Cumhuriyet Sci. J., 45(1) (2024) 130-134
DOI: https://doi.org/10.17776/csj.1359651

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Solvability of an Inverse Problem for an Elliptic-Type Equation

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Research Article

History
Received: 13/09/2023
Accepted: 26/02/2024

ABSTRACT

In this study, we consider an inverse problem of determining an unknown source function in the right-hand side of an elliptic equation which is ill-posed in the Hadamard sense. To investigate the solvability of the problem, we reduce it to a Dirichlet problem for a third-order partial differential equation with homogeneous boundary condition. Since the problem is linear, the proof of the uniqueness theorem is based on the Fredholm Alternative Theorem. We prove the existence of the solution to the problem by using the Galerkin method.

Keywords: Elliptic equation, Inverse problem, Solvability of the problem.

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Introduction

Elliptic equations are used to describe the behavior of electromagnetic fields, the propagation of waves in a medium and the motion of fluids. They appear in a wide range of applications in various fields of science and technology such as physics, engineering and computer science.

In this paper, we consider the equation

$$Pu \equiv a(x,\bar{y})\left(u_{xx}+\Delta_y u\right)+ku_x-b(x,\bar{y})u = c(x,\bar{y})f(x,\bar{y}) \qquad (1)$$

in the domain

$$Q=\{(x,y)|\ x\in(0,1),\ y_i\in(0,1), \qquad i=1,2,\dots,n\}$$

where $y=(\bar{y},y_n)$, $\bar{y}=(y_1,y_2,\dots,y_{n-1})$. The boundary of the domain is defined as

$$\partial Q=\sigma_0\cup\{y_n=0\}\cup\{y_n=1\},$$

$$\sigma_0=\partial Q/(\{y_n=0\}\cup\{y_n=1\}).$$

The coefficients of equation (1) satisfy the following conditions:

$$a(x,\bar{y})\in C^1(\bar{Q}), \qquad b(x,\bar{y}), \qquad c(x,\bar{y})\in C(\bar{Q}),$$
$$a(x,\bar{y}),\ b(x,\bar{y}),\ a'(x,\bar{y})>0,\ k<0.$$

We consider the following problem:

Problem 1. Find the functions $u(x,y)$ and $f(x,\bar{y})$ from equation (1), provided that the following conditions are given:

$$\frac{\partial u}{\partial y_n}|_{y_n=0}=u_0,\ \frac{\partial u}{\partial y_n}|_{y_n=1}=u_1,\ u|_{\sigma_0}=u_2, \qquad (2)$$

$$u|_{y_n=0}=u_3. \qquad (3)$$

A problem is said to be well posed if a unique solution exists which depends continuously on the data. Problem 1 is not a well posed problem. Some of the typical ill-posed problems for partial differential equations are the Cauchy problem for the Laplace equation, the Dirichlet problem for the wave equation and the initial-boundary value problem for the backward heat equation, [1].

The existence, uniqueness and stability of solution of various inverse problems for elliptic, hyperbolic and parabolic equations are studied in [2-5]. As for the solvability results for some ill-posed problems for other type equations, we refer to [6-10].

The first result of this paper is given below:

Theorem 1. Problem 1 has at most one solution (u,f) such that $u\in H^3(Q)$, $f\in H^1(Q)$.

In the proof, we shall use the Fredholm Alternative Theorem and show that the homogeneous problem has only zero solution.

Proof of Theorem 1.

It is sufficient to prove for $u\in C^k(\bar{Q})$, since $C^k(\bar{Q})$ is dense in $H^k(Q)$. Let us assume that $\left(u^{(1)},f^{(1)}\right)$, $\left(u^{(2)},f^{(2)}\right)$ are two solutions of problem (1)-(3) in the space $C^3(\bar{Q})\times C^1(\bar{Q})$.

Then we can write

$$a(x,\bar{y})\left(u_{xx}{}^{(i)}+\Delta_y u^{(i)}\right)+k u_x{}^{(i)}-b(x,\bar{y})u^{(i)}=c(x,\bar{y})f^{(i)}(x,\bar{y}), \tag{4}$$

$$u_{y_n}^{(i)}|_{y_n=0}=u_0,\ u_{y_n}^{(i)}|_{y_n=1}=u_1,\ u^{(i)}|_{\sigma_0}=u_2, \tag{5}$$

$$u^{(i)}|_{y_n=0}=u_3,\ i=1,2. \tag{6}$$

For $\tilde{u}=u^{(2)}-u^{(1)},\ \tilde{f}=f^{(2)}-f^{(1)}$, we have

$$a(x,\bar{y})\left(\tilde{u}_{xx}+\Delta_y\tilde{u}\right)+k\tilde{u}_x-b(x,\bar{y})\tilde{u}=c(x,\bar{y})\tilde{f}(x,\bar{y}) \tag{7}$$

$$\tilde{u}_{y_n}|_{y_n=0}=0,\ \tilde{u}_{y_n}|_{y_n=1}=0,\ \tilde{u}|_{\sigma_0}=0, \tag{8}$$

$$\tilde{u}|_{y_n=0}=0,\ i=1,2. \tag{9}$$

Taking the derivative of both sides of equation (7) with respect to y_n yields to

$$P\tilde{u}_{y_n}=0 \tag{10}$$

and taking $\tilde{u}_{y_n}=\hat{u}$, we obtain

$$P\hat{u}=0, \tag{11}$$

$$\hat{u}_{y_n}|_{y_n=0}=0,\ \hat{u}_{y_n}|_{y_n=1}=0,\ \hat{u}|_{\sigma_0}=0. \tag{12}$$

Dividing both sides of equation (11) by $a(x,\bar{y})$, we get

$$\hat{u}_{xx}+\Delta_y\hat{u}+\frac{k}{a(x,\bar{y})}\hat{u}_x-\frac{b(x,\bar{y})}{a(x,\bar{y})}\hat{u}=0, \tag{13}$$

$$\hat{u}|_{\partial Q}=0. \tag{14}$$

Multiplying equation (13) by $-\hat{u}$ and using the equalities

$$-\hat{u}_{xx}\hat{u}=-(\hat{u}_x\hat{u})_x+\hat{u}_x^2,$$
$$-\hat{u}_{y_iy_i}\hat{u}=-(\hat{u}_{y_i}\hat{u})_{y_i}+\hat{u}_{y_i}^2,\qquad i=1,2,\dots,n$$
$$-\hat{u}\frac{k}{a(x,\bar{y})}\hat{u}_x=-\frac{1}{2}\left(\frac{k}{a(x,\bar{y})}\hat{u}^2\right)_x-\frac{1}{2}\frac{k}{a'(x,\bar{y})}\hat{u}^2,$$

we see that

$$-\hat{u}P\hat{u}=\hat{u}_x^2+\sum_{i=1}^n\hat{u}_{y_i}^2+\left(\frac{b(x,\bar{y})}{a(x,\bar{y})}-\frac{1}{2}\frac{k}{a'(x,\bar{y})}\right)\hat{u}^2$$
$$-(\hat{u}_x\hat{u})_x-\sum_{i=1}^n(\hat{u}_{y_i}\hat{u})_{y_i}-\frac{1}{2}\left(\frac{k}{a(x,\bar{y})}\hat{u}^2\right)_x. \tag{15}$$

Since $a(x,\bar{y}),\ b(x,\bar{y}),\ a'(x,\bar{y})>0$ and $k<0$, we have

$$\hat{u}_x^2+\sum_{i=1}^n\hat{u}_{y_i}^2+\left(\frac{b(x,\bar{y})}{a(x,\bar{y})}-\frac{1}{2}\frac{k}{a'(x,\bar{y})}\right)\hat{u}^2>0.$$

If we integrate equality (15) over Q, we get

$$\int_Q\left[(\hat{u}_x\hat{u})_x+\sum_{i=1}^n(\hat{u}_{y_i}\hat{u})_{y_i}+\frac{1}{2}\left(\frac{k}{a(x,\bar{y})}\hat{u}^2\right)_x\right]dQ=\int_Q\left(\hat{u}_x^2+\sum_{i=1}^n\hat{u}_{y_i}^2+\left(\frac{b(x,\bar{y})}{a(x,\bar{y})}-\frac{1}{2}\frac{k}{a'(x,\bar{y})}\right)\hat{u}^2\right)dQ. \tag{16}$$

From the Ostrogradsky formula, we can write

$$\int_Q\left(\hat{u}_x^2+\sum_{i=1}^n\hat{u}_{y_i}^2+\left(\frac{b(x,\bar{y})}{a(x,\bar{y})}-\frac{1}{2}\frac{k}{a'(x,\bar{y})}\right)\hat{u}^2\right)dQ=\int_Q\left\{\left[(\hat{u}_x\hat{u})+\frac{1}{2}\left(\frac{k}{a(x,\bar{y})}\hat{u}^2\right)\right]n_x+\sum_{i=1}^n(\hat{u}_{y_i}\hat{u})\,n_{y_i}\right\}dS.$$

By $a(x,\bar{y}),\ b(x,\bar{y}),\ a'(x,\bar{y})>0,\ k<0$ and $\hat{u}|_{\sigma_0}=0$, we obtain

$$\int_Q\left(\hat{u}_x^2+\sum_{i=1}^n\hat{u}_{y_i}^2+\left(\frac{b(x,\bar{y})}{a(x,\bar{y})}-\frac{1}{2}\frac{k}{a'(x,\bar{y})}\right)\hat{u}^2\right)dQ=0 \tag{17}$$

which means that

$\hat{u}_x = 0,\ \hat{u}_{y_i} = 0$ and $\hat{u}|_{\sigma_0} = 0$ and thus $\hat{u} = 0$ in the domain Q. Moreover, $\tilde{u} = 0$ in Q from the equality $\tilde{u}_{y_n} = \hat{u}$ and condition (9). On the other hand, we see that $\tilde{f} = 0$ in Q from equation (7). Thus, $\tilde{u} = u^{(2)} - u^{(1)} = 0$ and $\tilde{f} = f^{(2)} - f^{(1)} = 0$ where imply $u^{(1)} = u^{(2)}$ and $f^{(1)} = f^{(2)}$. Therefore, Theorem 1 is proven.

In order to prove the existence of the solution of the problem, we reduce the problem to a homogeneous Dirichlet problem.

Taking the derivative of both sides of equation (1) with respect to y_n, we see that

$$Pu_{y_n} = 0. \tag{18}$$

In conditions (2), if we take $u_{y_n} = \hat{u}$, we can write

$$P\hat{u} = 0, \tag{19}$$

$$\hat{u}|_{y_n=0} = u_0,\ \hat{u}|_{y_n=1} = u_1,\ \hat{u}|_{\sigma_0} = u_{2y_n}. \tag{20}$$

Conditions (20) can be written in the form

$$\hat{u}|_{\partial Q} = \tilde{u}_0. \tag{21}$$

Assume that the functions u_0, u_1, u_2 are smooth enough on the boundary of the domain and

$$w \in C^2(\bar{Q}),\ w|_{\partial Q} = \tilde{u}_0.$$

With the help of the new unknown function $v = \hat{u} - w$, we have

$$-w_{xx} - \Delta_y w - \frac{k}{a(x,\bar{y})} w_x + \frac{b(x,\bar{y})}{a(x,\bar{y})} w = F(x,\bar{y})$$

and problem (19)-(21) becomes

$$Pv \equiv v_{xx} + \Delta_y v + \frac{k}{a(x,\bar{y})} v_x - \frac{b(x,\bar{y})}{a(x,\bar{y})} v = F(x,\bar{y}), \tag{22}$$

$$v|_{\partial Q} = 0. \tag{23}$$

Theorem 2. Assume that $\partial Q \in C^2(\bar{Q})$ and $F \in L^2(Q)$. Then problem (22)-(23) has a generalized solution in the Hilbert space $H_0^1(Q)$.

Proof of Theorem 2.

We first take an a priori estimate for the solution of problem (22)-(23). If we multiply (22) by $-v$, integrate over the domain Q and using the equalities (15)-(16), we obtain

$$\int_Q (v_x^2 + \sum_{i=1}^n v_{y_i}^2 + (\frac{b(x,\bar{y})}{a(x,\bar{y})} - \frac{1}{2}\frac{k}{a'(x,\bar{y})}) v^2) dQ = -\int_Q Fv dQ. \tag{24}$$

From the Cauchy-Bunyakovskii inequality, we get

$$-\int_Q Fv dQ \le \int_Q \left|\beta F \frac{1}{\beta} v\right| dQ \le \int_Q \beta^2 F^2 dQ + \int_Q \frac{1}{\beta^2} v^2 dQ.$$

By equality (24), we see that

$$\int_Q (v_x^2 + \int_Q (\sum_{i=1}^n v_{y_i}^2 + (\frac{b(x,\bar{y})}{a(x,\bar{y})} - \frac{1}{2}\frac{k}{a'(x,\bar{y})}) v^2) dQ - \frac{1}{\beta^2}\int_Q v^2 dQ \le \beta^2 \int_Q F^2 dQ. \tag{25}$$

Moreover, we apply the Rellich-Poincare inequality to the first term on the left side of inequality (25). Since

$$\int_Q v_x^2 dQ \ge C \int_Q v^2 dQ,\ \ (C = diamQ)$$

and by taking

$$C\int_Q v^2 dQ + \int_Q (\sum_{i=1}^n v_{y_i}^2 + (\frac{b(x,\bar{y})}{a(x,\bar{y})} - \frac{1}{2}\frac{k}{a'(x,\bar{y})}) v^2) dQ - \frac{1}{\beta^2}\int_Q v^2 dQ \le \beta^2 \int_Q F^2 dQ,$$

$$(C - \frac{1}{\beta^2}) \int_Q v^2 dQ + \int_Q (\sum_{i=1}^n v_{y_i}^2 + (\frac{b(x,\bar{y})}{a(x,\bar{y})} - \frac{1}{2}\frac{k}{a'(x,\bar{y})}) v^2) dQ \le \beta^2 \int_Q F^2 dQ,$$

we have

$$C_1 \int_Q v^2 dQ + \int_Q (\sum_{i=1}^n v_{y_i}^2 + (\frac{b(x,\bar{y})}{a(x,\bar{y})} - \frac{1}{2}\frac{k}{a'(x,\bar{y})}) v^2) dQ \le \beta^2 \int_Q F^2 dQ, \tag{26}$$

where $C_1 = C - \frac{1}{\beta^2} > 0$. Inequality (26) is an a priori estimate which we look for.

We apply the Galerkin method to problem (22)-(23). Let the functions $w_1(x,y), w_2(x,y), \dots, w_n(x,y), \dots$ be linearly independent and complete system of functions in $L^2(Q)$.

We assume that $w_i(x,y) = 0$ on ∂Q ($i = 1,2,\dots$) and $w_i(x,y) \in C^2(\bar{Q})$. There exists a system $\{w_1, w_2, \dots, w_n, \dots\}$ such that

$u_N(x,y) = \sum_{i=1}^{N} c_i w_i(x,y)$.

Since $w_i|_{\partial Q} = 0$, then we have $u_N|_{\partial Q} = 0$. Now, let us obtain the function $u_N(x,y)$ from the system of equations

$$\langle u_{N_{xx}} + \Delta_y u_N + \frac{k}{a(x,\bar{y})} u_{Nx} - \frac{b(x,\bar{y})}{a(x,\bar{y})} u_N, w_j(x,y)\rangle = \langle F(x,\bar{y}), w_j(x,y)\rangle,$$

$$\langle u_{N_{xx}} + \Delta_y u_N + \frac{k}{a(x,\bar{y})} u_{Nx} - \frac{b(x,\bar{y})}{a(x,\bar{y})} u_N - F, w_j(x,y)\rangle = 0, \ j = \overline{1,N}. \tag{27}$$

We will show that u_N which is the solution of system (27) converges to the exact solution of problem (22)-(23) when $N \to \infty$. Then we can write

$$\langle \sum_{i=1}^{N} (c_i w_i)_{xx} + \Delta_y \sum_{i=1}^{N} c_i w_i + \frac{k}{a(x,\bar{y})} \sum_{i=1}^{N} (c_i w_i)_x - \frac{b(x,\bar{y})}{a(x,\bar{y})} \sum_{i=1}^{N} c_i w_i - F, w_j(x,y)\rangle = 0, \ j = \overline{1,N}. \tag{28}$$

We will prove the homogeneous system

$$\langle \sum_{i=1}^{N} (c_i w_i)_{xx} + \Delta_y \sum_{i=1}^{N} c_i w_i + \frac{k}{a(x,\bar{y})} \sum_{i=1}^{N} (c_i w_i)_x - \frac{b(x,\bar{y})}{a(x,\bar{y})} \sum_{i=1}^{N} c_i w_i, w_j\rangle = 0, \ j = \overline{1,N} \tag{29}$$

has only the zero solution. If we multiply the jth equation of system (29) by $-c_j$ and add from 1 to N, we obtain

$$\langle u_{N_{xx}} + \Delta_y u_N + \frac{k}{a(x,\bar{y})} u_{Nx} - \frac{b(x,\bar{y})}{a(x,\bar{y})} u_N, -u_N\rangle = 0.$$

Then, from equality (17), we have

$$\int_Q (u_{Nx}^2 + \sum_{i=1}^{n} u_{Ny_i}^2 + (\frac{b(x,\bar{y})}{a(x,\bar{y})} - \frac{1}{2}\frac{k}{a'(x,\bar{y})}) u_N^2) dQ = 0.$$

In this case, $u_N = 0$ in the domain Q and therefore, we write

$$\sum_{i=1}^{N} c_i w_i(x,y) = 0.$$

Since $w_i(x,y)$ is linearly independent, we have $c_i = 0, (i = 1,2,\dots,N)$. Thus, the homogeneous system (28) has only zero solution. Therefore, there is only one solution for every $F(x,y)$ of system (28).

Now, we take an a priori estimate for $u_N(x,y)$. For this, if we multiply the jth equation of system (28) by $-c_j$ and add from 1 to N, we obtain

$$\langle u_{N_{xx}} + \Delta_y u_N + \frac{k}{a(x,\bar{y})} u_{Nx} - \frac{b(x,\bar{y})}{a(x,\bar{y})} u_N - F, -u_N\rangle = 0,$$

$$\langle u_{N_{xx}} + \Delta_y u_N + \frac{k}{a(x,\bar{y})} u_{Nx} - \frac{b(x,\bar{y})}{a(x,\bar{y})} u_N, -u_N\rangle = \langle F, -u_N\rangle. \tag{30}$$

By inequality (26), we see that

$$C_1 \int_Q u_N^2 dQ + \int_Q \left(\sum_{i=1}^{n} u_{Ny_i}^2 + \left(\frac{b(x,\bar{y})}{a(x,\bar{y})} - \frac{1}{2}\frac{k}{a'(x,\bar{y})}\right) u_N^2\right) dQ \le \beta^2 \int_Q F^2 dQ,$$

$$C_1 \int_Q u_N^2 dQ \le \beta^2 \int_Q F^2 dQ \tag{31}$$

and

$$\int_Q (\sum_{i=1}^{n} u_{Ny_i}^2 + (\frac{b(x,\bar{y})}{a(x,\bar{y})} - \frac{1}{2}\frac{k}{a'(x,\bar{y})}) u_N^2) dQ \le \beta^2 \int_Q F^2 dQ. \tag{32}$$

Since the right sides of inequalities (31) and (32) are independent of N, there is a constant C_2 independent of N such that

$$\int_Q u_N^2 dQ \le C_2. \tag{33}$$

Similarly, we get

$$\int_Q \sum_{i=1}^{n} u_{Ny_i}^2 \ dQ \le C_2, \ \int_Q (\frac{b(x,\bar{y})}{a(x,\bar{y})} - \frac{1}{2}\frac{k}{a'(x,\bar{y})}) u_N^2) dQ \le C_2. \tag{34}$$

Then, the sequence $\{u_N\}$ is bounded in the Hilbert space $H_0^1(Q)$. Since a bounded set in a Hilbert space is weakly compact, there is a weakly convergent subsequence of $\{u_N\}$. Let this subsequence be $\{u_N\}$ for the sake of simplicity. We can write system (28) in the form

$$\langle u_{N_{xx}} + \Delta_y u_N + \frac{k}{a(x,\bar{y})} u_{Nx} - \frac{b(x,\bar{y})}{a(x,\bar{y})} u_N, w_j\rangle = \langle F, w_j\rangle, \ j = \overline{1,N}. \tag{35}$$

By using the equalities

$$\Delta_y u_N \cdot w_j = \int_Q \Delta_y u_N w_j \ dQ = \int_Q \sum_{i=1}^{n} u_{Ny_i y_i} w_j \ dQ,$$

$u_{N_{xx}} \cdot w_j = (u_{N_x} w_j)_x - u_{N_x} w_{jx},$
$u_{Ny_iy_i} \cdot w_j = (u_{Ny_i} w_j)_{y_i} - u_{Ny_i} w_{jy_i},$
$\frac{k}{a(x,\bar{y})} u_{Nx} \cdot w_j = (\frac{k}{a(x,\bar{y})} u_N w_j)_x - \frac{k}{a(x,\bar{y})} u_N w_{jx} - \frac{k}{a'(x,\bar{y})} u_N w_j,$
$\langle u_{N_{xx}}, w_j \rangle = -\int_Q u_{N_x} w_{jx} \; dQ,$
$\langle \Delta_y u_N, w_j \rangle = -\int_Q \sum_{i=1}^n u_{Ny_i} w_{jy_i} \; dQ,$
$\langle \frac{k}{a(x,\bar{y})} u_{Nx}, w_j \rangle = -\int_Q \frac{k}{a(x,\bar{y})} u_N w_{jx} \; dQ - \int_Q \frac{k}{a'(x,\bar{y})} u_N w_j \; dQ$
in (35), we obtain
$-\int_Q u_{N_x} w_{jx} \; dQ - \int_Q \sum_{i=1}^n u_{Ny_i} w_{jy_i} \; dQ - \int_Q \frac{k}{a(x,\bar{y})} u_N w_{jx} \; dQ$
$-\int_Q \frac{k}{a'(x,\bar{y})} u_N w_j \; dQ - \int_Q \frac{k}{a(x,\bar{y})} u_N w_{jx} \; dQ - \int_Q \frac{b(x,\bar{y})}{a(x,\bar{y})} u_N \; w_j dQ$
$= \int_Q F \; w_j dQ.$

Taking the limit for $N \to \infty$, in the sense of the generalized function ($u_N \overset{H_0^1(Q)}{\rightharpoonup} v$ converges), we get

$$\langle v, P^* w_j \rangle = \langle F, w_j \rangle \tag{36}$$

or

$$\langle Pv - F, w_j \rangle = 0. \tag{37}$$

Since the system $\{w_j\}$ in (37) is complete in $L^2(Q)$, we can write

$$Pv - F = 0.$$

Then, v is a solution to equation (22). By $u_N \overset{H_0^1(Q)}{\rightharpoonup} v$ and $u_N|_{\sigma_0} = 0$, we have $u|_{\sigma_0} = 0$. Thus, problem (22)-(23) has a generalized solution in the space $H_0^1(Q)$.

Finally, by (32), we have

$$\int_Q (\sum_{i=1}^n v_{y_i}^2 + (\frac{b(x,\bar{y})}{a(x,\bar{y})} - \frac{1}{2} \frac{k}{a'(x,\bar{y})}) v^2) dQ \; \leq \beta^2 \int_Q F^2 dQ,$$

as $N \to \infty$ which show that the solution depends continuously on the data.

Conclusion

In this study, we deal with an inverse problem for an elliptic equation. We prove the uniqueness, existence and stability of the solution of the problem. Our main tools are Fredholm Alternative Theorem and Galerkin method.

Acknowledgments

The authors would like to sincerely thank the editor and the reviewers for the helpful comments and suggestions.

Conflicts of interest

There are no conflicts of interest in this work.

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