i}\frac{s(i,r+1)D_r(n-i)(r+1)!}{\alpha^i n!}$$

$$=\sum_{i=0}^{n}H(i-1,r,\alpha)\frac{D_{r-1}(n-i)}{(n-i)!},$$

and for $n>r$,

$$H(n,r,\alpha)$$

$$=\frac{(r+1)!}{n!}\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{i+r+1}\binom{n}{j}\binom{j}{i}\alpha^{-i}s(i,r+1)d_{j-i}.$$

Proof. By product of exponential generating functions, we have

$$\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}\frac{x^r e^{-x}}{(1-x)^{r+1}}$$

$$=\sum_{n=0}^{\infty}(-1)^{n+r+1}\alpha^{-n}s(n,r+1)(r+1)!\frac{x^n}{n!}\sum_{n=0}^{\infty}D_r(n)\frac{x^n}{n!}$$

$$=(-1)^{r+1}(r+1)!$$

$$\times\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(-1)^i s(i,r+1)\,D_r(n-i)\frac{x^n}{\alpha^i n!},$$

and from product of generating functions,

$$\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}\frac{x^r e^{-x}}{(1-x)^{r+1}}$$

$$=\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x}\frac{x^{r-1}e^{-x}}{(1-x)^r}x$$

$$=\sum_{n=0}^{\infty}H(n,r,\alpha)x^{n+1}\sum_{n=0}^{\infty}\frac{D_{r-1}(n)}{n!}x^n$$

$$=\sum_{n=0}^{\infty}H(n-1,r,\alpha)x^n\sum_{n=0}^{\infty}\frac{D_{r-1}(n)}{n!}x^n$$

$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{H(i-1,r,\alpha)D_{r-1}(n-i)}{(n-i)!}x^n.$$

Thus, comparing the coefficients on both sides, we get the desired result.

Similarly, considering

$$\sum_{n=0}^{\infty}H(n,r,\alpha)x^n$$

$$=\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x}=\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}\frac{e^{-x}}{1-x}e^x$$

$$=\sum_{n=0}^{\infty}(-1)^{n+r+1}\alpha^{-n}s(n,r+1)(r+1)!\frac{x^n}{n!}$$

$$\times\sum_{n=0}^{\infty}d_n\frac{x^n}{n!}\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$=(-1)^{r+1}(r+1)$$

$$\times\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{j}\frac{(-1)^i}{n!}\binom{n}{j}\binom{j}{i}\alpha^{-i}s(i,r+1)d_{j-i}\,x^n,$$

the proof of the other sum is obtained.

Conclusion

With the help of product of generating functions and then comparing the coefficients of x^n in the first and last series, some sums are obtained involving generalized harmonic, r-derangement, Cauchy numbers and some special numbers. In the future, it is aimed to find new sums with the help of the derivative operator.

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Conflicts of interest

There are no conflicts of interest in this work.

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