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Research Article

Kähler-Norden structures on Hom-Lie groups and Hom-Lie algebras

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Abstract

The aim of this paper is to describe two geometric notions, holomorphic Norden structures and Kähler-Norden structures on Hom-Lie groups, and study their relationships in the left invariant setting. We study Kähler-Norden structures with abelian complex structures and give the curvature properties of holomorphic Norden structures on Hom-Lie groups. Finally, we show that any left-invariant holomorphic Hom-Lie group is a flat (holomorphic Norden Hom-Lie algebra carries a Hom-Left-symmetric algebra) if its left-invariant complex structure (complex structure) is abelian.

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1. Introduction

An almost complex structure on a $2n$-dimensional manifold M is a $(1,1)$ tensor field J satisfying $J^2 = -Id$. A Kähler-Norden manifold can be considered as a triple (M, g, J) consisting of a smooth manifold M endowed with a Riemannian metric g and an almost complex structure J such that $\nabla J = 0$, where ∇ is the Levi-Civita connection of g and the metric g is assumed to be anti-Hermitian (Norden), i.e., $g(JX, JY) = -g(X, Y)$ for all vector fields X and Y on M. This kind of manifolds have been studied under the names of Kählerian manifolds with Norden (or B) metrics or anti-Kähler manifolds (see [7, 8, 26]). The group structure of Kähler-Norden manifolds is the complex orthogonal group $O(n, \mathbb{C})$ and these manifolds have recently become an important area of research, mainly because of their applications in theoretical physics. For example Andrzej Borowiec, Mauro Francaviglia, Marco Ferraris and Igor Volovich proved that there is a one-to-one correspondence between (Einstein) holomorphic-Riemannian manifolds and (Einstein) anti-Kählerian manifolds [1, 2]. The curvature properties of anti-Kähler-manifolds have been studied in [11, 20, 27]. Other strong motivation to study anti-Kähler manifolds comes from

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the spinor geometry and geometric analysis on anti-Kähler manifolds and the generalized geometry of anti-Hermitian manifolds as complex Lie algebroids over anti-Kähler manifolds [4,5,18]. When the manifold is a Lie group G, the metric and the complex structure are considered left-invariant, where they are both determined by their restrictions to the Lie algebra $\mathfrak{g}$ of G. In this situation, $(\mathfrak{g}, g_e, J_e)$ is called an anti-Kähler Lie algebra. Anti-Kähler geometry on Lie groups have been studied by Edison Alberto Fernández-Culma and Yamile Godoy [6].

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov in the study of σ-deformations of the Witt and Virasoro algebras in [9]. Indeed, some q-deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra, the Jacobi identity is twisted by a homomorphism, [9, 10]. Based on the close relation between the discrete, deformed vector fields and differential calculus, this algebraic structure plays an important role in various research fields [9,13,15–17,21]. Recently, many algebraic and geometric problems related to Hom-Lie algebras were raised and studied, such as infinite-dimensional Hom-Lie algebras, the classical Hom-Yang-Baxter equation [25], para-Kähler and complex and Kähler structures on Hom-Lie algebras [21,22], complex product structures on Hom-Lie algebras and Hom-left symmetric algebroids [19, 23] and Hom-Novikov algebras [28].

The concept of Hom-groups was introduced first by Laurent-Gengoux, Makhlouf and Teles as a non-associative analogue of a group in [14]. Then J. Jiang, S. Mishra, and Y. Sheng modified the axioms in the definition of Hom-group which is different from the one in [14], where is showed that if the structure map is invertible, then some of the axioms in original definition are redundant and can be obtained from the Hom-associativity condition. Also, they introduced the notion of a Hom-Lie group in [12]. In fact, a Hom-Lie group is considered as a Hom-group $(G, \diamond, e_\Phi, \Phi)$, where the underlying set G is a (real) smooth manifold, the Hom-group operations (such as the product and the inverse) are smooth maps, and the underlying structure map $\Phi : G \to G$ is a diffeomorphism. Also a Hom-Lie algebra is associated to a Hom-Lie group by considering the notion of left-invariant sections of the pullback bundle $\Phi^! TG$.

This work is intended as an attempt to study two geometric notions, holomorphic Norden structures and Kähler-Norden structures on Hom-Lie groups. A relationship is established in the left invariant setting. Moreover, we study Kähler-Norden structures with abelian complex structures and give the curvature properties of holomorphic Norden structures on Hom-Lie groups.

The paper is organized as follows. In Section 2, we recall the basics about Hom-algebras, Hom-Lie algebras, Hom-groups and Hom-Lie groups. In Section 3, we study complex and Norden structures, pseudo-Riemannian metric and connections on Hom-Lie groups and Hom-Lie algebras. Also, we give examples of these structures. Then we present the notion of a holomorphic tensor on them. In Section 4, we introduce left-invariant Kähler-Norden structures on Hom-Lie groups and Kähler-Norden structures on Hom-Lie algebras and provide examples of these structures. We also study Kähler-Norden structures on Hom-Lie groups and Hom-Lie algebras that complex structures are abelian. Moreover, we describe Twin Norden metric of Hom-Lie groups and Hom-Lie algebras. In Section 5, we introduce the notion of left-invariant holomorphic Norden Hom-Lie group (a holomorphic Norden Hom-Lie algebra). We show that there exists a one-to-one correspondence between left invariant Kähler-Norden Hom-Lie groups (Kähler-Norden Hom-Lie algebras) and left-invariant holomorphic Norden Hom-Lie groups (holomorphic Norden Hom-Lie algebras). Finally, in Section 6 we provide some properties of the Riemannian curvature tensor of a left-invariant holomorphic Norden Hom-Lie group (a holomorphic Norden Hom-Lie algebra). Also, we show that any left-invariant holomorphic Hom-Lie group is flat (holomorphic Norden Hom-Lie algebra carries a Hom-Left-symmetric algebra) if its left-invariant complex structure (complex structure) is abelian.

In this paper, we work over the real field $\mathbb{R}$ and the complex field $\mathbb{C}$, which we denote by $\mathbb{K}$.

2. Basic concepts on Hom-Lie groups

Definition 2.1. [21] A Hom-algebra is a triple $(V, \cdot, \phi_V)$ consisting of a linear space V, a bilinear map (product) $\cdot : V \times V \to V$ and an algebra morphism $\phi_V : V \to V$.

Definition 2.2. [15] A Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ consisting of a linear space $\mathfrak{g}$, a bilinear map (bracket) $[\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ and an algebra morphism $\phi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ satisfying the following conditions:

$$[u, v]_{\mathfrak{g}} = -[v, u]_{\mathfrak{g}}, \qquad \circlearrowleft_{u,v,w} [\phi_{\mathfrak{g}}(u), [v, w]_{\mathfrak{g}}]_{\mathfrak{g}} = 0,$$

for any $u, v, w \in \mathfrak{g}$, where $\circlearrowleft$ is the symbol of cyclic summation. The second equation is called Hom-Jacobi identity. The Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}})$ is called regular (involutive), if $\phi_{\mathfrak{g}}$ is non-degenerate (satisfies ${\phi_{\mathfrak{g}}}^2 = Id_{\mathfrak{g}}$).

The Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ is called *regular Hom-Lie algebra* (respectively, *involutive Hom-Lie algebra*), if $\phi_{\mathfrak{g}}$ is non-degenerate (respectively, satisfies ${\phi_{\mathfrak{g}}}^2 = Id_{\mathfrak{g}}$). It is known that a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ with $\phi_{\mathfrak{g}} = Id_{\mathfrak{g}}$ is a Hom-Lie algebra. We call $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ *proper Hom-Lie algebra* if $\phi_{\mathfrak{g}} \neq Id_{\mathfrak{g}}$.

Definition 2.3. [12] A Hom-group is a set G equipped with a product $\diamond : G\times G \to G$, a bijective map $\Phi : G \to G$ such that the following axioms are satisfied:
(i) $\Phi(a \diamond b) = \Phi(a) \diamond \Phi(b)$,
(ii) the product is Hom-associative, i.e., $\Phi(a) \diamond (b \diamond c) = (a \diamond b) \diamond \Phi(c)$,
(iii) there exists a unique Hom-unit $e_\Phi \in G$ such that $a \diamond e_\Phi = e_\Phi \diamond a = \Phi(a)$,
(iv) for each $a \in G$ there exists an element a^{-1} satisfying the condition
$a \diamond a^{-1} = a^{-1} \diamond a = e_\Phi$,
for any $a, b, c \in G$. We denote a Hom-group by $(G, \diamond, e_\Phi, \Phi)$.

Definition 2.4. [12] A real Hom-Lie group is a Hom-group $(G, \diamond, e_\Phi, \Phi)$, in which G is also a smooth real manifold, the map $\Phi : G \to G$ is a diffeomorphism, and the Hom-group operations (product and inversion) are smooth maps.

Let $(G, \diamond, e_\Phi, \Phi)$ be a Hom-Lie group. The pullback map $\Phi^* : C^\infty(G) \to C^\infty(G)$ is a morphism of the function ring $C^\infty(G)$, i.e.

$$\Phi^*(fg) = \Phi^*(f)\Phi^*(g), \quad \forall f, g \in C^\infty(G).$$

Let $A \to G$ be a vector bundle of rank n. Denote by $\Gamma(A)$ the $C^\infty(G)$-module of sections of $A \to G$. A *Hom-bundle* is a triple $(A \to G, \Phi, \phi_A)$ consisting of a vector bundle $A \to G$, a smooth map $\Phi : G \to G$ and an algebra morphism $\phi_A : \Gamma(A) \to \Gamma(A)$ satisfying

$$\phi_A(fx) = \Phi^*(f)\phi_A(x),$$

for any $x \in \Gamma(A)$ and $f \in C^\infty(G)$ (in this case, ϕ_A is called a Φ^*-function linear). Consider the tangent bundle TG of the manifold G, we denote by $\Phi^!TG$, the pullback bundle of the tangent bundle TG along the diffeomorphism $\Phi : G \to G$. The triple $(\Phi^!TG, \Phi, Ad_{\Phi^*})$ is an example of Hom-bundles, where $Ad_{\Phi^*} : \Gamma(\Phi^!TG) \to \Gamma(\Phi^!TG)$ is given by

$$Ad_{\Phi^*}(x) = \Phi^* \circ x \circ (\Phi^*)^{-1},$$

for any $x \in \Gamma(\Phi^!TG)$ [3]. Note that $\Gamma(\Phi^!TG)$ can be identified with $Der_{\Phi^*,\Phi^*}(C^\infty(G))$, i.e.

$$x(fg) = x(f)\Phi^*(g) + \Phi^*(f)x(g), \qquad \forall x \in \Gamma(\Phi^!TG), \forall f, g \in C^\infty(G).$$

The linear map $Ad_{\Phi^*}:\Gamma(\Phi^!TG)\to\Gamma(\Phi^!TG)$ given in a Hom-bundle $(\Phi^!TG\to G,\Phi,Ad_{\Phi^*})$ can be extended to a linear map from $\Gamma(\wedge^k\Phi^!TG)$ to $\Gamma(\wedge^k\Phi^!TG)$ for which we use the same notation Ad_{Φ^*} via

$$Ad_{\Phi^*}(x)=Ad_{\Phi^*}(x_1)\wedge\ldots\wedge Ad_{\Phi^*}(x_k),\qquad \forall x=x_1\wedge\ldots\wedge x_k\in\Gamma(\wedge^k\Phi^!TG).$$

If we denote the inverses of Φ and Ad_{Φ^*} by Φ^{-1} and $Ad_{(\Phi^*)^{-1}}$, respectively, it is easy to see that

$$Ad_{(\Phi^*)^{-1}}(fX)=(\Phi^*)^{-1}(f)Ad_{(\Phi^*)^{-1}}(X),\quad \forall f\in C^\infty(G),x\in\Gamma(\wedge^k\Phi^!TG).$$

So $(\Phi^!TG,\Phi^{-1},Ad_{(\Phi^*)^{-1}})$ is a Hom-bundle. In the sequel, we denote by $\Phi^!T^*G$ the dual of the pullback bundle $\Phi^!TG$. We consider $Ad^\dagger_{\Phi^*}:\Gamma(\wedge^k\Phi^!T^*G)\to\Gamma(\wedge^k\Phi^!T^*G)$ that is defined by

$$(Ad^\dagger_{\Phi^*}(\xi))(x)=\Phi^*\xi(Ad_{(\Phi^*)^{-1}}(x)),\qquad \forall x\in\Gamma(\wedge^k\Phi^!TG),\xi\in\Gamma(\wedge^k\Phi^!T^*G).$$

From the above equation, we can conclude the following

$$Ad^\dagger_{\Phi^*}(f\xi)=\Phi^*(f)Ad^\dagger_{\Phi^*}(\xi).$$

Therefore $(\wedge^k\Phi^!T^*G,\Phi,Ad^\dagger_{\Phi^*})$ is a Hom-bundle.

Theorem 2.5. *Let G be a smooth manifold. Then $(\Gamma(\Phi^!TG),[\cdot,\cdot]_\Phi,Ad_{\Phi^*})$ is a Hom-Lie algebra, where the Hom-Lie bracket $[\cdot,\cdot]_\Phi$ and the map $\phi:\Gamma(\Phi^!TG)\to\Gamma(\Phi^!TG)$ are defined as follows:*

$$\begin{aligned}\phi(x)&=Ad_{\Phi^*}(x)=\Phi^*\circ x\circ(\Phi^{-1})^*,\\ [x,y]_\Phi&=\Phi^*\circ x\circ(\Phi^{-1})^*\circ y\circ(\Phi^{-1})^*-\Phi^*\circ y\circ(\Phi^{-1})^*\circ x\circ(\Phi^{-1})^*,\end{aligned}$$

for any $x,y\in\Gamma(\Phi^!TG)$.

The above Theorem is Theorem 3.6 of [12], in which the authors consider the following relations for Ad_{Φ^*} and $[\cdot,\cdot]_\Phi$:

$$\begin{aligned}\phi(x)=&(\Phi^{-1})^*\circ x\circ\Phi^*=Ad_{(\Phi^{-1})^*},\\ [x,y]_\Phi=&(\Phi^{-1})^*\circ x\circ(\Phi^{-1})^*\circ y\circ\Phi^*-(\Phi^{-1})^*\circ y\circ(\Phi^{-1})^*\circ x\circ\Phi^*.\end{aligned}$$

Definition 2.6. [12] Let $(G,\diamond,e_\Phi,\Phi)$ be a Hom-Lie group. A smooth section $x\in\Gamma(\Phi^!TG)$ is called left-invariant if x satisfies the following condition

$$x_a=(l_a\circ\Phi^{-1})_{*e_\Phi}(x_{e_\Phi}),$$

or

$$x(f)(a)=x(f\circ l_a\circ\Phi^{-1})(e_\Phi),$$

for any $a\in G$ and $f\in C^\infty(G)$, where $l_a:G\to G$ is a smooth map given by $l_ab=a\diamond b$ for any $b\in G$.

Theorem 2.7. [12] *The space $\Gamma_L(\Phi^!TG)$ of left-invariant sections of the pullback bundle $\Phi^!TG$ is a Hom-Lie subalgebra of the Hom-Lie algebra $(\Gamma(\Phi^!TG),[\cdot,\cdot]_\Phi,\phi)$.*

Remark 2.8. Let $(G,\diamond,e_\Phi,\Phi)$ be a Hom-Lie group and $\mathfrak{g}^!$ be the fibre of e_Φ in the pullback bundle $\Phi^!TG$. Then $\Phi^!T_{e_\Phi}G=\mathfrak{g}^!$ and also $\mathfrak{g}^!$ is in one-to-one correspondence with $\Gamma_L(\Phi^!TG)$ (see Lemma 3.11 in [12], for more details). Moreover, define a bracket $[\cdot,\cdot]_{\mathfrak{g}^!}$ and a vector space isomorphism $\phi_{\mathfrak{g}^!}:\mathfrak{g}^!\to\mathfrak{g}^!$ by

$$\begin{aligned}[x(e_\Phi),y(e_\Phi)]_{\mathfrak{g}^!}=&[x,y]_\Phi(e_\Phi),\\ \phi_{\mathfrak{g}^!}(x(e_\Phi))=&(\phi(x))(e_\Phi),\end{aligned}$$

for all $x,y\in\Gamma_L(\Phi^!TG)$. The triple $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!})$ is a Hom-Lie algebra that is isomorphic to the Hom-Lie algebra $(\Gamma_L(\Phi^!TG),[\cdot,\cdot]_\Phi,\phi)$.

Definition 2.9. Let $(G, \diamond, e_\Phi, \Phi)$ be a Hom-Lie group. A smooth section $\omega \in \Gamma(\Phi^! T^* G)$ is called left-invariant if

$$\omega_a = (l_a \circ \Phi^{-1})^*{}_{e_\Phi}(\omega_{e_\Phi}),$$

for any $a \in G$, where $((l_a \circ \Phi^{-1})^* \omega)(x) = \omega((l_a \circ \Phi^{-1})_* x)$. We denote by $\Gamma_L(\Phi^! T^* G)$, the space of these left-invariant sections.

A *tensor* T *of type* (p, q) *or a* (p, q)*-tensor* on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$ is a Φ^*-function linear mapping

$$T : \underbrace{\Gamma(\Phi^! T^* G) \times \cdots \times \Gamma(\Phi^! T^* G)}_{p} \times \underbrace{\Gamma(\Phi^! TG) \times \cdots \times \Gamma(\Phi^! TG)}_{q} \to C^\infty(G).$$

So, considering $y_1, \ldots, y_q \in \Gamma(\Phi^! TG)$ and $\theta^1, \cdots, \theta^p \in \Gamma(\Phi^! T^* G)$, $T(\theta^1, \cdots, \theta^p, y_1, \cdots, y_q)$ is a differentiable function on G and that T is Φ^*-function linear in each argument, i.e.

$$\begin{aligned} T(\theta^1, \cdots, \theta^p, y_1, \cdots, fx + y, \cdots, y_q) =& \Phi^*(f) T(\theta^1, \cdots, \theta^p, y_1, \cdots, x, \cdots, y_q) \\ &+ T(\theta^1, \cdots, \theta^p, y_1, \cdots, y, \cdots, y_q), \end{aligned}$$

and

$$\begin{aligned} T(\theta^1, \cdots, f\theta + \eta \cdots, \theta^p, y_1, \cdots, y_q) =& \Phi^*(f) T(\theta^1, \cdots, \theta \cdots, \theta^p, y_1, \cdots, y_q) \\ &+ T(\theta^1, \cdots, \eta \cdots, \theta^p, y_1, \cdots, y_q), \end{aligned}$$

for any $x, y \in \Gamma(\Phi^! TG)$ and $\theta, \eta \in \Gamma(\Phi^! T^* G)$. We denote the set of tensors of type (p, q) by $\mathcal{T}^p_q(\Gamma(\Phi^! TG))$. Also, the set of tensors of type (p, q) on $(\mathfrak{g}^!, [\cdot, \cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!})$ is denoted by $\mathcal{T}^p_q(\mathfrak{g}^!)$.

Lemma 2.10. *Let* $(G, \diamond, e_\Phi, \Phi)$ *be a Hom-Lie group. There is an isomorphism between* $\mathcal{T}^{p+1}_q(\Phi^! TG)$ *and the space of* Φ^**-function linear maps*

$$\underbrace{\Gamma(\Phi^! T^* G) \times \cdots \times \Gamma(\Phi^! T^* G)}_{p} \times \underbrace{\Gamma(\Phi^! TG) \times \cdots \times \Gamma(\Phi^! TG)}_{q} \to \Gamma(\Phi^! TG),$$

which we denote this space by $L_{p,q}(\Phi^! T^* G, \Phi^! TG; \Phi^! TG)$.

Proof. We consider the map $\Psi : L_{p,q}(\Phi^! T^* G, \Phi^! TG; \Phi^! TG) \to \mathcal{T}^{p+1}_q(\Phi^! TG)$ given by

$$\begin{aligned} \Psi(F)(\omega, \theta^1, \cdots, \theta^p, x_1, \cdots, x_q) =& \Phi^*(\omega(F(Ad^\dagger_{(\Phi^*)^{-2}}(\theta^1), \cdots, Ad^\dagger_{(\Phi^*)^{-2}}(\theta^p), Ad_{(\Phi^*)^{-2}}(x_1) \\ &, \cdots, Ad_{(\Phi^*)^{-2}}(x_q)))). \end{aligned}$$

It is easy to see that Ψ is an isomorphism. □

Definition 2.11. A tensor $T \in \mathcal{T}^p_q(\Gamma(\Phi^! TG))$ is called left-invariant if we have

$$\begin{aligned} T((l_a \circ \Phi^{-1})^*{}_{e_\Phi}((\theta^1)_{e_\Phi}), \cdots, (l_a \circ \Phi^{-1})^*{}_{e_\Phi}((\theta^p)_{e_\Phi}), (l_a \circ \Phi^{-1})_{*_{e_\Phi}}((x_1)_{e_\Phi}) \\ , \cdots, (l_a \circ \Phi^{-1})_{*_{e_\Phi}}((x_q)_{e_\Phi})) \\ = T((\theta^1)_{e_\Phi}, \cdots, (\theta^p)_{e_\Phi}, (x_1)_{e_\Phi}, \cdots, (x_q)_{e_\Phi}), \end{aligned}$$

for any $\theta^1, \cdots, \theta^p \in \Gamma(\Phi^! T^* G)$ and $x_1, \cdots, x_q \in \Gamma(\Phi^! TG)$.

Remark 2.12. If T is a left-invariant tensor of type (p, q) on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$, then using Definitions 2.6 and 2.9 we get

$$T((\theta^1)_a, \cdots, (\theta^p)_a, (x_1)_a, \cdots, (x_q)_a), = T((\theta^1)_{e_\Phi}, \cdots, (\theta^p)_{e_\Phi}, (x_1)_{e_\Phi}, \cdots, (x_q)_{e_\Phi}), \quad \forall a \in G,$$

where $\theta^1, \cdots, \theta^p \in \Gamma_L(\Phi^! T^* G)$ and $x_1, \cdots, x_q \in \Gamma_L(\Phi^! TG)$. So, the restriction of T to left-invariant sections is constant.

3. Norden structures on Hom-Lie algebras

Definition 3.1. An almost complex structure on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$ is a $(1,1)$-tensor field $J : \Phi^! TG \to \Phi^! TG$ such that $(Ad_{\Phi^*} \circ J)^2 = -Id_{\Phi^! TG}$ and $Ad_{\Phi^*} \circ J = J \circ Ad_{\Phi^*}$. Moreover, if for all $x \in \Gamma_L(\Phi^! TG)$ we have $J \circ x \in \Gamma_L(\Phi^! TG)$, then J is called left-invariant almost complex structure. The left-invariant almost complex structure is integrable (left-invariant complex structure) if the Nijenhuis torsion $N_{Ad_{\Phi^*} \circ J}$ of $Ad_{\Phi^*} \circ J$ defined by

$$\begin{aligned} N_{\phi_{\mathfrak{g}^!} \circ J}(x, y) &= [(Ad_{\Phi^*} \circ J)x, (Ad_{\Phi^*} \circ J)y] - Ad_{\Phi^*} \circ J[(\phi_{\mathfrak{g}^!} \circ J)x, y] \\ &\quad - Ad_{\Phi^*} \circ J[x, (Ad_{\Phi^*} \circ J)y] - [x, y], \end{aligned} \tag{3.1}$$

for any $x, y \in \Gamma_L(\Phi^! TG)$, vanishes.

It easy to see that if $(\varphi^*)^2 = Id_{C^\infty(M)}$, then $N_{\phi_A \circ J}(fX, Y) = \varphi^*(f) N_{\phi_A \circ J}(X, Y)$, for any $f \in C^\infty(M)$, i.e., $N_{\phi_A \circ J}$ is a bilinear φ^*-function. In the sequel we assume that $(\varphi^*)^2 = Id_{C^\infty(M)}$. In addition, for simplicity, we often set $N := N_{\phi_A \circ J}$. If $N = 0$, then the almost complex structure is called *complex).*

According to Remark 2.8, we can define a left-invariant almost complex structure on a Hom-Lie group as follows:

Definition 3.2. A left invariant almost complex structure on the Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$ (or an almost complex structure on a Hom-Lie algebra $(\mathfrak{g}^!, [\cdot, \cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!})$), is an isomorphism $J : \mathfrak{g}^! \to \mathfrak{g}^!$ that satisfies $(\phi_{\mathfrak{g}^!} \circ J)^2 = -Id_{\mathfrak{g}^!}$ and $\phi_{\mathfrak{g}^!} \circ J = J \circ \phi_{\mathfrak{g}^!}$.

We denote an almost complex Hom-Lie algebra by $(\mathfrak{g}^!, [\cdot, \cdot], \phi_{\mathfrak{g}^!}, J)$. If the n-dimensional Hom-Lie algebra $\mathfrak{g}^!$ admits an almost complex structure J, then

$$(det(\phi_{\mathfrak{g}^!} \circ J))^2 = det((\phi_{\mathfrak{g}^!} \circ J)^2) = det(-Id_{\mathfrak{g}^!}) = (-1)^n,$$

which implies that n is even.

Definition 3.3. Let $(G, \diamond, e_\Phi, \Phi)$ be a Hom-Lie group. A $(0,2)$-tensor symmetric and non-degenerate $\langle \cdot, \cdot \rangle : \Gamma(\Phi^! TG) \times \Gamma(\Phi^! TG) \to C^\infty(G)$ is called a pseudo-Riemannian metric if it is a left-invariant tensor, i.e.

$$\langle Ad_{\Phi^*}(x), Ad_{\Phi^*}(y) \rangle = \Phi^* \langle x, y \rangle,$$

for any $x, y \in \Gamma(\Phi^! TG)$. Moreover, $\langle \cdot, \cdot \rangle$ is called a left-invariant pseudo-Riemannian metric if

$$\langle (l_a \circ \Phi^{-1})_{*_{e_\Phi}}(x_{e_\Phi}), (l_a \circ \Phi^{-1})_{*_{e_\Phi}}(y_{e_\Phi}) \rangle_a = \langle x_{e_\Phi}, y_{e_\Phi} \rangle_{e_\Phi},$$

for all $x, y \in \Gamma(\Phi^! TG)$.

Remark 3.4. According to Remark 2.8 and Definition 3.3, we conclude that a left-invariant pseudo-Riemannian metric on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$ (or a pseudo-Riemannian metric on a Hom-Lie algebra $(\mathfrak{g}^!, [\cdot, \cdot], \phi_{\mathfrak{g}^!})$) is a Φ^*-function linear symmetric non-degenerate form $\langle \cdot, \cdot \rangle$ such that

$$\langle \phi_{\mathfrak{g}^!}(x), \phi_{\mathfrak{g}^!}(y) \rangle = \langle x, y \rangle,$$

for any $x, y \in \mathfrak{g}^!$. In this case, $(G, \diamond, e_\Phi, \Phi, \langle \cdot, \cdot \rangle)$ (or $(\mathfrak{g}^!, [\cdot, \cdot], \phi_{\mathfrak{g}^!}), \langle \cdot, \cdot \rangle)$ is called a left-invariant pseudo-Riemannian Hom-Lie group (or a pseudo-Riemannian Hom-Lie algebra). Also, if $\phi^2_{\mathfrak{g}^!} = Id_{\mathfrak{g}^!}$, then we have

$$\langle \phi_{\mathfrak{g}^!}(x), y \rangle = \langle x, \phi_{\mathfrak{g}^!}(y) \rangle,$$

for any $x, y \in \mathfrak{g}^!$. Changing $\mathfrak{g}^!$ to $\mathfrak{g}$ and $\phi_{\mathfrak{g}^!}$ to $\phi_{\mathfrak{g}}$, we obtain the definition of a pseudo-Riemannian metric on a Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ [21].

Definition 3.5. A linear connection on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$ is a map

$$\begin{cases} \nabla : \Gamma(\Phi^! TG) \times \Gamma(\Phi^! TG)) \to \Gamma(\Phi^! TG)) \\ \qquad (x, y) \longrightarrow \nabla_x y \end{cases},$$

with the following properties:
i) $\nabla_{x+y} z = \nabla_x z + \nabla_y z$,
ii) $\nabla_{fx} y = \Phi^*(f)\nabla_x y$,
iii) $\nabla_x(y + z) = \nabla_x y + \nabla_x z$,
iv) $\nabla_x(fy) = \Phi^*(f)\nabla_x y + Ad_{\Phi^*}(x)(f)Ad_{\Phi^*}y$,
for any $f \in C^\infty(G), x, y, z \in \Gamma(\Phi^! TG)$.

Theorem 3.6. *Let $(G, \diamond, e_\Phi, \Phi, \langle \cdot, \cdot \rangle)$ be a pseudo-Riemannian Hom-Lie group. Then there exists a unique connection ∇ on it which is characterized by the following properties:*
(i) ∇ is symmetric, i.e.,

$$[x, y]_\Phi = \nabla_x y - \nabla_y x,$$

(ii) ∇ is compatible with $\langle \cdot, \cdot \rangle$, i.e.

$$Ad_{\Phi^*}(x) < y, z >=< \nabla_x y, Ad_{\Phi^*}(z) > + < Ad_{\Phi^*}(y), \nabla_x z >,$$

for any $x, y, z \in \Gamma(\Phi^! TG)$.
This connection is called the Hom-Levi-Civita connection.

Proof. Considering ∇ as

$$\begin{aligned} 2\langle \nabla_x y, Ad_{\Phi^*}(z)\rangle =& Ad_{\Phi^*}(x)\langle y, z\rangle + Ad_{\Phi^*}(y)\langle z, x\rangle - Ad_{\Phi^*}(z)\langle x, y\rangle \\ &+ \langle [x, y]_\Phi, Ad_{\Phi^*}(z)\rangle + \langle [z, x]_\Phi, Ad_{\Phi^*}(y)\rangle + \langle [z, y]_\Phi, Ad_{\Phi^*}(x)\rangle, \end{aligned} \tag{3.2}$$

it is easy to see that ∇ satisfies the properties of Definition 3.5. Also, the above equation gives us

$$\langle \nabla_x y - \nabla_y x, Ad_{\Phi^*}(z)\rangle = \langle [x, y]_\Phi, Ad_{\Phi^*}(z)\rangle,$$

which implies (i). In a similar way, we can deduce (ii). Finally (3.2) shows the uniqueness of ∇. □

Formula (3.2) is called Koszul's formula.

Definition 3.7. A linear connection ∇ on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$ is called left-invariant if for all $x, y \in \Gamma_L(\Phi^! TG)$ we have $\nabla_x y \in \Gamma_L(\Phi^! TG)$.

Remark 3.8. Let $\langle \cdot, \cdot \rangle$ be a left-invariant pseudo-Riemannian metric on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$. Then using Remark 2.12, $\langle x, y\rangle$ is a constant for all $x, y \in \Gamma_L(\Phi^! TG)$, and so we get $Ad_{\Phi^*}(z)\langle x, y\rangle = 0$. Therefore Koszul's formula reduces to the following

$$2\langle \nabla_x y, Ad_{\Phi^*}(z)\rangle = \langle [x, y]_\Phi, Ad_{\Phi^*}(z)\rangle + \langle [z, x]_\Phi, Ad_{\Phi^*}(y)\rangle + \langle [z, y]_\Phi, Ad_{\Phi^*}(x)\rangle,$$

for any $x, y, z \in \Gamma_L(\Phi^! TG)$.

Remark 3.9. According to Definitions 3.5 and 3.7, we can rewrite the definition of left-invariant connection on a Hom-Lie group (or a connection on a Hom-Lie algebra $(\mathfrak{g}^!, [\cdot, \cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!})$) as follows:
A left-invariant connection on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$ (or a connection on a Hom-Lie algebra $(\mathfrak{g}^!, [\cdot, \cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!})$) is a linear map

$$\begin{cases} \nabla : \mathfrak{g}^! \times \mathfrak{g}^! \to \mathfrak{g}^! \\ \quad (x, y) \longrightarrow \nabla_x y \end{cases},$$

satisfying the following properties

$$\nabla_{fx} y = \Phi^*(f)\nabla_x y, \qquad \nabla_x(fy) = \Phi^*(f)\nabla_x y + \phi_{\mathfrak{g}^!}(x)(f)\phi_{\mathfrak{g}^!}(y),$$

for any $f \in C^\infty(G)$ and $x, y, z \in \mathfrak{g}^!$. Also, from Theorem 3.6 and Remark 3.8 we deduce that the linear connection ∇ given by

$$2\langle \nabla_x y, \phi_{\mathfrak{g}^!}(z)\rangle = \langle [x,y]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}(z)\rangle + \langle [z,y]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}(x)\rangle + \langle [z,x]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}(y)\rangle, \quad \forall x,y,z \in \mathfrak{g}^!, \tag{3.3}$$

is the unique left-invariant connection (or the unique connection) on a left-invariant pseudo-Riemannian Hom-Lie group $(G, \diamond, e_\Phi, \Phi, \langle\cdot,\cdot\rangle)$ (or pseudo-Riemannian Hom-Lie algebra $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}))$ such that

$$(i)\ \ [x,y]_{\mathfrak{g}^!} = \nabla_x y - \nabla_y x, \qquad (ii)\ \ \langle \nabla_x y, \phi_{\mathfrak{g}^!}(z)\rangle = -\langle \phi_{\mathfrak{g}^!}(y), \nabla_x z\rangle, \tag{3.4}$$

for any $x, y, z \in \mathfrak{g}^!$.
This connection is called the Hom-Levi-Civita connection on $(G, \diamond, e_\Phi, \Phi, \langle\cdot,\cdot\rangle)$ (or $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}), \langle\cdot,\cdot\rangle)$. Changing $\mathfrak{g}^!$ to $\mathfrak{g}$ and $\phi_{\mathfrak{g}^!}$ to $\phi_{\mathfrak{g}}$, we obtain the definition of a linear connection (also, Hom–Levi-Civita connection) for any arbitrary Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, \langle\cdot,\cdot\rangle)$.

The total covariant derivative of a (p,q)-tensor T on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi, \langle\cdot,\cdot\rangle)$ is a tensor of order $(p, q+1)$ given by

$$\begin{aligned}(\nabla_x T)(\theta^1, \cdots, \theta^p, y_1, \cdots, y_q) &= \nabla_x T(\theta^1, \cdots, \theta^p, y_1, \cdots, y_q) \\ &- \sum_{i=1}^{p} T(Ad^\dagger_{\Phi^*}(\theta^1), \cdots, \nabla_x \theta^i, \cdots, Ad^\dagger_{\Phi^*}(\theta^p), Ad_{\Phi^*}(y_1), \cdots, Ad_{\Phi^*}(y_q)) \\ &- \sum_{i=1}^{q} T(Ad^\dagger_{\Phi^*}(\theta^1), \cdots, Ad^\dagger_{\Phi^*}(\theta^p), Ad_{\Phi^*}(y_1), \cdots, \nabla_x y_i, \cdots, Ad_{\Phi^*}(y_q)),\end{aligned} \tag{3.5}$$

for all $\theta^1, \cdots, \theta^p \in \Gamma(\Phi^! T^*G)$ and $x_1, \cdots, x_q \in \Gamma(\Phi^! TG)$. If ∇ is a left-invariant connection on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi, \langle\cdot,\cdot\rangle)$, then using Remark 2.12, $T(\theta^1, \cdots, \theta^p, y_1, \cdots, y_q)$ is constant, for all $\theta^1, \cdots, \theta^p \in \Gamma(\Phi^! T^*G)$ and $x_1, \cdots, x_q \in \Gamma(\Phi^! TG)$. So,

$$\nabla_x T(\theta^1, \cdots, \theta^p, y_1, \cdots, y_q) = x(T(\theta^1, \cdots, \theta^p, y_1, \cdots, y_q)) = 0.$$

Therefore, (3.5) reduces to

$$\begin{aligned}(\nabla_x T)(\theta^1, \cdots, \theta^p, y_1, \cdots, y_q) &= -\sum_{i=1}^{p} T(Ad^\dagger_{\Phi^*}(\theta^1), \cdots, \nabla_x \theta^i, \cdots, Ad^\dagger_{\Phi^*}(\theta^p), Ad_{\Phi^*}(y_1) \\ , \cdots, Ad_{\Phi^*}(y_q)) &- \sum_{i=1}^{q} T(Ad^\dagger_{\Phi^*}(\theta^1), \cdots, Ad^\dagger_{\Phi^*}(\theta^p), Ad_{\Phi^*}(y_1), \cdots, \nabla_x y_i, \cdots, Ad_{\Phi^*}(y_q)).\end{aligned} \tag{3.6}$$

Remark 3.10. Given $(\mathfrak{g}^!)^*$ as the dual space of $\mathfrak{g}^!$, we denote by $\phi^*_{\mathfrak{g}^!}$ the dual map of the endomorphism $\phi_{\mathfrak{g}^!}$. In this case, according to (3.6), the total covariant derivative of $T \in \mathcal{T}^p_q(\mathfrak{g}^!)$ is a tensor of order $(p, q+1)$ such that

$$\begin{aligned}(\nabla_x T)(\theta^1, \cdots, \theta^p, y_1, \cdots, y_q) &= -\sum_{i=1}^{p} T(\phi^*_{\mathfrak{g}^!}(\theta^1), \cdots, \nabla_x \theta^i, \cdots, \phi^*_{\mathfrak{g}^!}(\theta^p), \phi_{\mathfrak{g}^!}(y_1) \\ , \cdots, \phi_{\mathfrak{g}^!}(y_q)) &- \sum_{i=1}^{q} T(\phi^*_{\mathfrak{g}^!}(\theta^1), \cdots, \phi^*_{\mathfrak{g}^!}(\theta^p), \phi_{\mathfrak{g}^!}(y_1), \cdots, \nabla_x y_i, \cdots, \phi_{\mathfrak{g}^!}(y_q)),\end{aligned} \tag{3.7}$$

for all $\theta^1, \cdots, \theta^p \in (\mathfrak{g}^!)^*$ and $x_1, \cdots, x_q \in \mathfrak{g}^!$.

Definition 3.11. Let $(G, \diamond, e_\Phi, \Phi, J)$ be a left-invariant almost complex Hom-Lie group(or $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J)$ be an almost complex Hom-Lie algebra). A left-invariant pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ on G (or a pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}^!$) is called Norden metric (anti-Hermitian, B-metric) if

$$\langle (\phi_{\mathfrak{g}^!} \circ J)x, (\phi_{\mathfrak{g}^!} \circ J)y\rangle = -\langle x, y\rangle,$$

or equivalenty

$$\langle(\phi_{\mathfrak{g}^!} \circ J)x, y\rangle = \langle x, (\phi_{\mathfrak{g}^!} \circ J)y\rangle, \tag{3.8}$$

for all $x, y \in \mathfrak{g}^!$. In this case, we say that $(G, \diamond, e_\Phi, \Phi, J, \langle\cdot,\cdot\rangle)$ is a left-invariant almost Norden Hom-Lie group (or, $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle\cdot,\cdot\rangle)$ is an almost Norden Hom-Lie algebra). Moreover, if $\phi_{\mathfrak{g}^!} \circ J$ is integrable, then the pair $(J, \langle\cdot,\cdot\rangle)$ is called left-invariant Norden structure on G (or, Norden structure on $\mathfrak{g}^!$).

Remark 3.12. All concepts defined on the Hom-Lie algebra $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!})$ of a Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$ hold for an arbitrary Hom-Lie algebra. So, we study these concepts on Hom-Lie algebras as examples.

Example 3.13. We consider a 4-dimensional linear space $\mathfrak{g}$ with a basis $\{e_1, e_2, e_3, e_4\}$. We define the bracket and linear map $\phi_\mathfrak{g}$ on $\mathfrak{g}$ as follows

$$[e_1, e_4]_\mathfrak{g} = ae_2, \quad [e_2, e_3]_\mathfrak{g} = ae_1,$$

and

$$\phi_\mathfrak{g}(e_1) = -e_2, \quad \phi_\mathfrak{g}(e_2) = -e_1, \quad \phi_\mathfrak{g}(e_3) = e_4, \quad \phi_\mathfrak{g}(e_4) = e_3.$$

The above bracket is not a Lie bracket on $\mathfrak{g}$ if $a \neq 0$, because

$$[e_2, [e_3, e_4]_\mathfrak{g}]_\mathfrak{g} + [e_3, [e_4, e_2]_\mathfrak{g}]_\mathfrak{g} + [e_4, [e_2, e_3]_\mathfrak{g}]_\mathfrak{g} = [e_4, ae_1]_\mathfrak{g} + [e_3, ae_1]_\mathfrak{g} = -ae_2.$$

It is easy to see that

$$[\phi_\mathfrak{g}(e_1), \phi_\mathfrak{g}(e_4)]_\mathfrak{g} = -ae_1 = \phi_\mathfrak{g}([e_1, e_4]_\mathfrak{g}), \quad [\phi_\mathfrak{g}(e_2), \phi_\mathfrak{g}(e_3)]_\mathfrak{g} = -ae_2 = \phi_\mathfrak{g}([e_2, e_3]_\mathfrak{g}),$$

i.e. $\phi_\mathfrak{g}$ is the algebra morphism. Also, we have

$$[\phi_\mathfrak{g}(e_i), [e_j, e_k]_\mathfrak{g}]_\mathfrak{g} + [\phi_\mathfrak{g}(e_j), [e_k, e_i]_\mathfrak{g}]_\mathfrak{g} + [\phi_\mathfrak{g}(e_k), [e_i, e_j]_\mathfrak{g}]_\mathfrak{g} = 0, \quad i, j, k = 1, 2, 3, 4.$$

Thus $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \phi_\mathfrak{g})$ is a Hom-Lie algebra. We define the metric $\langle\cdot,\cdot\rangle$ of $\mathfrak{g}$ as follows

$$\begin{bmatrix} A & B & C & D \\ B & A & -D & -C \\ C & -D & -A & B \\ D & -C & B & -A \end{bmatrix}.$$

It follows that $\langle\phi_\mathfrak{g}(e_i), \phi_\mathfrak{g}(e_j)\rangle = \langle e_i, e_j\rangle$, for all $i, j = 1, 2, 3, 4$. Hence (i) of (3.4) holds and so $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \phi_\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is a pseudo-Riemannian Hom-Lie algebra. Assume that the isomorphism J is defined as

$$J(e_1) = -e_3, \quad J(e_2) = e_4, \quad J(e_3) = e_1, \quad J(e_4) = -e_2.$$

Then we have

$$(J \circ \phi_\mathfrak{g})e_1 = -e_4 = (\phi_\mathfrak{g} \circ J)e_1, \quad (J \circ \phi_\mathfrak{g})e_2 = e_3 = (\phi_\mathfrak{g} \circ J)e_2,$$
$$(J \circ \phi_\mathfrak{g})e_3 = -e_2 = (\phi_\mathfrak{g} \circ J)e_3, \quad (J \circ \phi_\mathfrak{g})e_4 = e_1 = (\phi_\mathfrak{g} \circ J)e_4.$$

Thus J is an almost complex structure on $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \phi_\mathfrak{g})$. Also, since $N_{\phi_\mathfrak{g} \circ J}(e_i, e_j) = 0$, for all $i, j = 1, 2, 3, 4$, then J is an complex structure on $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \phi_\mathfrak{g})$. Moreover, we get $\langle(\phi_\mathfrak{g} \circ J)(e_i), (\phi_\mathfrak{g} \circ J)(e_j)\rangle = -\langle e_i, e_j\rangle$, for all $i, j = 1, 2, 3, 4$. Thus $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \phi_\mathfrak{g}, J, \langle\cdot,\cdot\rangle)$ is a Norden Hom-Lie algebra.

Example 3.14. We consider a 4-dimensional Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \phi_\mathfrak{g})$ with a basis $\{e_1, e_2, e_3, e_4\}$ such that bracket $[\cdot,\cdot]_\mathfrak{g}$ and linear map $\phi_\mathfrak{g}$ on $\mathfrak{g}$ are defined by

$$[e_1, e_4]_\mathfrak{g} = ae_1 + ae_2, \quad [e_2, e_3]_\mathfrak{g} = ae_1 + ae_2, \quad [e_3, e_4]_\mathfrak{g} = -ae_3 + ae_4,$$

and

$$\phi_\mathfrak{g}(e_1) = e_2, \quad \phi_\mathfrak{g}(e_2) = e_1, \quad \phi_\mathfrak{g}(e_3) = e_4, \quad \phi_\mathfrak{g}(e_4) = e_3.$$

The above bracket is not a Lie bracket on $\mathfrak{g}$ if $a \neq 0$. Now, we consider the metric $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ defined as

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It can be checked easily that $\langle \phi_{\mathfrak{g}}(e_i), \phi_{\mathfrak{g}}(e_j) \rangle = \langle e_i, e_j \rangle$, for all $i, j = 1, 2, 3, 4$. If the isomorphism J is determined as

$$J(e_1) = e_4, \quad J(e_2) = e_3, \quad J(e_3) = -e_2, \quad J(e_4) = -e_1,$$

a simple computation shows that

$$\begin{aligned} (J \circ \phi_{\mathfrak{g}})e_1 &= e_3 = (\phi_{\mathfrak{g}} \circ J)e_1, \quad (J \circ \phi_{\mathfrak{g}})e_2 = e_4 = (\phi_{\mathfrak{g}} \circ J)e_2, \\ (J \circ \phi_{\mathfrak{g}})e_3 &= -\, e_1 = (\phi_{\mathfrak{g}} \circ J)e_3, \quad (J \circ \phi_{\mathfrak{g}})e_4 = -e_2 = (\phi_{\mathfrak{g}} \circ J)e_4. \end{aligned}$$

So J is an almost complex structure on $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$. Also, we see that

$$\langle (\phi_{\mathfrak{g}} \circ J)(e_i), (\phi_{\mathfrak{g}} \circ J)(e_j) \rangle = \langle e_i, e_j \rangle, \quad \forall i, j = 1, 2, 3, 4.$$

Hence $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, J, \langle \cdot, \cdot \rangle)$ is an almost Norden Hom-Lie algebra. It is easy to see that $N(e_1, e_2) \neq 0$. Thus, J is not integrable and consequently $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, J, \langle \cdot, \cdot \rangle)$ is not a Norden Hom-Lie algebra.

Example 3.15. Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ be a 4-dimensional Hom-Lie algebra with a basis $\{e_1, e_2, e_3, e_4\}$ and where the bracket and linear map $\phi_{\mathfrak{g}}$ on $\mathfrak{g}$ are given by

$$[e_1, e_2]_{\mathfrak{g}} = -\, e_3, \quad [e_1, e_3]_{\mathfrak{g}} = e_2, \quad [e_2, e_4]_{\mathfrak{g}} = e_2, \quad [e_3, e_4]_{\mathfrak{g}} = e_3,$$

and

$$\phi_{\mathfrak{g}}(e_1) = e_1, \quad \phi_{\mathfrak{g}}(e_2) = -e_2, \quad \phi_{\mathfrak{g}}(e_3) = -e_3, \quad \phi_{\mathfrak{g}}(e_4) = e_4.$$

We define the metric $\langle \cdot, \cdot \rangle$ of $\mathfrak{g}$ by

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & -B & 0 \\ 0 & 0 & 0 & -A \end{bmatrix}.$$

It is easy to see that $\langle \phi_{\mathfrak{g}}(e_i), \phi_{\mathfrak{g}}(e_j) \rangle = \langle e_i, e_j \rangle$, for all $i, j = 1, 2, 3, 4$. Thus (i) of (3.4) holds and so $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, \langle \cdot, \cdot \rangle)$ is a pseudo-Riemannian Hom-Lie algebra. Define the isomorphism J as

$$J(e_1) = -e_4, \quad J(e_2) = -e_3, \quad J(e_3) = e_2, \quad J(e_4) = e_1,$$

we deduce that

$$\begin{aligned} (J \circ \phi_{\mathfrak{g}})e_1 &= -\, e_4 = (\phi_{\mathfrak{g}} \circ J)e_1, \quad (J \circ \phi_{\mathfrak{g}})e_2 = e_3 = (\phi_{\mathfrak{g}} \circ J)e_2, \\ (J \circ \phi_{\mathfrak{g}})e_3 &= -\, e_2 = (\phi_{\mathfrak{g}} \circ J)e_3, \quad (J \circ \phi_{\mathfrak{g}})e_4 = e_1 = (\phi_{\mathfrak{g}} \circ J)e_4. \end{aligned}$$

Thus J is an almost complex structure on $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$. Also, we have

$$N_{\phi_{\mathfrak{g}} \circ J}(e_i, e_j) = 0, \quad \forall i, j = 1, 2, 3, 4,$$

i.e. J is an complex structure on $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$. It is easy to check that $\langle (\phi_{\mathfrak{g}} \circ J)(e_i), (\phi_{\mathfrak{g}} \circ J)(e_j) \rangle = -\langle e_i, e_j \rangle$, for all $i, j = 1, 2, 3, 4$, and so $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, J, \langle \cdot, \cdot \rangle)$ is a Norden Hom-Lie algebra.

Theorem 3.16. *Let $(J, \langle\cdot,\cdot\rangle)$ be a left-invariant almost Norden structure on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$ (or, an almost Norden structure on a Hom-Lie algebra $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!})$). Then we have*

$$\langle \phi_{\mathfrak{g}^!}(y), (\nabla_x(\phi_{\mathfrak{g}^!}\circ J))z\rangle = \langle(\nabla_x(\phi_{\mathfrak{g}^!}\circ J))y, \phi_{\mathfrak{g}^!}(z)\rangle,$$

for any $x, y, z \in \mathfrak{g}^!$, where ∇ is the left-invariant Hom-Levi-Civita connection on G (or, the Hom-Levi-Civita connection on $\mathfrak{g}^!$) and $(\nabla_x(\phi_{\mathfrak{g}^!}\circ J))y = \nabla_x(\phi_{\mathfrak{g}^!}\circ J)y - (\phi_{\mathfrak{g}^!}\circ J)\nabla_x y$.

Proof. From (ii) of (3.4), we have

$$\langle \nabla_x(\phi_{\mathfrak{g}^!}\circ J)y, \phi_{\mathfrak{g}^!}(z)\rangle = -\langle \phi_{\mathfrak{g}^!}((\phi_{\mathfrak{g}^!}\circ J)y), \nabla_x z\rangle,$$
$$\langle \nabla_x y, \phi_{\mathfrak{g}^!}((\phi_{\mathfrak{g}^!}\circ J)z)\rangle = -\langle \phi_{\mathfrak{g}^!}(y), \nabla_x(\phi_{\mathfrak{g}^!}\circ J)z\rangle.$$

Since $\langle(\phi_{\mathfrak{g}^!}\circ J)\cdot,\cdot\rangle = \langle\cdot,(\phi_{\mathfrak{g}^!}\circ J)\cdot\rangle$, from the above equations it follows that

$$\langle \nabla_x(\phi_{\mathfrak{g}^!}\circ J)y, \phi_{\mathfrak{g}^!}(z)\rangle = -\langle \phi_{\mathfrak{g}^!}(y), (\phi_{\mathfrak{g}^!}\circ J)(\nabla_x z)\rangle,$$
$$\langle (\phi_{\mathfrak{g}^!}\circ J)(\nabla_x y), \phi_{\mathfrak{g}^!}(z)\rangle = -\langle \phi_{\mathfrak{g}^!}(y), \nabla_x(\phi_{\mathfrak{g}^!}\circ J)z\rangle.$$

Substracting the above equations completes the proof. □

3.1. Holomorphic tensors on Hom-Lie groups and Hom-Lie algebras

We introduce the notion of a holomorphic tensor on a Hom-Lie group and a Hom-Lie algebra.

A left-invariant complex Hom-Lie group (or, a complex Hom-Lie algebra) is a left-invariant almost complex Hom-Lie group $(G, \diamond, e_\Phi, \Phi, J)$ (or, an almost complex Hom-Lie algebra $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J)$) such that the tensor J is integrable. In this case, the left-invariant almost complex structure (or, the almost complex structure) J is called a left-invariant complex structure (or, a complex structure). On the other hand, in order that an almost complex structure be integrable, it is necessary and sufficient that for any $x, y \in \mathfrak{g}^!$, we consider a left-invariant connection ∇ such that the symmetric condition be held, i.e.

$$[x, y]_{\mathfrak{g}^!} = \nabla_x y - \nabla_y x,$$

and $\phi_{\mathfrak{g}^!}\circ J$ is invariant with respect to ∇, that is $\nabla(\phi_{\mathfrak{g}^!}\circ J) = 0$ or

$$\nabla_x(\phi_{\mathfrak{g}^!}\circ J) = (\phi_{\mathfrak{g}^!}\circ J)\nabla_x.$$

Notice that condition $\nabla_x(\phi_{\mathfrak{g}^!}\circ J)(y) = (\phi_{\mathfrak{g}^!}\circ J)\nabla_x y$ is equivalent to

$$\nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)}(\phi_{\mathfrak{g}^!}\circ J)(y) = (\phi_{\mathfrak{g}^!}\circ J)\nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)} y,$$

and

$$\nabla_x y = -(\phi_{\mathfrak{g}^!}\circ J)\nabla_x(\phi_{\mathfrak{g}^!}\circ J)(y).$$

Also (3.1) and two last equations imply

$$\begin{aligned} N_{\phi_{\mathfrak{g}^!}\circ J}(x,y) =& \nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)}(\phi_{\mathfrak{g}^!}\circ J)(y) - \nabla_{(\phi_{\mathfrak{g}^!}\circ J)(y)}(\phi_{\mathfrak{g}^!}\circ J)(x) - (\phi_{\mathfrak{g}^!}\circ J)(\nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)} y \\ &- \nabla_y(\phi_{\mathfrak{g}^!}\circ J)(x)) - (\phi_{\mathfrak{g}^!}\circ J)(\nabla_x(\phi_{\mathfrak{g}^!}\circ J)(y) - \nabla_{(\phi_{\mathfrak{g}^!}\circ J)(y)} x) - \nabla_x y - \nabla_y x = 0. \end{aligned} \tag{3.9}$$

It is known that the integrability of J is equivalent to the vanishing of the Nijenhuis tensor $N_{\phi_{\mathfrak{g}^!}\circ J}$. So, J is integrable.

We can give another equivalent definition of left-invariant complex Hom-Lie group $(G, \diamond, e_\Phi, \Phi, J)$ (or, complex Hom-Lie algebras $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J)$) in holomorphic terms. Given a tensor $\omega \in \mathcal{T}^0_q(\mathfrak{g}^!)$, the purity means that for any $x_1, \cdots, x_q \in \mathfrak{g}^!$ the following condition should be held

$$\omega((\phi_{\mathfrak{g}^!}\circ J)x_1, x_2, \cdots, x_q) = \omega(x_1, (\phi_{\mathfrak{g}^!}\circ J)x_2, \cdots, x_q) = \cdots = \omega(x_1, x_2, \cdots, (\phi_{\mathfrak{g}^!}\circ J)x_q).$$

We define an operator

$$\Phi_{\phi_{\mathfrak{g}^!}\circ J} : \mathcal{T}^0_q(\mathfrak{g}^!) \longrightarrow \mathcal{T}^0_{q+1}(\mathfrak{g}^!),$$

applied to the pure tensor $\omega \in \mathcal{T}_q^0(\mathfrak{g}^!)$ by

$$(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\ \omega)(x, y_1, y_2, \cdots, y_q) = \sum_{i=1}^{q} \omega(\phi_{\mathfrak{g}^!}(y_1), \cdots, \phi_{\mathfrak{g}^!}(y_{i-1}), [y_i, (\phi_{\mathfrak{g}^!} \circ J)x]_{\mathfrak{g}^!} \\ - (\phi_{\mathfrak{g}^!} \circ J)[y_i, x]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}(y_{i+1}), \cdots, \phi_{\mathfrak{g}^!}(y_q)), \tag{3.10}$$

for any $x, y_1, \cdots, y_q \in \mathfrak{g}^!$. The pure tensor ω is called holomorphic if J is a complex structure on $\mathfrak{g}^!$ and

$$\Phi_{\phi_{\mathfrak{g}^!}\circ J}\ \omega = 0.$$

According to Definition 3.11, we see that the Norden metric $\langle\cdot,\cdot\rangle$ on the almost complex Hom-Lie algebra $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J)$ is a pure tensor with respect to J.

4. Left-invariant Kähler-Norden Hom-Lie groups (Kähler-Norden Hom-Lie algebras)

In this section, we introduce left-invariant Kähler-Norden structures on Hom-Lie groups (Kähler-Norden structures on Hom-Lie algebras) and provide examples of these structures.

Definition 4.1. A left-invariant Kähler Hom-Lie group (or, a Kähler-Norden Hom-Lie algebra) is a left-invariant almost Norden Hom-Lie group $(G, \diamond, e_\Phi, \Phi, J, \langle\cdot,\cdot\rangle)$ (or, an almost Norden Hom-Lie algebra $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle\cdot,\cdot\rangle))$ such that $\phi_{\mathfrak{g}^!} \circ J$ is invariant with respect to the left-invariant Hom-Levi-Civita connection ∇, i.e.

$$\nabla(\phi_{\mathfrak{g}^!} \circ J) = 0. \tag{4.1}$$

Notice that (3.9) implies that the structure J introduced in Definition 4.1 is integrable.

Remark 4.2. The above definition holds for an arbitrary Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \phi)$.

Example 4.3. Consider the 4-dimensional Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ introduced in Example (3.15). We study its Kählerian-Norden property. Let ∇ be the Hom-Levi-Civita connection. From Koszul's formula given by (3.3) we deduce that $\langle\nabla_{e_i} e_j, \phi_{\mathfrak{g}}(e_k)\rangle = 0$, for all $i, j, k = 1, 2, 3, 4$, except

$$\nabla_{e_2} e_1 = e_3, \quad \nabla_{e_2} e_2 = -\frac{B}{A} e_4, \quad \nabla_{e_2} e_3 = -\frac{B}{A} e_1, \quad \nabla_{e_2} e_4 = e_2,$$
$$\nabla_{e_3} e_1 = -e_2, \quad \nabla_{e_3} e_2 = -\frac{B}{A} e_1, \quad \nabla_{e_3} e_3 = \frac{B}{A} e_4, \quad \nabla_{e_3} e_4 = e_3.$$

One can see that $\phi_{\mathfrak{g}} \circ J$ is invariant with respect to this Hom-Levi-Civita connection, i.e (4.1) holds. So $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, J, \langle\cdot,\cdot\rangle)$ is a Kähler-Norden Hom-Lie algebra.

Example 4.4. Let $\{e_1, \cdots, e_6\}$ be a basis of a 6-dimensional Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$, where

$$[e_3, e_5]_{\mathfrak{g}} = -\, e_2, \quad [e_3, e_6]_{\mathfrak{g}} = e_1, \quad [e_4, e_5]_{\mathfrak{g}} = e_1, \quad [e_4, e_6]_{\mathfrak{g}} = e_2, \quad [e_5, e_6]_{\mathfrak{g}} = e_3,$$

and

$$\phi_{\mathfrak{g}}(e_1) = -e_1, \quad \phi_{\mathfrak{g}}(e_2) = -e_2, \quad \phi_{\mathfrak{g}}(e_3) = e_3, \quad \phi_{\mathfrak{g}}(e_4) = e_4, \quad \phi_{\mathfrak{g}}(e_5) = -e_5, \quad \phi_{\mathfrak{g}}(e_6) = -e_6.$$

We define an isomorphism J on $\mathfrak{g}$ by

$$J(e_1) = -e_2, \quad J(e_2) = e_1, \quad J(e_3) = e_4, \quad J(e_4) = -e_3, \quad J(e_5) = e_6, \quad J(e_6) = -e_5,$$

which implies that

$$(J \circ \phi_{\mathfrak{g}})e_1 = e_2 = (\phi_{\mathfrak{g}} \circ J)e_1, \qquad (J \circ \phi_{\mathfrak{g}})e_2 = -e_1 = (\phi_{\mathfrak{g}} \circ J)e_2,$$
$$(J \circ \phi_{\mathfrak{g}})e_3 = e_4 = (\phi_{\mathfrak{g}} \circ J)e_3, \qquad (J \circ \phi_{\mathfrak{g}})e_4 = -e_3 = (\phi_{\mathfrak{g}} \circ J)e_4,$$
$$(J \circ \phi_{\mathfrak{g}})e_5 = -\, e_6 = (\phi_{\mathfrak{g}} \circ J)e_5, \quad (J \circ \phi_{\mathfrak{g}})e_6 = e_5 = (\phi_{\mathfrak{g}} \circ J)e_6.$$

Thus J is an almost complex structure on $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$. On the other hand, $N_{\phi_{\mathfrak{g}}\circ J}(e_i, e_j) = 0$, for all $i, j = 1, \cdots, 6$, so J is a complex structure. Consider the metric $\langle\cdot,\cdot\rangle$ of $\mathfrak{g}$ by

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{A}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{A}{2} \\ 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & -A & 0 & 0 \\ \frac{A}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{A}{2} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\langle\phi_{\mathfrak{g}}(e_i), \phi_{\mathfrak{g}}(e_j)\rangle = \langle e_i, e_j\rangle$, for all $i, j = 1, \cdots, 6$. So $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, \langle\cdot,\cdot\rangle)$ is a pseudo-Riemannian Hom-Lie algebra. Also, since $\langle(\phi_{\mathfrak{g}} \circ J)(e_i), (\phi_{\mathfrak{g}} \circ J)(e_j)\rangle = -\langle e_i, e_j\rangle$, for all $i, j = 1, \cdots, 6$, thus $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, J, \langle\cdot,\cdot\rangle)$ is a Norden Hom-Lie algebra. Using (3.3), we obtain $\langle e_i \cdot e_j, \phi_{\mathfrak{g}}(e_k)\rangle = 0$, for all $i, j, k = 1, \cdots, 6$, except

$$\begin{aligned} &\nabla_{e_5} e_3 = e_2, \qquad \nabla_{e_5} e_4 = -e_1, \qquad \nabla_{e_5} e_5 = \frac{1}{2} e_4, \qquad \nabla_{e_5} e_6 = \frac{1}{2} e_3, \\ &\nabla_{e_6} e_3 = -e_1, \qquad \nabla e_6 e_4 = -e_2, \qquad \nabla_{e_6} e_5 = -\frac{1}{2} e_3, \qquad \nabla_{e_6} e_6 = \frac{1}{2} e_4, \end{aligned} \tag{4.2}$$

where ∇ denote the Hom-Levi-Civita connection on $\mathfrak{g}$. Obviously (4.1) holds, therefore $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, J, \langle\cdot,\cdot\rangle)$ is a Kähler-Norden Hom-Lie algebra.

Proposition 4.5. *Let $(G, \diamond, e_\Phi, \Phi, J, \langle\cdot,\cdot\rangle)$ be a left-invariant almost Norden Hom-Lie group (or, $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle\cdot,\cdot\rangle)$ be an almost Norden Hom-Lie algebra). Then $(G, \diamond, e_\Phi, \Phi, J, \langle\cdot,\cdot\rangle)$ (or, $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle\cdot,\cdot\rangle)$) is a left-invariant Kähler-Norden Hom-Lie group (or, a Kähler-Norden Hom-Lie algebra) if and only if*

$$(\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}(\phi_{\mathfrak{g}^!} \circ J))y = -(\phi_{\mathfrak{g}^!} \circ J)(\nabla_x(\phi_{\mathfrak{g}^!} \circ J))y, \tag{4.3}$$

for any $x, y \in \mathfrak{g}^!$.

Proof. Consider $\alpha(x, y, z) = \langle(\nabla_x(\phi_{\mathfrak{g}^!} \circ J))y, \phi_{\mathfrak{g}^!}(z)\rangle$. Theorem 3.16 implies that α is symmetric in the last two variables

$$\alpha(x, y, z) = \alpha(x, z, y). \tag{4.4}$$

Using the definition of α, we have

$$\alpha((\phi_{\mathfrak{g}^!} \circ J)x, y, z) = \langle(\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}(\phi_{\mathfrak{g}^!} \circ J))y, \phi_{\mathfrak{g}^!}(z)\rangle.$$

Conditions (3.8), (4.3) and the above equation imply

$$\alpha((\phi_{\mathfrak{g}^!} \circ J)x, y, z) = -\langle(\phi_{\mathfrak{g}^!} \circ J)(\nabla_x(\phi_{\mathfrak{g}^!} \circ J))y, \phi_{\mathfrak{g}^!}(z)\rangle = -\langle(\nabla_x(\phi_{\mathfrak{g}^!} \circ J))y, (\phi_{\mathfrak{g}^!} \circ J)(\phi_{\mathfrak{g}^!}(z))\rangle,$$

from which it follows

$$\alpha((\phi_{\mathfrak{g}^!} \circ J)x, y, z) = -\alpha(x, y, (\phi_{\mathfrak{g}^!} \circ J)z). \tag{4.5}$$

On the other hand, we have

$$\alpha((\phi_{\mathfrak{g}^!} \circ J)x, y, z) = -\langle(\phi_{\mathfrak{g}^!} \circ J)(\nabla_x(\phi_{\mathfrak{g}^!} \circ J))y, \phi_{\mathfrak{g}^!}(z)\rangle.$$

The above equation is equivalent to

$$\alpha((\phi_{\mathfrak{g}^!} \circ J)x, y, z) = \langle(\nabla_x(\phi_{\mathfrak{g}^!} \circ J))(\phi_{\mathfrak{g}^!} \circ J)y, \phi_{\mathfrak{g}^!}(z)\rangle,$$

which gives us

$$\alpha((\phi_{\mathfrak{g}^!} \circ J)x, y, z) = \alpha(x, (\phi_{\mathfrak{g}^!} \circ J)y, z). \tag{4.6}$$

By (4.4), (4.5) and (4.6), we get

$$\begin{aligned} \alpha((\phi_{\mathfrak{g}^!} \circ J)x, y, z) =& \alpha((\phi_{\mathfrak{g}^!} \circ J)x, z, y) = \alpha(x, (\phi_{\mathfrak{g}^!} \circ J)z, y) \\ =& \alpha(x, y, (\phi_{\mathfrak{g}^!} \circ J)z) = -\alpha((\phi_{\mathfrak{g}^!} \circ J)x, y, z). \end{aligned}$$

From the above equation and the non-degenerate condition of $\langle\cdot,\cdot\rangle$, we obtain $\nabla(\phi_{\mathfrak{g}^!}\circ J)=0$. $\square$

4.1. Left-invariant abelian structures on Hom-Lie groups (abelian complex structures on Hom-Lie algebras)

Let $(G,\diamond,e_\Phi,\Phi,J)$ be a left-invariant almost complex Hom-Lie group (or, $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J)$ be an almost complex Hom-Lie algebra). The left-invariant almost complex J (or, the almost complex structure J) is called abelian if

$$[(\phi_{\mathfrak{g}^!}\circ J)x,(\phi_{\mathfrak{g}^!}\circ J)y]_{\mathfrak{g}^!}=[x,y]_{\mathfrak{g}^!}, \tag{4.7}$$

for any $x,y\in\mathfrak{g}^!$.

Lemma 4.6. *A left-invariant almost Norden Hom-Lie group $(G,\diamond,e_\Phi,\Phi,J,\langle\cdot,\cdot\rangle)$ (or, an almost Norden Hom-Lie algebra $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J,\langle\cdot,\cdot\rangle)$) is a left-invariant Kähler-Norden Hom-Lie group (or, a Kähler-Norden Hom-Lie algebra) if for any $x\in\mathfrak{g}^!$, the following condition is satisfied*

$$\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}=-(\phi_{\mathfrak{g}^!}\circ J)\nabla_x. \tag{4.8}$$

Moreover, the complex structure J is abelian.

Proof. From (4.8), it follows

$$\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}(\phi_{\mathfrak{g}^!}\circ J)=-(\phi_{\mathfrak{g}^!}\circ J)\nabla_x(\phi_{\mathfrak{g}^!}\circ J),$$

and

$$(\phi_{\mathfrak{g}^!}\circ J)\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}=\nabla_x.$$

Subtracting the above two equations, we get (4.3). Now, let $x,y\in\mathfrak{g}^!$. Using (i) of (3.4), we obtain

$$[(\phi_{\mathfrak{g}^!}\circ J)x,(\phi_{\mathfrak{g}^!}\circ J)y]_{\mathfrak{g}^!}=\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}(\phi_{\mathfrak{g}^!}\circ J)y-\nabla_{(\phi_{\mathfrak{g}^!}\circ J)y}(\phi_{\mathfrak{g}^!}\circ J)x.$$

Applying (4.1) and (4.8) in the above equation yields

$$\begin{aligned}[(\phi_{\mathfrak{g}^!}\circ J)x,(\phi_{\mathfrak{g}^!}\circ J)y]_{\mathfrak{g}^!}&=-(\phi_{\mathfrak{g}^!}\circ J)(\nabla_x(\phi_{\mathfrak{g}^!}\circ J)y)+(\phi_{\mathfrak{g}^!}\circ J)(\nabla_y(\phi_{\mathfrak{g}^!}\circ J)x)\\&=\nabla_xy-\nabla_yx=[x,y]_{\mathfrak{g}^!}.\end{aligned}$$

$\square$

Proposition 4.7. *Let $(G,\diamond,e_\Phi,\Phi,J,\langle\cdot,\cdot\rangle)$ be a left-invariant Kähler-Norden Hom-Lie group (or, $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J,\langle\cdot,\cdot\rangle)$ be a Kähler-Norden Hom-Lie algebra). If the left-invariant complex structure J (or, the complex structure J) is abelian, then (4.8) holds.*

Proof. For any $x,y,z\in\mathfrak{g}^!$, setting $\alpha(x,y,z)=\langle\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}y+(\phi_{\mathfrak{g}^!}\circ J)\nabla_xy,\phi_{\mathfrak{g}^!}(z)\rangle$ and using (ii) of (3.4), we get

$$\alpha(x,y,z)=-\langle\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}z,\phi_{\mathfrak{g}^!}(y)\rangle+\langle\nabla_xy,(\phi_{\mathfrak{g}^!}\circ J)\phi_{\mathfrak{g}^!}(z)\rangle=-\langle\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}z+\nabla_x(\phi_{\mathfrak{g}^!}\circ J)z,\phi_{\mathfrak{g}^!}(y)\rangle.$$

According to (4.1), the above equation gives

$$\alpha(x,y,z)=-\langle\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}z+(\phi_{\mathfrak{g}^!}\circ J)(\nabla_xz),\phi_{\mathfrak{g}^!}(y)\rangle,$$

from which it follows

$$\alpha(x,y,z)=-\alpha(x,z,y). \tag{4.9}$$

As J is abelian, from (i) of (3.4) and (4.7), we have

$$(\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}+(\phi_{\mathfrak{g}^!}\circ J)\nabla_x)y=(\nabla_{(\phi_{\mathfrak{g}^!}\circ J)y}+(\phi_{\mathfrak{g}^!}\circ J)\nabla_y)x.$$

Hence α is symmetric in the first two variables

$$\alpha(x, y, z) = \alpha(y, x, z). \tag{4.10}$$

By (4.9) and (4.10), we conclude that

$$\alpha(x, y, z) = -\alpha(x, z, y) = -\alpha(z, x, y) = \alpha(z, y, x).$$

The last equation and (4.9) imply

$$\alpha(x, y, z) = \alpha(y, x, z) = \alpha(z, x, y) = \alpha(x, z, y).$$

From (4.9) and the above equation, we deduce that $\alpha = 0$. Hence the proof completed. $\square$

Proposition 4.8. *Assume that $(G, \diamond, e_\Phi, \Phi, J, \langle \cdot, \cdot \rangle)$ is a left-invariant Kähler-Norden Hom-Lie group (or, $(\mathfrak{g}^!, [\cdot, \cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle \cdot, \cdot \rangle)$ is a Kähler-Norden Hom-Lie algebra) such that the left-invariant complex structure J (or, the complex structure J) is abelian. Then for the left-invariant Hom-Levi-Civita connection ∇ (or, the Hom-Levi-Civita connection ∇), we have*

$$2\nabla_x y = [x, y]_{\mathfrak{g}^!} - (\phi_{\mathfrak{g}^!} \circ J)[x, (\phi_{\mathfrak{g}^!} \circ J)y]_{\mathfrak{g}^!}, \tag{4.11}$$

for any $x, y \in \mathfrak{g}^!$.

Proof. Using (i) of (3.4) and (4.1), we have

$$\begin{aligned}
[(\phi_{\mathfrak{g}^!} \circ J)x, y]_{\mathfrak{g}^!} - (\phi_{\mathfrak{g}^!} \circ J)[x, y]_{\mathfrak{g}^!} &= \nabla_{(\phi_{\mathfrak{g}^!} \circ J)x} y - \nabla_y (\phi_{\mathfrak{g}^!} \circ J)x - (\phi_{\mathfrak{g}^!} \circ J)(\nabla_x y - \nabla_y x) \\
&= \nabla_{(\phi_{\mathfrak{g}^!} \circ J)x} y - (\phi_{\mathfrak{g}^!} \circ J)(\nabla_x y).
\end{aligned}$$

Applying (4.7) and using $(\phi_{\mathfrak{g}} \circ J)^2 = -Id_{\mathfrak{g}^!}$ in the above equation, we obtain the assertion. $\square$

Example 4.9. We consider the Kähler-Norden Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, J, \langle \cdot, \cdot \rangle)$ defined in Example 4.3. As

$$\begin{aligned}
&[(\phi_{\mathfrak{g}} \circ J)e_1, (\phi_{\mathfrak{g}} \circ J)e_2]_{\mathfrak{g}} = e_3 \neq [e_1, e_2]_{\mathfrak{g}}, && [(\phi_{\mathfrak{g}} \circ J)e_1, (\phi_{\mathfrak{g}} \circ J)e_3]_{\mathfrak{g}} = -e_2 \neq [e_1, e_3]_{\mathfrak{g}}, \\
&[(\phi_{\mathfrak{g}} \circ J)e_2, (\phi_{\mathfrak{g}} \circ J)e_4]_{\mathfrak{g}} = -e_2 \neq [e_2, e_4]_{\mathfrak{g}}, && [(\phi_{\mathfrak{g}} \circ J)e_3, (\phi_{\mathfrak{g}} \circ J)e_4]_{\mathfrak{g}} = -e_3 \neq [e_3, e_4]_{\mathfrak{g}},
\end{aligned}$$

so the complex structure J is not abelian.

Example 4.10. For the Kähler-Norden Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, J, \langle \cdot, \cdot \rangle)$ in Example 4.4, we have

$$\begin{aligned}
&[(\phi_{\mathfrak{g}} \circ J)e_3, (\phi_{\mathfrak{g}} \circ J)e_5]_{\mathfrak{g}} = -e_2 = [e_3, e_5]_{\mathfrak{g}}, && [(\phi_{\mathfrak{g}} \circ J)e_3, (\phi_{\mathfrak{g}} \circ J)e_6]_{\mathfrak{g}} = e_1 = [e_3, e_6]_{\mathfrak{g}}, \\
&[(\phi_{\mathfrak{g}} \circ J)e_4, (\phi_{\mathfrak{g}} \circ J)e_5]_{\mathfrak{g}} = e_1 = [e_4, e_5]_{\mathfrak{g}}, && [(\phi_{\mathfrak{g}} \circ J)e_4, (\phi_{\mathfrak{g}} \circ J)e_6]_{\mathfrak{g}} = e_2 = [e_4, e_6]_{\mathfrak{g}},
\end{aligned}$$
$$[(\phi_{\mathfrak{g}} \circ J)e_5, (\phi_{\mathfrak{g}} \circ J)e_6]_{\mathfrak{g}} = e_3 = [e_5, e_6]_{\mathfrak{g}},$$

which give that J satisfies (4.7), i.e. the complex structure J is abelian. We get that $[e_i, e_j]_{\mathfrak{g}} - (\phi_{\mathfrak{g}} \circ J)[e_i, (\phi_{\mathfrak{g}} \circ J)e_j]_{\mathfrak{g}} = 0$, for all $i, j = 1, \cdots, 6$, except

$$\begin{aligned}
&[e_5, e_3]_{\mathfrak{g}} - (\phi_{\mathfrak{g}} \circ J)[e_5, (\phi_{\mathfrak{g}} \circ J)e_3]_{\mathfrak{g}} = 2e_2, && [e_5, e_4]_{\mathfrak{g}} - (\phi_{\mathfrak{g}} \circ J)[e_5, (\phi_{\mathfrak{g}} \circ J)e_4]_{\mathfrak{g}} = -2e_1, \\
&[e_5, e_5]_{\mathfrak{g}} - (\phi_{\mathfrak{g}} \circ J)[e_5, (\phi_{\mathfrak{g}} \circ J)e_5]_{\mathfrak{g}} = e_4, && [e_5, e_6]_{\mathfrak{g}} - (\phi_{\mathfrak{g}} \circ J)[e_5, (\phi_{\mathfrak{g}} \circ J)e_6]_{\mathfrak{g}} = e_3, \\
&[e_6, e_3]_{\mathfrak{g}} - (\phi_{\mathfrak{g}} \circ J)[e_6, (\phi_{\mathfrak{g}} \circ J)e_3]_{\mathfrak{g}} = -2e_1, && [e_6, e_4]_{\mathfrak{g}} - (\phi_{\mathfrak{g}} \circ J)[e_6, (\phi_{\mathfrak{g}} \circ J)e_4]_{\mathfrak{g}} = -2e_2, \\
&[e_6, e_5]_{\mathfrak{g}} - (\phi_{\mathfrak{g}} \circ J)[e_6, (\phi_{\mathfrak{g}} \circ J)e_5]_{\mathfrak{g}} = -e_3, && [e_6, e_6]_{\mathfrak{g}} - (\phi_{\mathfrak{g}} \circ J)[e_6, (\phi_{\mathfrak{g}} \circ J)e_6]_{\mathfrak{g}} = e_4.
\end{aligned}$$

Applying (4.2) and the above equations, we conclude that (4.11) holds.

4.2. Twin Norden metric of Hom-Lie groups and Hom-Lie algebras

In this section, we restrict ourselves to the case where $\langle\cdot,\cdot\rangle$ is a left-invariant Norden metric (or, a Norden metric) on a left-invariant almost complex Hom-Lie group $(G,\diamond,e_\Phi,\Phi)$ (or, an almost complex Hom-Lie algebra $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!})$). So we can apply J to obtain a new left-invariant Norden metric (or, a Norden metric) $\ll\cdot,\cdot\gg$ associated with the left-invariant Norden metric (or, the Norden metric) $\langle\cdot,\cdot\rangle$ of the left-invariant almost Norden Hom-Lie group $(G,\diamond,e_\Phi,\Phi,J,\langle\cdot,\cdot\rangle)$ (or, the almost Norden Hom-Lie algebra $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J,\langle\cdot,\cdot\rangle)$) defined by

$$\ll x,y\gg=\langle(\phi_{\mathfrak{g}^!}\circ J)x,y\rangle,$$

for any $x,y\in\mathfrak{g}^!$. In this case, the pure tensor $\ll\cdot,\cdot\gg$ is called the twin metric of $\langle\cdot,\cdot\rangle$.

Proposition 4.11. *Let $(J,\langle\cdot,\cdot\rangle)$ be a left-invariant almost Norden structure on a Hom-Lie group $(G,\diamond,e_\Phi,\Phi)$ (or, an almost Norden structure on a Hom-Lie algebra $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!})$) and $\ll\cdot,\cdot\gg=\langle(\phi_{\mathfrak{g}^!}\circ J)\cdot,\cdot\rangle$. Then the following identities hold*

(i) $(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\ll\cdot,\cdot\gg)(x,y,z)=(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\langle\cdot,\cdot\rangle)(x,(\phi_{\mathfrak{g}^!}\circ J)y,z)+\langle N_{\phi_{\mathfrak{g}^!}\circ J}(x,y),\phi_{\mathfrak{g}^!}(z)\rangle$,

(ii) $(\nabla_x\ll\cdot,\cdot\gg)(y,z)=(\nabla_x\langle\cdot,\cdot\rangle)((\phi_{\mathfrak{g}^!}\circ J)y,z)+\langle(\nabla_x(\phi_{\mathfrak{g}^!}\circ J))y,\phi_{\mathfrak{g}^!}(z)\rangle$,

for any $x,y,z\in\mathfrak{g}^!$.

Proof. Using (3.10), we have

$$\begin{aligned}(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\ll\cdot,\cdot\gg)(x,y,z)=&\ll[y,(\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!}-(\phi_{\mathfrak{g}^!}\circ J)[y,x]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!}(z)\gg\\&+\ll\phi_{\mathfrak{g}^!}(y),[z,(\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!}-(\phi_{\mathfrak{g}^!}\circ J)[z,x]_{\mathfrak{g}^!}\gg.\end{aligned}$$

The above equation leads to

$$\begin{aligned}(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\ll\cdot,\cdot\gg)(x,y,z)=&\langle(\phi_{\mathfrak{g}^!}\circ J)[y,(\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!}-(\phi_{\mathfrak{g}^!}\circ J)[y,x]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!}(z)\rangle\\&+\langle(\phi_{\mathfrak{g}^!}\circ J)\phi_{\mathfrak{g}^!}(y),[z,(\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!}-(\phi_{\mathfrak{g}^!}\circ J)[z,x]_{\mathfrak{g}^!}\rangle.\end{aligned}$$

By adding and subtracting $\langle(\phi_{\mathfrak{g}^!}\circ J)[x,(\phi_{\mathfrak{g}^!}\circ J)(y)]_{\mathfrak{g}^!}+[(\phi_{\mathfrak{g}^!}\circ J)x,(\phi_{\mathfrak{g}^!}\circ J)(y)]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!}(z)\rangle$ in the above equation, we get (i). From (3.7), we have

$$(\nabla_x\ll\cdot,\cdot\gg)(y,z)=-\ll\nabla_x y,\phi_{\mathfrak{g}^!}(z)\gg-\ll\phi_{\mathfrak{g}^!}(y),\nabla_x z\gg,$$

from which it follows

$$\begin{aligned}(\nabla_x\ll\cdot,\cdot\gg)(y,z)=&-\langle(\phi_{\mathfrak{g}^!}\circ J)(\nabla_x y),\phi_{\mathfrak{g}^!}(z)\rangle-\langle(\phi_{\mathfrak{g}^!}\circ J)(\phi_{\mathfrak{g}^!}(y)),\nabla_x z\rangle\\=&-\langle(\phi_{\mathfrak{g}^!}\circ J)(\nabla_x y)+\nabla_x(\phi_{\mathfrak{g}^!}\circ J)y-\nabla_x(\phi_{\mathfrak{g}^!}\circ J)y,\phi_{\mathfrak{g}^!}(z)\rangle\\&-\langle(\phi_{\mathfrak{g}^!}\circ J)(\phi_{\mathfrak{g}^!}(y)),\nabla_x z\rangle.\end{aligned}$$

Applying (3.7) in the last equation, yields (ii). The proof is completed. □

As an immediate consequence of formula (i), we obtain that in a left-invariant Norden Hom-Lie group $(G,\diamond,e_\Phi,\Phi,J,\langle\cdot,\cdot\rangle)$ (or, a Norden Hom-Lie algebra $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J,\langle\cdot,\cdot\rangle)$), $\Phi_{\phi_{\mathfrak{g}^!}\circ J}\langle\cdot,\cdot\rangle=0$ if and only if $\Phi_{\phi_{\mathfrak{g}^!}\circ J}\ll\cdot,\cdot\gg=0$.

5. Left-invariant holomorphic Norden Hom-lie groups (holomorphic Norden Hom-Lie algebras)

In this Section, we give the definition of a left-invariant holomorphic Norden Hom-Lie group (or, a holomorphic Norden Hom-Lie algebra). We show that there exists a one-to-one correspondence between left-invariant Kähler-Norden Hom-Lie groups (or, Kähler-Norden Hom-Lie algebras) and left-invariant holomorphic Norden Hom-Lie groups (or, holomorphic Norden Hom-Lie algebras).

Proposition 5.1. *Let $(G, \diamond, e_\Phi, \Phi, J, \langle\cdot,\cdot\rangle)$ be a left-invariant almost Norden Hom-Lie group (or, $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle\cdot,\cdot\rangle)$ be an almost Norden Hom-Lie algebra). Then*

$$(i)\quad (\Phi_{\phi_{\mathfrak{g}^!}\circ J}\langle\cdot,\cdot\rangle)(x,y,z) = \langle(\nabla_y(\phi_{\mathfrak{g}^!}\circ J))x, \phi_{\mathfrak{g}^!}(z)\rangle + \langle\phi_{\mathfrak{g}^!}(y), (\nabla_z(\phi_{\mathfrak{g}^!}\circ J))x\rangle - \langle(\nabla_x(\phi_{\mathfrak{g}^!}\circ J))(y), \phi_{\mathfrak{g}^!}z\rangle,$$

$$(ii)\quad (\Phi_{\phi_{\mathfrak{g}^!}\circ J}\langle\cdot,\cdot\rangle)(x,y,z) + (\Phi_{\phi_{\mathfrak{g}^!}\circ J}\langle\cdot,\cdot\rangle)(z,y,x) = 2\langle(\nabla_y(\phi_{\mathfrak{g}^!}\circ J))x, \phi_{\mathfrak{g}^!}(z)\rangle,$$

for any $x,y,z \in \mathfrak{g}^!$, where ∇ is the left-invariant Hom-Levi-Civita connection on G (or, the Hom-Levi-Civita connection on $\mathfrak{g}^!$).

Proof. From (3.10), we have

$$(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\langle\cdot,\cdot\rangle)(x,y,z) = \langle[y, (\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!} - (\phi_{\mathfrak{g}^!}\circ J)[y,x]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}(z)\rangle + \langle\phi_{\mathfrak{g}^!}(y), [z, (\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!} - (\phi_{\mathfrak{g}^!}\circ J)[z,x]_{\mathfrak{g}^!}\rangle.$$

Applying (i) of (3.4) in the above equation yields

$$\begin{aligned}(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\langle\cdot,\cdot\rangle)(x,y,z) =& \langle\nabla_y(\phi_{\mathfrak{g}^!}\circ J)(x) - \nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)}y - (\phi_{\mathfrak{g}^!}\circ J)(\nabla_y x - \nabla_x y), \phi_{\mathfrak{g}^!}(z)\rangle \\ &+ \langle\phi_{\mathfrak{g}^!}(y), \nabla_z(\phi_{\mathfrak{g}^!}\circ J)(x) - \nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)}z - (\phi_{\mathfrak{g}^!}\circ J)(\nabla_z x - \nabla_x z)\rangle.\end{aligned} \tag{5.1}$$

On the other hand, from (ii) of (3.4) and Definition 3.11 it follows

$$\langle\nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)}y, \phi_{\mathfrak{g}^!}(z)\rangle = -\langle\phi_{\mathfrak{g}^!}(y), \nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)}z\rangle,$$
$$\langle\phi_{\mathfrak{g}^!}(y), (\phi_{\mathfrak{g}^!}\circ J)(\nabla_x z)\rangle = -\langle\nabla_x(\phi_{\mathfrak{g}^!}\circ J)y, \phi_{\mathfrak{g}^!}(z)\rangle.$$

Setting two last equations in (5.1), we get (i). Similarly, we have

$$(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\langle\cdot,\cdot\rangle)(z,y,x) = \langle(\nabla_y(\phi_{\mathfrak{g}^!}\circ J))z, \phi_{\mathfrak{g}^!}(x)\rangle + \langle\phi_{\mathfrak{g}^!}(y), (\nabla_x(\phi_{\mathfrak{g}^!}\circ J))z - \langle(\nabla_z(\phi_{\mathfrak{g}^!}\circ J))y, \phi_{\mathfrak{g}^!}(x)\rangle.$$

The above equation and Theorem 3.16 imply (ii). □

Assume that $(J, \langle\cdot,\cdot\rangle)$ is a left-invariant almost Norden structure on a Hom-Lie group $(G, \diamond, e_\Phi, \Phi)$ (or, an almost Norden structure on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!})$). We say that $(G, \diamond, e_\Phi, \Phi, J, \langle\cdot,\cdot\rangle)$ (or, $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle\cdot,\cdot\rangle)$) is a left-invariant almost *holomorphic* Norden Hom-Lie group (or, an almost *holomorphic* Norden Hom-Lie algebra) if

$$\Phi_{\phi_{\mathfrak{g}^!}\circ J}\langle\cdot,\cdot\rangle = 0,$$

i.e.

$$\begin{aligned}&\langle[y, (\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}(z)\rangle + \langle\phi_{\mathfrak{g}^!}(y), [z, (\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!}\rangle \\ =&\langle(\phi_{\mathfrak{g}^!}\circ J)[y,x]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}(z)\rangle + \langle\phi_{\mathfrak{g}^!}(y), (\phi_{\mathfrak{g}^!}\circ J)[z,x]_{\mathfrak{g}^!}\rangle,\end{aligned} \tag{5.2}$$

for any $x,y,z \in \mathfrak{g}^!$. Also, if J is integrable, $(G, \diamond, e_\Phi, \Phi, J, \langle\cdot,\cdot\rangle)$ is called a left-invariant holomorphic Norden Hom-Lie group (or, $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle\cdot,\cdot\rangle)$ is called a holomorphic Norden Hom-Lie algebra).

Example 5.2. We consider the Norden Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, J, \langle\cdot,\cdot\rangle)$ introduced in Example 3.13. Setting $x = y = w = e_1$ in (5.2), implies $A = 0$. Considering $x = y = e_1$ and $w = e_2$, we get $B = 0$. Also, putting $x = y = e_1$ and $w = e_3$, yields $D = 0$. Moreover, from $x = y = e_1$ and $w = e_4$, we conclude $C = 0$. This is a contradiction with the non-degeneracy of the metric $\langle\cdot,\cdot\rangle$ and so $(\mathfrak{g}, [\cdot,\cdot], \phi_{\mathfrak{g}}, J, \langle\cdot,\cdot\rangle)$ is not a holomorphic Norden Hom-Lie algebra.

Example 5.3. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, J, \langle,\rangle)$ be a 4-dimensional Norden Hom-Lie algebra defined in Example 3.15. It results that all of

$$\begin{aligned}&\langle[e_i, (\phi_{\mathfrak{g}}\circ J)(e_j)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(e_k)\rangle + \langle\phi_{\mathfrak{g}}(e_i), [e_k, (\phi_{\mathfrak{g}}\circ J)(e_j)]_{\mathfrak{g}}\rangle \\ &= \langle(\phi_{\mathfrak{g}}\circ J)[e_i, e_j]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(e_k)\rangle + \langle\phi_{\mathfrak{g}}(e_i), (\phi_{\mathfrak{g}}\circ J)[e_k, e_j]_{\mathfrak{g}}\rangle, \quad i,j,k = 1,2,3,4,\end{aligned}$$

are zero except

$$
\begin{aligned}
&\langle[e_1,(\phi_{\mathfrak{g}}\circ J)e_2]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_2)\rangle+\langle\phi_{\mathfrak{g}}(e_1),[e_2,(\phi_{\mathfrak{g}}\circ J)e_2]_{\mathfrak{g}}\rangle\\
&\quad=-B=\langle(\phi_{\mathfrak{g}}\circ J)[e_1,e_2]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_2)\rangle+\langle\phi_{\mathfrak{g}}(e_1),(\phi_{\mathfrak{g}}\circ J)[e_2,e_2]_{\mathfrak{g}}\rangle,\\
&\langle[e_1,(\phi_{\mathfrak{g}}\circ J)e_3]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_3)\rangle+\langle\phi_{\mathfrak{g}}(e_1),[e_3,(\phi_{\mathfrak{g}}\circ J)e_3]_{\mathfrak{g}}\rangle\\
&\quad=B=\langle(\phi_{\mathfrak{g}}\circ J)[e_1,e_3]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_3)\rangle+\langle\phi_{\mathfrak{g}}(e_1),(\phi_{\mathfrak{g}}\circ J)[e_3,e_3]_{\mathfrak{g}}\rangle,\\
&\langle[e_2,(\phi_{\mathfrak{g}}\circ J)e_1]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_2)\rangle+\langle\phi_{\mathfrak{g}}(e_2),[e_2,(\phi_{\mathfrak{g}}\circ J)e_1]_{\mathfrak{g}}\rangle\\
&\quad=2B=\langle(\phi_{\mathfrak{g}}\circ J)[e_2,e_1]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_2)\rangle+\langle\phi_{\mathfrak{g}}(e_2),(\phi_{\mathfrak{g}}\circ J)[e_2,e_1]_{\mathfrak{g}}\rangle,\\
&\langle[e_2,(\phi_{\mathfrak{g}}\circ J)e_2]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_1)\rangle+\langle\phi_{\mathfrak{g}}(e_2),[e_1,(\phi_{\mathfrak{g}}\circ J)e_2]_{\mathfrak{g}}\rangle\\
&\quad=-B=\langle(\phi_{\mathfrak{g}}\circ J)[e_2,e_2]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_1)\rangle+\langle\phi_{\mathfrak{g}}(e_2),(\phi_{\mathfrak{g}}\circ J)[e_1,e_2]_{\mathfrak{g}}\rangle,\\
&\langle[e_2,(\phi_{\mathfrak{g}}\circ J)e_3]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_4)\rangle+\langle\phi_{\mathfrak{g}}(e_2),[e_4,(\phi_{\mathfrak{g}}\circ J)e_3]_{\mathfrak{g}}\rangle\\
&\quad=-B=\langle(\phi_{\mathfrak{g}}\circ J)[e_2,e_3]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_4)\rangle+\langle\phi_{\mathfrak{g}}(e_2),(\phi_{\mathfrak{g}}\circ J)[e_4,e_3]_{\mathfrak{g}}\rangle,\\
&\langle[e_2,(\phi_{\mathfrak{g}}\circ J)e_4]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_3)\rangle+\langle\phi_{\mathfrak{g}}(e_2),[e_3,(\phi_{\mathfrak{g}}\circ J)e_4]_{\mathfrak{g}}\rangle\\
&\quad=2B=\langle(\phi_{\mathfrak{g}}\circ J)[e_2,e_4]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_3)\rangle+\langle\phi_{\mathfrak{g}}(e_2),(\phi_{\mathfrak{g}}\circ J)[e_3,e_4]_{\mathfrak{g}}\rangle,\\
&\langle[e_3,(\phi_{\mathfrak{g}}\circ J)e_1]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_3)\rangle+\langle\phi_{\mathfrak{g}}(e_3),[e_3,(\phi_{\mathfrak{g}}\circ J)e_1]_{\mathfrak{g}}\rangle\\
&\quad=-2B=\langle(\phi_{\mathfrak{g}}\circ J)[e_3,e_1]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_3)\rangle+\langle\phi_{\mathfrak{g}}(e_3),(\phi_{\mathfrak{g}}\circ J)[e_3,e_1]_{\mathfrak{g}}\rangle,\\
&\langle[e_3,(\phi_{\mathfrak{g}}\circ J)e_2]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_4)\rangle+\langle\phi_{\mathfrak{g}}(e_3),[e_4,(\phi_{\mathfrak{g}}\circ J)e_2]_{\mathfrak{g}}\rangle\\
&\quad=-B=\langle(\phi_{\mathfrak{g}}\circ J)[e_3,e_2]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_4)\rangle+\langle\phi_{\mathfrak{g}}(e_3),(\phi_{\mathfrak{g}}\circ J)[e_4,e_2]_{\mathfrak{g}}\rangle,\\
&\langle[e_3,(\phi_{\mathfrak{g}}\circ J)e_3]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_1)\rangle+\langle\phi_{\mathfrak{g}}(e_3),[e_1,(\phi_{\mathfrak{g}}\circ J)e_3]_{\mathfrak{g}}\rangle\\
&\quad=B=\langle(\phi_{\mathfrak{g}}\circ J)[e_3,e_3]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_1)\rangle+\langle\phi_{\mathfrak{g}}(e_3),(\phi_{\mathfrak{g}}\circ J)[e_1,e_3]_{\mathfrak{g}}\rangle,\\
&\langle[e_3,(\phi_{\mathfrak{g}}\circ J)e_4]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_2)\rangle+\langle\phi_{\mathfrak{g}}(e_3),[e_2,(\phi_{\mathfrak{g}}\circ J)e_4]_{\mathfrak{g}}\rangle\\
&\quad=2B=\langle(\phi_{\mathfrak{g}}\circ J)[e_3,e_4]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_2)\rangle+\langle\phi_{\mathfrak{g}}(e_3),(\phi_{\mathfrak{g}}\circ J)[e_2,e_4]_{\mathfrak{g}}\rangle,\\
&\langle[e_4,(\phi_{\mathfrak{g}}\circ J)e_2]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_3)\rangle+\langle\phi_{\mathfrak{g}}(e_4),[e_3,(\phi_{\mathfrak{g}}\circ J)e_2]_{\mathfrak{g}}\rangle\\
&\quad=-B=\langle(\phi_{\mathfrak{g}}\circ J)[e_4,e_2]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_3)\rangle+\langle\phi_{\mathfrak{g}}(e_4),(\phi_{\mathfrak{g}}\circ J)[e_3,e_2]_{\mathfrak{g}}\rangle,\\
&\langle[e_4,(\phi_{\mathfrak{g}}\circ J)e_3]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_2)\rangle+\langle\phi_{\mathfrak{g}}(e_4),[e_2,(\phi_{\mathfrak{g}}\circ J)e_3]_{\mathfrak{g}}\rangle\\
&\quad=-B=\langle(\phi_{\mathfrak{g}}\circ J)[e_4,e_3]_{\mathfrak{g}},\phi_{\mathfrak{g}}(e_2)\rangle+\langle\phi_{\mathfrak{g}}(e_4),(\phi_{\mathfrak{g}}\circ J)[e_2,e_3]_{\mathfrak{g}}\rangle,
\end{aligned}
$$

which imply that, (5.2) holds. Thus $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}},J,\langle\cdot,\cdot\rangle)$ is a holomorphic Norden Hom-Lie algebra.

Corollary 5.4. *A left-invariant almost Norden Hom-Lie group $(G,\diamond,e_\Phi,\Phi,J,\langle\cdot,\cdot\rangle)$ (or, an almost Norden Hom-Lie algebra $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J,\langle\cdot,\cdot\rangle)$) is holomorphic if and only if the structure $\phi_{\mathfrak{g}^!}\circ J$ is invariant with respect to the left-invariant Hom-Levi-Civita connection (or, the Hom-Levi-Civita connection) ∇, i.e.*

$$\nabla(\phi_{\mathfrak{g}^!}\circ J)=0.$$

Applying Corollary 5.4, we deduce

Proposition 5.5. *Let $(G,\diamond,e_\Phi,\Phi,J)$ be a left-invariant almost complex Hom-Lie group (or, $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J)$ be an almost complex Hom-Lie algebra) equipped with a left-invariant pseudo-Riemannian metric (or, a pseudo-Riemannian metric) $\langle\cdot,\cdot\rangle$ and a left-invariant connection (or, connection) ∇. Then the following statements are equivalent:*
a) $(G,\diamond,e_\Phi,\Phi,J,\langle\cdot,\cdot\rangle)$ is a left-invariant Kähler-Norden Hom-Lie group
(or, $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J,\langle\cdot,\cdot\rangle)$ is a Kähler-Norden Hom-Lie algebra),
b) $(G,\diamond,e_\Phi,\Phi,J,\langle\cdot,\cdot\rangle)$ is a left-invariant holomorphic Norden Hom-Lie group
(or, $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J,\langle\cdot,\cdot\rangle)$ is a holomorphic Norden Hom-Lie algebra).

Corollary 5.6. *On a left-invariant Kähler-Norden Hom-Lie group* $(G, \diamond, e_\Phi, \Phi, J, \langle\cdot,\cdot\rangle)$ *(or, a Kähler-Norden Hom-Lie algebra* $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle\cdot,\cdot\rangle)$*), we have*

$$\Phi_{\phi_{\mathfrak{g}^!}\circ J}\langle\cdot,\cdot\rangle = 0 = \Phi_{\phi_{\mathfrak{g}^!}\circ J} \ll \cdot,\cdot \gg .$$

Corollary 5.7. *Let* $(G, \diamond, e_\Phi, \Phi, J, \langle\cdot,\cdot\rangle)$ *be a left-invariant Kähler-Norden Hom-Lie group (or,* $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle\cdot,\cdot\rangle)$ *be a Kähler-Norden Hom-Lie algebra) and* $\ll \cdot,\cdot \gg = \langle(\phi_{\mathfrak{g}^!}\circ J)\cdot,\cdot\rangle$. *Then the following statements hold:*
a) $(G, \diamond, e_\Phi, \Phi, J, \ll \cdot,\cdot \gg)$ *is a left-invariant Kähler-Norden Hom-Lie group*
(or, $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \ll \cdot,\cdot \gg)$ *is a Kähler-Norden Hom-Lie algebra),*
b) $(G, \diamond, e_\Phi, \Phi, J, \ll \cdot,\cdot \gg)$ *is a left-invariant holomorphic Norden Hom-Lie group (or,* $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \ll \cdot,\cdot \gg)$ *is a holomorphic Norden Hom-Lie algebra).*

6. Curvature tensors in holomorphic Norden Hom-Lie algebras

In this section, we provide some properties of the Riemannian curvature tensor of a left-invariant holomorphic Norden Hom-Lie group (or, a holomorphic Norden Hom-Lie algebra). Also, we show that any left-invariant holomorphic Hom-Lie group is flat (or, holomorphic Norden Hom-Lie algebra carries a Hom-Left-symmetric algebra) if its left-invariant complex structure (or, complex structure) is abelian.

Let $(G, \diamond, e_\Phi, \Phi, J, \langle\cdot,\cdot\rangle)$ be a left-invariant Kähler-Norden Hom-Lie group (or, $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle\cdot,\cdot\rangle)$ be a Kähler-Norden Hom-Lie algebra). The (1,3)-curvature tensor $\mathcal{K}$ of G (or, $\mathfrak{g}^!$) is defined by

$$\mathcal{K}(x,y) := \nabla_{\phi_{\mathfrak{g}^!}(x)}\circ\nabla_y - \nabla_{\phi_{\mathfrak{g}^!}(y)}\circ\nabla_x - \nabla_{[x,y]_{\mathfrak{g}^!}}\circ\phi_{\mathfrak{g}^!},$$

for any $x, y \in \mathfrak{g}^!$. Also, left-invariant Kähler-Norden Hom-Lie group is said to be flat if its curvature tensor vanishes identically.

Theorem 6.1. *[24] The curvature tensor* R *of a pseudo-Riemannian Hom-Lie algebra* $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, \langle\cdot,\cdot\rangle)$ *has the following properties*

$$\begin{aligned}
&\langle\mathcal{K}(x,y)z,w\rangle + \langle\mathcal{K}(y,z)x,w\rangle + \langle\mathcal{K}(z,x)y,w\rangle = 0,\\
&\langle\mathcal{K}(x,y)z,w\rangle = -\langle\mathcal{K}(y,x)z,w\rangle,\\
&\langle\mathcal{K}(x,y)z,\phi^2_{\mathfrak{g}^!}(w)\rangle = -\langle\mathcal{K}(x,y)w,\phi^2_{\mathfrak{g}^!}(z)\rangle,\\
&\langle\mathcal{K}(x,y)z,\phi^2_{\mathfrak{g}^!}(w)\rangle = \langle\mathcal{K}(z,w)x,\phi^2_{\mathfrak{g}^!}(y)\rangle,
\end{aligned}$$

for any $x, y, z, w \in \mathfrak{g}^!$.

Lemma 6.2. *For an involutive pseudo-Riemannian Hom-Lie algebra* $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, \langle\cdot,\cdot\rangle)$, *the curvature tensor* $\mathcal{K}$ *satisfies*

$$(\nabla_x\mathcal{K})(y,z,w) + (\nabla_y\mathcal{K})(z,x,w) + (\nabla_z\mathcal{K})(x,y,w) = 0, \qquad \forall x,y,z,w \in \mathfrak{g}^!. \tag{6.1}$$

Proof. By (3.7), we have

$$\begin{aligned}
(\nabla_x\mathcal{K})(y,z,w) =& \nabla_x\mathcal{K}(y,z)w - \mathcal{K}(\nabla_x y,\phi_{\mathfrak{g}^!}(z))\phi_{\mathfrak{g}^!}(w) - \mathcal{K}(\phi_{\mathfrak{g}^!}(y),\nabla_x z)\phi_{\mathfrak{g}^!}(w)\\
&- \mathcal{K}(\phi_{\mathfrak{g}^!}(y),\phi_{\mathfrak{g}^!}(z))\nabla_x w.
\end{aligned}$$

Using the above equation and the part (i) of (3.4), we obtain

$$\begin{aligned}
&(\nabla_x\mathcal{K})(y,z,w) + (\nabla_y\mathcal{K})(z,x,w) + (\nabla_z\mathcal{K})(x,y,w) = \nabla_x\mathcal{K}(y,z)w \qquad (6.2)\\
&- \mathcal{K}([x,y]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!}(z))\phi_{\mathfrak{g}^!}(w) - \mathcal{K}(\phi_{\mathfrak{g}^!}(y),\phi_{\mathfrak{g}^!}(z))\nabla_x w + \nabla_y\mathcal{K}(z,x)w - \mathcal{K}([y,z]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!}(x))\phi_{\mathfrak{g}^!}(w)\\
&- \mathcal{K}(\phi_{\mathfrak{g}^!}(z),\phi_{\mathfrak{g}^!}(x))\nabla_y w + \nabla_z\mathcal{K}(x,y)w - \mathcal{K}([z,x]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!}(y))\phi_{\mathfrak{g}^!}(w) - \mathcal{K}(\phi_{\mathfrak{g}^!}(x),\phi_{\mathfrak{g}^!}(y))\nabla_z w.
\end{aligned}$$

On the other hand

$$\nabla_x\mathcal{K}(y,z)w = \nabla_x\nabla_{\phi_{\mathfrak{g}^!}(y)}\nabla_z w - \nabla_x\nabla_{\phi_{\mathfrak{g}^!}(z)}\nabla_y w - \nabla_x\nabla_{[y,z]_{\mathfrak{g}^!}}\phi_{\mathfrak{g}^!}(w).$$

Applying Hom-Jacobi identity and using the last equation in the (6.2), we obtain the assertion. $\square$

The condition $\nabla(\phi_{\mathfrak{g}^!}\circ J)=0$, in a left-invariant Kähler-Norden Hom-Lie group $(G,\diamond,e_\Phi,\Phi,J,\langle\cdot,\cdot\rangle)$ (or, a Kähler-Norden Hom-Lie algebra $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J,\langle\cdot,\cdot\rangle)$), leads to

$$\mathcal{K}(x,y)(\phi_{\mathfrak{g}^!}\circ J)z=(\phi_{\mathfrak{g}^!}\circ J)\mathcal{K}(x,y)z.$$

We set

$$\mathcal{K}(x,y,z,w)=\langle\mathcal{K}(x,y)z,w\rangle. \tag{6.3}$$

If $\phi_{\mathfrak{g}^!}$ is involutive, then

$$\begin{aligned}\mathcal{K}(x,y,(\phi_{\mathfrak{g}^!}\circ J)z,w)&=\mathcal{K}(x,y,z,(\phi_{\mathfrak{g}^!}\circ J)w),\\ \mathcal{K}((\phi_{\mathfrak{g}^!}\circ J)x,y,z,w)&=\mathcal{K}(x,(\phi_{\mathfrak{g}^!}\circ J)y,z,w),\end{aligned}$$

i.e. $\mathcal{K}$ is pure with respect to z and w, and also pure with respect to x and y. Consider $\ll\cdot,\cdot\gg=\langle(\phi_{\mathfrak{g}^!}\circ J)\cdot,\cdot\rangle$ and $\widetilde{\mathcal{K}}$ as the curvature tensor of $\ll\cdot,\cdot\gg$. By means of the Proposition 4.11, $\mathcal{K}=\widetilde{\mathcal{K}}$ and considering $\widetilde{\mathcal{K}}(x,y,z,w)=\ll\widetilde{\mathcal{K}}(x,y)z,w\gg$, we get

$$\begin{aligned}\widetilde{\mathcal{K}}(x,y,z,w)&=\ll\widetilde{\mathcal{K}}(x,y)z,w\gg=\langle(\phi_{\mathfrak{g}^!}\circ J)\widetilde{\mathcal{K}}(x,y)z,w\rangle\\ &=\langle\widetilde{\mathcal{K}}(x,y)z,(\phi_{\mathfrak{g}^!}\circ J)w\rangle=\langle\mathcal{K}(x,y)z,(\phi_{\mathfrak{g}^!}\circ J)w\rangle=\mathcal{K}(x,y,z,(\phi_{\mathfrak{g}^!}\circ J)w),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathcal{K}}(z,w,x,y)&=\ll\widetilde{\mathcal{K}}(z,w)x,y\gg=\langle(\phi_{\mathfrak{g}^!}\circ J)\widetilde{\mathcal{K}}(z,w)x,y\rangle\\ &=\langle\widetilde{\mathcal{K}}(z,w)x,(\phi_{\mathfrak{g}^!}\circ J)y\rangle=\langle\mathcal{K}(z,w)x,(\phi_{\mathfrak{g}^!}\circ J)y\rangle=\mathcal{K}(z,w,x,(\phi_{\mathfrak{g}^!}\circ J)y).\end{aligned}$$

Notice that $\widetilde{\mathcal{K}}(x,y,z,w)=\widetilde{\mathcal{K}}(z,w,x,y)$, thus the above equations imply

$$\mathcal{K}(x,y,z,(\phi_{\mathfrak{g}^!}\circ J)w)=\mathcal{K}(z,w,x,(\phi_{\mathfrak{g}^!}\circ J)y),$$

which means that $\mathcal{K}(x,y,z,w)$ is pure with respect to y and w. Thus the pseudo-Riemannian curvature tensor $\mathcal{K}(x,y,z,w)$ is pure.

Actually, we proved the following result.

Proposition 6.3. *The pseudo-Riemannian curvature tensor of an involutive left-invariant holomorphic Norden Hom-Lie group (or, an involutive holomorphic Norden Hom-Lie algebra) is pure.*

Theorem 6.4. *In an involutive left-invariant holomorphic Norden Hom-Lie group $(G,\diamond,e_\Phi,\Phi,J,\langle\cdot,\cdot\rangle)$ (or, an involutive holomorphic Norden Hom-Lie algebra $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J,\langle\cdot,\cdot\rangle)$), the curvature tensor is a holomorphic tensor, i.e.*

$$\Phi_{\phi_{\mathfrak{g}^!}\circ J}\mathcal{K}=0.$$

Proof. According to (3.10), we have

$$\begin{aligned}(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\mathcal{K})(x,y,z,w,t)&=\mathcal{K}([y,(\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!}-(\phi_{\mathfrak{g}^!}\circ J)[y,x]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!}(z),\phi_{\mathfrak{g}^!}(w),\phi_{\mathfrak{g}^!}(t))\\ &\quad+\mathcal{K}(\phi_{\mathfrak{g}^!}(y),[z,(\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!}-(\phi_{\mathfrak{g}^!}\circ J)[z,x]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!}(w),\phi_{\mathfrak{g}^!}(t))\\ &\quad+\mathcal{K}(\phi_{\mathfrak{g}^!}(y),\phi_{\mathfrak{g}^!}(z),[w,(\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!}-(\phi_{\mathfrak{g}^!}\circ J)[w,x]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!}(t))\\ &\quad+\mathcal{K}(\phi_{\mathfrak{g}^!}(y),\phi_{\mathfrak{g}^!}(z),\phi_{\mathfrak{g}^!}(w),[t,(\phi_{\mathfrak{g}^!}\circ J)(x)]_{\mathfrak{g}^!}-(\phi_{\mathfrak{g}^!}\circ J)[t,x]_{\mathfrak{g}^!}).\end{aligned}$$

From condition $\nabla(\phi_{\mathfrak{g}^!} \circ J) = (\phi_{\mathfrak{g}^!} \circ J)\nabla$ and the last equation, one gets

$$\begin{aligned}(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\mathcal{K})(x,y,z,w,t) = &\mathcal{K}(-\nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)}y + (\phi_{\mathfrak{g}^!} \circ J)(\nabla_x y), \phi_{\mathfrak{g}^!}(z), \phi_{\mathfrak{g}^!}(w), \phi_{\mathfrak{g}^!}(t))\\ &+ \mathcal{K}(\phi_{\mathfrak{g}^!}(y), -\nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)}z + (\phi_{\mathfrak{g}^!} \circ J)(\nabla_x z), \phi_{\mathfrak{g}^!}(w), \phi_{\mathfrak{g}^!}(t))\\ &+ \mathcal{K}(\phi_{\mathfrak{g}^!}(y), \phi_{\mathfrak{g}^!}(z), -\nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)}w + (\phi_{\mathfrak{g}^!} \circ J)(\nabla_x w), \phi_{\mathfrak{g}^!}(t))\\ &+ \mathcal{K}(\phi_{\mathfrak{g}^!}(y), \phi_{\mathfrak{g}^!}(z), \phi_{\mathfrak{g}^!}(w), -\nabla_{(\phi_{\mathfrak{g}^!}\circ J)(x)}t + (\phi_{\mathfrak{g}^!} \circ J)(\nabla_x t)).\end{aligned}$$

Using again (3.10) in the above equation, it follows

$$(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\mathcal{K})(x,y,z,w,t) = (\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}\mathcal{K})(y,z,w,t) - (\nabla_x\mathcal{K})((\phi_{\mathfrak{g}^!} \circ J)y,z,w,t).$$

Applying (3.7), (6.3) and $\nabla(\phi_{\mathfrak{g}^!} \circ J) = 0$ in the above equation, we get

$$(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\mathcal{K})(x,y,z,w,t) = \langle(\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}\mathcal{K})(y,z)w,t\rangle - \langle(\phi_{\mathfrak{g}^!} \circ J)(\nabla_x\mathcal{K})(y,z)w,t\rangle.$$

From (6.1) and the last equation, we obtain

$$\begin{aligned}(\Phi_{\phi_{\mathfrak{g}^!}\circ J}\mathcal{K})(x,y,z,w,t) = &- \langle(\nabla_y\mathcal{K})(z,(\phi_{\mathfrak{g}^!} \circ J)x)w,t\rangle - \langle(\nabla_z\mathcal{K})((\phi_{\mathfrak{g}^!} \circ J)x,y)w,t\rangle\\ &- \langle(\phi_{\mathfrak{g}^!} \circ J)(\nabla_x\mathcal{K})(y,z)w,t\rangle.\end{aligned}$$

On the other hand, since the curvature tensor is pure and $\nabla(\phi_{\mathfrak{g}^!} \circ J) = 0$, it is easy to see that

$$\begin{aligned}(\nabla_{(\phi_{\mathfrak{g}^!}\circ J)x}\mathcal{K})(y,z,w,t) = &- \langle(\phi_{\mathfrak{g}^!} \circ J)(\nabla_y\mathcal{K})(z,x)w,t\rangle - \langle(\phi_{\mathfrak{g}^!} \circ J)(\nabla_z\mathcal{K})(x,y)w,t\rangle\\ &- \langle(\phi_{\mathfrak{g}^!} \circ J)(\nabla_x\mathcal{K})(y,z)w,t\rangle.\end{aligned}$$

Using (6.1) and the last equations, we complete the proof. □

Proposition 6.5. *Let $(G, \diamond, e_\Phi, \Phi, J, \langle\cdot,\cdot\rangle)$ be a left-invariant holomorphic Norden Hom-Lie group (or, $(\mathfrak{g}^!, [\cdot,\cdot]_{\mathfrak{g}^!}, \phi_{\mathfrak{g}^!}, J, \langle,\rangle)$ be a holomorphic Norden Hom-Lie algebra) such that J is a left-invariant abelian complex structure (or, an abelian complex structure). Then we have*

$$\nabla_{\phi_{\mathfrak{g}^!}(x)} \circ \nabla_y = \nabla_{\phi_{\mathfrak{g}^!}(y)} \circ \nabla_x,$$

for any $x, y \in \mathfrak{g}^!$.

Proof. Using Proposition 4.8, we can write

$$\begin{aligned}(\nabla_{\phi_{\mathfrak{g}^!}(x)}\nabla_y)z =&\frac{1}{2}\nabla_{\phi_{\mathfrak{g}^!}(x)}([y,z]_{\mathfrak{g}^!} - (\phi_{\mathfrak{g}} \circ J)[y,(\phi_{\mathfrak{g}^!} \circ J)z]_{\mathfrak{g}^!})\\ =&\frac{1}{4}([\phi_{\mathfrak{g}^!}(x),[y,z]_{\mathfrak{g}^!} - (\phi_{\mathfrak{g}} \circ J)[y,(\phi_{\mathfrak{g}^!} \circ J)z]_{\mathfrak{g}^!}]_{\mathfrak{g}^!} - (\phi_{\mathfrak{g}^!} \circ J)[\phi_{\mathfrak{g}^!}(x)\\ &,(\phi_{\mathfrak{g}^!} \circ J)[y,z]_{\mathfrak{g}^!}]_{\mathfrak{g}^!} - (\phi_{\mathfrak{g}^!} \circ J)[\phi_{\mathfrak{g}^!}(x),[y,(\phi_{\mathfrak{g}^!} \circ J)z]_{\mathfrak{g}^!}]_{\mathfrak{g}^!}).\end{aligned}$$

Applying (4.7) yields

$$\begin{aligned}(\nabla_{\phi_{\mathfrak{g}^!}(x)}\nabla_y)z =&\frac{1}{4}([\phi_{\mathfrak{g}^!}(x),[y,z]_{\mathfrak{g}^!}]_{\mathfrak{g}^!} - [(\phi_{\mathfrak{g}^!} \circ J)(\phi_{\mathfrak{g}^!}(x)),[(\phi_{\mathfrak{g}^!} \circ J)y,z]_{\mathfrak{g}^!}]_{\mathfrak{g}^!}\\ &+ (\phi_{\mathfrak{g}^!} \circ J)[(\phi_{\mathfrak{g}^!} \circ J))\phi_{\mathfrak{g}^!}(x)),[(\phi_{\mathfrak{g}^!} \circ J)y,(\phi_{\mathfrak{g}^!} \circ J)z]_{\mathfrak{g}^!}]_{\mathfrak{g}^!}\\ &- (\phi_{\mathfrak{g}^!} \circ J)[\phi_{\mathfrak{g}^!}(x),[y,(\phi_{\mathfrak{g}^!} \circ J)z]_{\mathfrak{g}^!}]_{\mathfrak{g}^!}).\end{aligned}$$

Similarly, we get

$$\begin{aligned}(\nabla_{\phi_{\mathfrak{g}^!}(y)}\nabla_x)z =&\frac{1}{4}([\phi_{\mathfrak{g}^!}(y),[x,z]_{\mathfrak{g}^!}]_{\mathfrak{g}^!} - [(\phi_{\mathfrak{g}^!} \circ J)(\phi_{\mathfrak{g}^!}(y)),[(\phi_{\mathfrak{g}^!} \circ J)x,z]_{\mathfrak{g}^!}]_{\mathfrak{g}^!}\\ &+ (\phi_{\mathfrak{g}^!} \circ J)[(\phi_{\mathfrak{g}^!} \circ J))\phi_{\mathfrak{g}^!}(y)),[(\phi_{\mathfrak{g}^!} \circ J)x,(\phi_{\mathfrak{g}^!} \circ J)z]_{\mathfrak{g}^!}]_{\mathfrak{g}^!}\\ &- (\phi_{\mathfrak{g}^!} \circ J)[\phi_{\mathfrak{g}^!}(y),[x,(\phi_{\mathfrak{g}^!} \circ J)z]_{\mathfrak{g}^!}]_{\mathfrak{g}^!}).\end{aligned}$$

Subtracting two above equations and using the Hom-Jacobi identity, we obtain

$$(\nabla_{\phi_{\mathfrak{g}^!}(x)}\nabla_y - \nabla_{\phi_{\mathfrak{g}^!}(y)}\nabla_x)z = \frac{1}{4}(-[\phi_{\mathfrak{g}^!}(z),[x,y]_{\mathfrak{g}^!}]_{\mathfrak{g}^!} + [\phi_{\mathfrak{g}^!}(z),[(\phi_{\mathfrak{g}^!}\circ J)x,(\phi_{\mathfrak{g}^!}\circ J)y]_{\mathfrak{g}^!}]_{\mathfrak{g}^!}$$
$$-(\phi_{\mathfrak{g}^!}\circ J)[(\phi_{\mathfrak{g}^!}\circ J))\phi_{\mathfrak{g}^!}(z)),[(\phi_{\mathfrak{g}^!}\circ J)x,(\phi_{\mathfrak{g}^!}\circ J)y]_{\mathfrak{g}^!}]_{\mathfrak{g}^!} + (\phi_{\mathfrak{g}^!}\circ J)[\phi_{\mathfrak{g}^!}(z),[x,(\phi_{\mathfrak{g}^!}\circ J)y]_{\mathfrak{g}^!}]_{\mathfrak{g}^!}).$$

From (4.7) and the last equation, we deduce the assertion. □

Now, we discuss Hom-Left-symmetric algebra structures on holomorphic Norden Hom-Lie algebras. A Hom-Left-symmetric algebra is a Hom-algebra $(V,\cdot,\phi_V)$ such that

$$ass_{\phi_V}(u,v,w) = ass_{\phi_V}(v,u,w), \tag{6.4}$$

where

$$ass_{\phi_V}(u,v,w) = (u\cdot v)\cdot\phi_V(w) - \phi_V(u)\cdot(v\cdot w),$$

for any $u,v,w\in V$. This relation is equivalent to the vanishing of the curvature of $(V,\cdot,\phi_V)$ with the *commutator* on V is given by $[u,v]_V = u\cdot v - v\cdot u$.

A Hom-algebra $(V,\cdot,\phi_V)$ is called Hom-Lie admissible algebra if its commutator bracket satisfies the Hom-Jacobi identity. For a Hom-Lie admissible algebra we have $\circlearrowleft_{u,v,w} \mathcal{K}(u,v)w = 0$. If $(V,\cdot,\phi_V)$ is a Hom-Lie admissible algebra, then $(V,[\cdot,\cdot]_V,\phi_V)$ is a Hom-Lie algebra, where $[\cdot,\cdot]_V$ is the commutator bracket.

Proposition 6.6. *A Hom-Left-symmetric algebra is a Hom-Lie admissible algebra.*

Theorem 6.7. *Any left-invariant holomorphic Norden Hom-Lie group $(G,\diamond,e_\Phi,\Phi,J,\langle\cdot,\cdot\rangle)$ is a flat Hom-Lie group (or, holomorphic Norden Hom-Lie algebra $(\mathfrak{g}^!,[\cdot,\cdot]_{\mathfrak{g}^!},\phi_{\mathfrak{g}^!},J,\langle\cdot,\cdot\rangle)$ carries a Hom-Left-symmetric algebra) if J is an abelian complex structure.*

Proof. In one hand, we show that $\mathcal{K}(x,y)z = 0$, for any $x,y,z\in\mathfrak{g}^!$. Using (4.1) and (4.8), we get that

$$\mathcal{K}((\phi_{\mathfrak{g}^!}\circ J)x,(\phi_{\mathfrak{g}^!}\circ J)y)z = -\mathcal{K}(x,y)z.$$

On the other hand, Proposition 6.5 implies

$$\mathcal{K}((\phi_{\mathfrak{g}^!}\circ J)x,(\phi_{\mathfrak{g}^!}\circ J)y)z = -\nabla_{[(\phi_{\mathfrak{g}^!}\circ J)x,(\phi_{\mathfrak{g}^!}\circ J)y]_{\mathfrak{g}^!}}\phi_{\mathfrak{g}^!} = -\nabla_{[x,y]_{\mathfrak{g}^!}}\phi_{\mathfrak{g}^!} = \mathcal{K}(x,y)z.$$

That completes the proof. □

Example 6.8. We consider a holomorphic Norden Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\phi_{\mathfrak{g}},J,\langle\cdot,\cdot\rangle)$ defined in Example 4.10. For the Hom-Levi-Civita connection ∇, we get

$$\nabla_{\nabla_{e_i}e_j}\phi_{\mathfrak{g}}(e_k) - \nabla_{\phi_{\mathfrak{g}}(e_i)}\nabla_{e_j}e_k = 0 = \nabla_{\nabla_{e_j}e_i}\phi_{\mathfrak{g}}(e_k) - \nabla_{\phi_{\mathfrak{g}}(e_j)}\nabla_{e_i}e_k, \quad \forall i,j,k = 1,\cdots,6,$$

except

$$\nabla_{\nabla_{e_5}e_5}\phi_{\mathfrak{g}}(e_5) - \nabla_{\phi_{\mathfrak{g}}(e_5)}\nabla_{e_5}e_5 = -\frac{1}{2}e_1 = \nabla_{\nabla_{e_5}e_5}\phi_{\mathfrak{g}}(e_5) - \nabla_{\phi_{\mathfrak{g}}(e_5)}\nabla_{e_5}e_5,$$
$$\nabla_{\nabla_{e_5}e_5}\phi_{\mathfrak{g}}(e_6) - \nabla_{\phi_{\mathfrak{g}}(e_5)}\nabla_{e_5}e_6 = \frac{1}{2}e_2 = \nabla_{\nabla_{e_5}e_5}\phi_{\mathfrak{g}}(e_6) - \nabla_{\phi_{\mathfrak{g}}(e_5)}\nabla_{e_5}e_6,$$
$$\nabla_{\nabla_{e_5}e_6}\phi_{\mathfrak{g}}(e_5) - \nabla_{\phi_{\mathfrak{g}}(e_5)}\nabla_{e_6}e_5 = -\frac{1}{2}e_2 = \nabla_{\nabla_{e_6}e_5}\phi_{\mathfrak{g}}(e_5) - \nabla_{\phi_{\mathfrak{g}}(e_6)}\nabla_{e_5}e_5,$$
$$\nabla_{\nabla_{e_5}e_6}\phi_{\mathfrak{g}}(e_6) - \nabla_{\phi_{\mathfrak{g}}(e_5)}\nabla_{e_6}e_6 = -\frac{1}{2}e_1 = \nabla_{\nabla_{e_6}e_5}\phi_{\mathfrak{g}}(e_6) - \nabla_{\phi_{\mathfrak{g}}(e_6)}\nabla_{e_5}e_6,$$
$$\nabla_{\nabla_{e_6}e_6}\phi_{\mathfrak{g}}(e_5) - \nabla_{\phi_{\mathfrak{g}}(e_6)}\nabla_{e_6}e_5 = \frac{1}{2}e_1 = \nabla_{\nabla_{e_6}e_6}\phi_{\mathfrak{g}}(e_5) - \nabla_{\phi_{\mathfrak{g}}(e_6)}\nabla_{e_6}e_5,$$
$$\nabla_{\nabla_{e_6}e_6}\phi_{\mathfrak{g}}(e_6) - \nabla_{\phi_{\mathfrak{g}}(e_6)}\nabla_{e_6}e_6 = -\frac{1}{2}e_2 = \nabla_{\nabla_{e_6}e_6}\phi_{\mathfrak{g}}(e_6) - \nabla_{\phi_{\mathfrak{g}}(e_6)}\nabla_{e_6}e_6.$$

The above equations imply that (6.4) holds. Therefore the holomorphic Norden Hom-Lie algebra $(\mathfrak{g}^!,[\cdot,\cdot],\phi_{\mathfrak{g}^!},J,\langle\cdot,\cdot\rangle)$ has a Hom-Left-symmetric algebra structure.

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