NEW TYPE INTEGRAL INEQUALITIES FOR p-QUASI CONVEX FUNCTIONS

Huriye KADAKAL[1], İmdat İŞCAN[2] and Mahir KADAKAL[2*]

[1]*Institute of Science, Ordu University, Ordu, Turkey,*
[2]*Department of Mathematics, Faculty of Arts and Sciences,Giresun University, 28200, Giresun, Turkey.*

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Abstract

In this paper, By using an identity for differentiable functions, we obtain some new type integral inequalities for the class of functions whose derivatives in absolutely value at certain powers are p-quasi convex. Also, we give some applications to special means of positive real numbers

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p-QUASI KONVEKS FONKSİYONLAR İÇİN YENİ TİP İNTEGRAL EŞİTSİZLİKLER

Özet

Bu çalışmada, diferansiyellenebilir fonksiyonlar için bir özdeşlik kullanılarak, türevlerinin mutlak değerlerinin belirli kuvvetleri p-quasi-konveks olan fonksiyonların sınıfı için bazı yeni tip integral eşitsizlikler elde ediyoruz. Aynı zamanda, pozitif reel sayıların özel ortalamaları için bazı uygulamalar veriyoruz.

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Anahtar Kelimeler: *p*-konveks fonksiyon, *p*-quasi konveks fonksiyon, Ostrowski tip eşitsizlik, hipergeometrik fonksiyon

**mahirkadakal@gmail.com*

1. Introduction

Definition 1.1 *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature.

Definition 1.2 *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex if the inequality*

$$f(tx+(1-t)y) \leq \max\{f(x), f(y)\}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Remark 1.1 *Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (Ion 2007).*

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

In recent years, much attention have been given to theory of convexity because of its great utility in various fields of pure and applied sciences. Many researchers have extended and generalized the classical concepts of convex functions in various directions using novel and innovative techniques. For more information, see (Dragomir et al 1995, Fang & Shi 2014, İşcan 2014; 2016; 2016, Kunt & İşcan 2016, Matkowski 2003/2004, Ostrowski 1938, Varošanec 2007).

In (İşcan 2014), the author gave the definition of harmonically convex function as follow and established Hermite-Hadamard's inequality for harmonically convex functions.

Definition 1.3 *Let $I \subseteq \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y)+(1-t)f(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

Definition 1.4 *Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

The above inequalities are sharp.

Definition 1.5 *A function* $f : I \subseteq (0,\infty) \to [0,\infty)$ *is said to be harmonically quasi-convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le \max\{f(x), f(y)\}$$

for all $x, y \in I$ *and* $t \in [0,1]$.

Remark 1.2 *Any harmonically convex function on* $I \subseteq (0,\infty)$ *is a harmonically quasi-convex function, but not conversely.*

In (İşcan 2016), the definition of *p*-convex function is given a different as follows:

Definition 1.6 *Let* $I \subset (0,\infty)$ *be a real interval and* $p \in \mathbb{R}\setminus\{0\}$*. A function* $f : I \to \mathbb{R}$ *is said to be a p-convex function, if*

$$f\left(\left[\alpha x^p + (1-\alpha) y^p\right]^{\frac{1}{p}}\right) \le \alpha f(x) + (1-\alpha) f(y)$$

for all $x, y \in I$ *and* $\alpha \in [0,1]$.

According to Definition 6, It can be easily seen that for $p = 1$ and $p = -1$, *p*-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.

Definition 1.7 (İşcan 2016) $I \subset (0,\infty)$ *be a real interval and* $p \in \mathbb{R}\setminus\{0\}$*. A function* $f : I \to \mathbb{R}$ *is said to be p-quasi-convex, if*

$$f\left(\left[\alpha x^p + (1-\alpha) y^p\right]^{\frac{1}{p}}\right) \le \max\{f(x), f(y)\} \tag{1.3}$$

for all $x, y \in I$ *and* $\alpha \in [0,1]$*. If the inequality in (1.3) is reversed, then f is said to be p-quasi-concave.*

In (Fang & Shi 2014, Theorem 5) and (İşcan 2016), Hermite-Hadamard's inequality for p-convex functions is given as follow:

Theorem 1.1 *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a* p*-convex function,* $p \in \mathbb{R}\setminus\{0\}$*, and* $a, b \in I$ *with* $a < b$*. If* $f \in L[a,b]$ *then we have*

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.4}$$

In order to prove our main results we need the following Lemma (İşcan et al 2017):

Lemma 1.1 *Let* $f : I \subseteq (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *and* $a, b \in I$ *with* $a < b$ *and* $p \in \mathbb{R}\setminus\{0\}$*. If* $f' \in L[a,b]$ *then*

$$\frac{b^p f(b) - a^p f(a)}{b^p - a^p} - \frac{p}{b^p - a^p}\int_a^b \frac{f(u)}{u^{1-p}}\,du = \frac{1}{p}\int_0^1 A_{t,p}^{\frac{1}{p}} f'\left(A_{t,p}^{\frac{1}{p}}\right) dt,$$

where $A_{t,p} = tb^p + (1-t)a^p$.

By using Lemma 1.1, İşcan obtained some new integral inequalities for p -convex functions in (İşcan 2016). In this work, we established some new type integral inequalities for the class of functions whose derivatives in absolutely value at certain powers are p -quasi-convex. Therefore we also obtained some new integral inequalities for quasi-convex and harmonically quasi-convex functions in special case of obtained inequalities.

For some results related to p -convex functions and its generalizations, we refer the reader to see (Fang & Shi 2014, İşcan 2016; 2016, Noor & Noor 2015).

Throughout this paper we will use the following notations: Let $0<a<b$, we will denote with

$$A(a,b):=\frac{a+b}{2},\qquad G(a,b):=\sqrt{ab},\qquad L(a,b):=\frac{b-a}{\ln b-\ln a},$$

$$L_p=L_p(a,b):=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},\ p\in\mathbb{R}\setminus\{-1,0\},$$

the arithmetic, geometric, logarithmic and p-logarithmic respectively.

2. Main results

Theorem 2.1 *Let* $f:I\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I°, $a,b\in I^\circ$ *with* $a<b$, $p\in\mathbb{R}\setminus\{0\}$ *and* $f'\in L[a,b]$. *If* $|f'|^q$ *is p-quasi-convex on* $[a,b]$ *for* $q\geq 1$, *then for all* $x\in[a,b]$, *we have*

$$\left|\frac{b^p f(b)-a^p f(a)}{b^p-a^p}-\frac{p}{b^p-a^p}\int_a^b\frac{f(u)}{u^{1-p}}du\right|\leq\frac{C_p(a,b)}{|p|}\max\{|f'(a)|,|f'(b)|\}. \tag{2.1}$$

Proof. From Lemma 1.1, Power mean integral inequality and the p-quasi convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}\left|\frac{b^p f(b)-a^p f(a)}{b^p-a^p}-\frac{p}{b^p-a^p}\int_a^b\frac{f(u)}{u^{1-p}}du\right| &\leq\frac{1}{|p|}\left(\int_0^1 A_{t,p}^{\frac{1}{p}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 A_{t,p}^{\frac{1}{p}}\left|f'\left(A_{t,p}^{1/p}\right)\right|^q dt\right)^{\frac{1}{q}}\\ &\leq\frac{1}{|p|}\left(\int_0^1 A_{t,p}^{\frac{1}{p}}dt\right)\max\{|f'(a)|,|f'(b)|\}\\ &=\frac{1}{|p|}C_p(a,b)\max\{|f'(a)|,|f'(b)|\}.\end{aligned}$$

Here, it is easily seen that the following equality holds:

$$C_p(a,b)=\int_0^1 A_{t,p}^{\frac{1}{p}}dt=\begin{cases}A(a,b), & p=1\\ G^2(a,b)/L(a,b), & p=-1\\ L_p^p(a,b)/L_{p-1}^{p-1}(a,b), & p\in\mathbb{R}\setminus\{-1,0,1\}\end{cases}$$

Hence, we obtain the desired result. This completes the proof.

Corollary 2.1 *In Theorem 2.1,*

(i) *If we take* $p=1$, *then we have the following inequality when* $|f'|^q$ *is convex on* $[a,b]$:

$$\left|\frac{bf(b)-af(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \le A(a,b)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}.$$

(ii) *If we take* $p=-1$, *then we have the following inequality when* $|f'|^q$ *is harmonically convex on* $[a,b]$:

$$\left|\frac{af(b)-bf(a)}{a-b}-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| \le \frac{G^2(a,b)}{L(a,b)}\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}.$$

Theorem 2.2 *Let* $f:I\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I°, $a,b\in I^\circ$ *with* $a<b$, *and* $f'\in L[a,b]$. *If* $|f'|^q$ *is p-quasi-convex on* $[a,b]$ *for* $q>1, \frac{1}{r}+\frac{1}{q}=1$, *then*

$$\left|\frac{b^p f(b)-a^p f(a)}{b^p-a^p}-\frac{p}{b^p-a^p}\int_a^b \frac{f(u)}{u^{1-p}}du\right| \le \frac{D_{p,r}(a,b)}{|p|}\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}. \tag{2.2}$$

Proof. From Lemma 1.1 and Hölder's inequality and the *p*-quasi-convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}\left|\frac{b^p f(b)-a^p f(a)}{b^p-a^p}-\frac{p}{b^p-a^p}\int_a^b \frac{f(u)}{u^{1-p}}du\right| &\le \frac{1}{|p|}\left(\int_0^1 A_{t,p}^{\frac{r}{p}}dt\right)^{\frac{1}{r}}\left(\int_0^1 \left|f'\left(A_{t,p}^{1/p}\right)\right|^q dt\right)^{\frac{1}{q}}\\ &\le \frac{1}{|p|}\left(\int_0^1 A_{t,p}^{\frac{r}{p}}dt\right)^{\frac{1}{r}}\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\\ &= \frac{1}{|p|}\left(\int_0^1 A_{t,p}^{\frac{r}{p}}dt\right)^{\frac{1}{r}}\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\\ &\le \frac{1}{|p|}D_{p,r}(a,b)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\end{aligned}$$

Here, it is easily seen that the following equality holds:

$$D_{p,r}(a,b)=\left(\int_0^1 A_{t,p}^{\frac{r}{p}}dt\right)^{\frac{1}{r}}=\begin{cases}L_r(a,b), & p=1\\ G^2(a,b)/L_{-r}(a,b), & p=-1\\ L_{p+r-1}^{\frac{p+r-1}{r}}(a,b)/L_{p-1}^{\frac{p-1}{r}}(a,b), & p\in\mathbb{R}\setminus\{-1,0,1\}\end{cases}$$

Hence, we obtain the desired result. This completes the proof. This completes the proof.

Corollary 2.2 *In Theorem 2.2,*

(i) *If we take $p=1$, then we have the following inequality when $|f'|^q$ is quasi-convex on $[a,b]$:*

$$\left|\frac{bf(b)-af(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right|\le L_r(a,b)\max\left\{|f'(a)|,|f'(b)|\right\}.$$

(ii) *If we take $p=-1$, then we have the following inequality when $|f'|^q$ is harmonically quasi-convex on $[a,b]$:*

$$\left|\frac{af(b)-bf(a)}{a-b}-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right|\le \frac{G^2(a,b)}{L_{-r}(a,b)}\max\left\{|f'(a)|,|f'(b)|\right\}$$

3. Some applications for special means

Proposition 3.1 *Let $0<a<b$ and $p\in\mathbb{R}\setminus\left\{-\frac{1}{2},-1,0\right\}$. Then we have the following inequality*

$$\begin{cases} L_{2p}^{2p}(a,b)\le b^p L_p^p(a,b) & \text{for} \quad p>0 \\ L_{2p}^{2p}(a,b)\le a^p L_p^p(a,b) & \text{for} \quad p<0 \end{cases}$$

Proof. The assertion follows from the inequality (2.1) in Theorem 2.1, for $f:(0,\infty)\to\mathbb{R},\ f(x)=\frac{x^{p+1}}{p+1}$.

Proposition 3.2 *Let $0<a<b, r>1$ and $p\in\mathbb{R}\setminus\left\{-\frac{1}{2},-r,1-r,0,1\right\}$. Then we have the following inequality*

$$\begin{cases} \dfrac{L_{2p}^{2p}(a,b)}{L_{p-1}^{p-1}(a,b)}\le b^p \dfrac{L_{p+r-1}^{\frac{p+r-1}{r}}(a,b)}{L_{p-1}^{\frac{p-1}{r}}(a,b)} & \text{for} \quad p>0 \\ \dfrac{L_{2p}^{2p}(a,b)}{L_{p-1}^{p-1}(a,b)}\le a^p \dfrac{L_{p+r-1}^{\frac{p+r-1}{r}}(a,b)}{L_{p-1}^{\frac{p-1}{r}}(a,b)} & \text{for} \quad p<0 \end{cases}$$

Proof. The assertion follows from the inequality (2.2) in Theorem 2.2, for $f:(0,\infty)\to\mathbb{R},\ f(x)=\frac{x^{p+1}}{p+1}$.

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