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On Deferred Statistical and Strong Deferred Cesàro Convergences of Sequences With Respect to A Modulus Function

Cemel Belen [1,a], Mustafa Yıldırım[2,b,*]

[1] *Department of Mathematics Education, Ordu University, 52200, Ordu, Türkiye.*
[2]*Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University, Sivas, Türkiye*
**Corresponding author*

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ABSTRACT

Let f be any modulus function. We prove that the classes of strongly deferred Cesàro convergent sequences defined by f and deferred statistical convergent sequences are equivalent if the sequence is f-deferred uniformly integrable. Some converse inclusions are obtained when the modulus function f is compatible. Finally, for any compatible modulus f, we prove that any sequence is f-strongly deferred Cesàro convergent if and ony if it is deferred f-statistically convergent and deferred uniformly integrable.

Keywords: Deferred statistical convergence, Strong deferred convergence, Uniformly integrable sequence, Compatible modulus function.

[a] cbelen52@gmail.com https://orcid.org/0000-0002-8832-1524
[b] yildirim@cumhuriyet.edu.tr https://orcid.org/0000-0002-8880-5457

Introduction

Statistical convergence was first introduced by Fast [1] and also independently by Buck [2] and Schoenberg [3] for real and complex sequences, but the rapid developments started after the papers of Šalát [4] and Fridy [5].

Strong Cesàro convergence with respect to a modulus function was introduced by Maddox [6]. Connor [7] extended this idea by replacing Cesàro matrix with a non-negative regular matrix A and proved that A-statistical convergence includes strong A-summability with respect to a modulus and further these notions are equivalent for bounded sequences. Connor also established the relationship between statistical convergence and strong Cesàro convergence [8]: A real sequence is strongly convergent if and only if it is statistical convergent and bounded. Khan and Orhan [9] extended this result by replacing the boundedness condition with a strictly weaker condition so-called uniform integrability.

By using any modulus function f, Aizpuru and coworkers [10] introduced the concept of f-statistical convergence. León-Saavedra et. al. [11] defined the notion of f-strongly convergence by means of modulus functions. They proved that if a sequence is f-strongly convergent then it is f-statistically convergent and uniformly integrable, and the converse statement is true when f is compatible modulus function. Such type of modulus functions are those for which the concepts of statistical convergence and f-statistical convergence are equivalent.

Motivated by Agnew [12], Küçükaslan and Yılmaztürk [13] defined and studied on the the concept of deferred statistical convergence. Later this concept was improved by Gupta and Bhardwaj [14] with the help of modulus functions. They also introduced the notion of strongly deferred Cesàro convergence of sequences defined by modulus function f and investigated its relation with deferred f-statistical convergence. We refer to [15-22] for additional different works on deferred statistical convergence.

In the present paper, we investigate the relationship between strongly deferred Cesàro convergent sequences defined by a modulus function and deferred statistically convergent sequences. We prove that these two classes are equivalent in the context of f-deferred uniformly integrable sequences. Later we define f-strongly deferred Cesàro convergence of a real sequence and examines its relation with strongly deferred Cesàro convergence. If f is any modulus function, f-strongly deferred Cesàro convergence (deferred f-statistical convergence) implies strongly deferred Cesàro convergence (deferred statistical convergence), but not conversely. We prove that converse statements are true when f is compatible modulus function. Finally, for any compatible modulus f, we prove that any sequence is f-strongly deferred Cesàro convergent if and ony if it is deferred f-statistically convergent and deferred uniformly integrable.

Materials and Methods

Let $\mathbb{N}$ be set of positive integers and $x = (x_k)$ be sequence of real numbers. Then x is statistically convergent to the number L (in short $x \in S$) provided for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |x_k - L| \ge \varepsilon\}| = 0.$$

Suppose that (p_n) and (q_n) are the sequences of non-negative integers with $p_n < q_n$ and $q_n \to \infty$ (as $n \to \infty$). We say that (x_k) is strongly deferred Cesàro convergent to L if

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} |x_k - L| = 0$$

(see [12]). In this paper we prefer the notation $w_{p,q}$ for the set of all strongly deferred Cesàro convergent sequences.

Any sequence (x_k) is said to be deferred statistical convergent to L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{k \in \mathbb{N}: p_n < k \le q_n, |x_k - L| \ge \varepsilon\}| = 0.$$

In this case we write $S_{p,q}$-$\lim x = L$ and the set of all deferred statistically convergent sequences will be denoted by $S_{p,q}$ (see [13]). Throughout the paper we will use the notation $E_{\varepsilon,p,q}$ instead of the set $\{k \in \mathbb{N}: p_n < k \le q_n, |x_k - L| \ge \varepsilon\}$. If we choose $q_n = n$ and $p_n = 0$ for all n, then $S_{p,q}$ coincides with S.

Note that $w_{p,q} \subset S_{p,q}$ and $w_{p,q} \cap \ell_\infty = S_{p,q} \cap \ell_\infty$, also if the sequence $\left(\frac{p_n}{q_n - p_n}\right)$ is bounded then $S \subset S_{p,q}$, where ℓ_∞ is the set of all bounded sequences.

Any function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with the following properties is called a modulus function;

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \le f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$,
3. f is increasing,
4. f, is continuous from the right at zero [23].

$f(x) = x^p$ $(0 < p \le 1)$ and $f(x) = \frac{x}{1+x}$ are some examples of a modulus function. A modulus function can be bounded or unbounded.

Let f be any modulus function. A sequence $x = (x_k)$ is said to be f-statically convergent to L if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{f(|\{k \le n: |x_k - L| \ge \varepsilon\}|)}{f(n)} = 0$$

(see, [10]). It is also known from [10] that any f-statistically convergent sequence is also statistically convergent but not conversely. We remark here that if f is bounded modulus function, then these definitions hold only for trivial cases (for empty set and constant sequences). So, throughout the paper, we only consider the unbounded modulus functions.

In [14], Gupta and Bhardwaj defined the notion of deferred f-statistical convergence and strongly deferred Cesàro convergence with respect to f as follows:

Let f be any modulus function and $x = (x_k)$ be any real sequence. Then, x is said to be deferred f-statistically convergent to L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{f(q_n - p_n)} f(|\{k: p_n < k \le q_n, |x_k - L| \ge \varepsilon\}|) = 0$$

and if

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} f(|x_k - L|) = 0$$

then x is said to be strongly deferred Cesàro convergent to L with respect to f. The sets of all deferred f-statistically convergent and all strongly deferred Cesàro convergent with respect to f will be denoted by $S_{p,q}^f$ and $w_{p,q}^f$, respectively. We know from [14] that the inclusion $S_{p,q}^f \subset S_{p,q}$ is strict.

Now let $A = (a_{nk})$, $n, k \in \mathbb{N}$, be any non-negative regular matrix, i.e. that transforms any convergent sequence into a convergent sequence with the same limit. Any real sequence $x = (x_k)$ is A-statistically convergent to L if

$$\lim_{n\to\infty} \sum_{k:|x_k - L| \ge \varepsilon} a_{nk} = 0$$

for each $\varepsilon > 0$. Also x is said to be A-strong convergent if

$$\lim_{n\to\infty} \sum_{k:=1}^{\infty} a_{nk}|x_k - L| = 0.$$

Khan and Orhan [9] characterized A-strong convergence and A-statistical convergence through A-uniform integrable sequences. A real sequence $x = (x_k)$ is called A-uniformly integrable if

$$\lim_{c\to\infty} \sup_n \sum_{|x_k| \ge c} a_{nk}|x_k| = 0.$$

Khan and Orhan proved that a sequence is A-strongly convergent if and only if it is A-statistically convergent and A-uniformly integrable. Replacing the matrix A with $D_{p,q} := (d_{nk})$, where

$$d_{nk} = \begin{cases} \dfrac{1}{q_n - p_n}, & p_n < k \le q_n, \\ 0, & \text{otherwise} \end{cases}$$

we obtain the the following result.

Theorem 2.1 *[9]Let $x = (x_k)$ be a real sequence. Then the following are equivalent.*

(i) x is strongly deferred Cesàro convergent to L.

(ii) x is deferred statistically convergent to L and $D_{p,q}$-uniformly integrable

Main Results

In this section, we first characterize the sets $w_{p,q}^f$ and $S_{p,q}$ via deferred uniformly integrable sequences with respect to a modulus function. For this, we define the following idea motivated by [9].

Definition 3.1 Let f be any modulus function. Then a sequence (x_k) is said to be f-$D_{p,q}$-uniformly integrable if

$$\lim_{c\to\infty}\sup_n \frac{1}{q_n-p_n}\sum_{\substack{k=p_n+1\\ f(|x_k|)\geq c}}^{q_n} f(|x_k|)=0.$$

Theorem 3.1 Let f be any modulus function and $x=(x_k)$ be a real sequence. Then the following are equivalent.

(i) x is strongly deferred Cesàro convergent to L with respect to f.

(ii) x is deferred statistically convergent to L and f-$D_{p,q}$-uniformly integrable.

Proof. $(i)\Rightarrow(ii)$. Let $x\in w_{p,q}^f$ with limit L, that is

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\sum_{k=p_n+1}^{q_n} f(|x_k-L|)=0.$$

Let $E_{\varepsilon,p,q}=\{k: p_n<k\leq q_n, |x_k-L|\geq\varepsilon\}$ for any given $\varepsilon>0$. Then we have

$$\frac{1}{q_n-p_n}\sum_{k=p_n+1}^{q_n} f(|x_k-L|) \geq \frac{1}{q_n-p_n}\sum_{k\in E_{\varepsilon,p,q}} f(|x_k-L|)$$

$$\geq \frac{f(\varepsilon)}{q_n-p_n}|\{k: p_n<k\leq q_n, |x_k-L|\geq\varepsilon\}|,$$

since f is increasing. Letting limit for $n\to\infty$ in this inequality, we get $S_{p,q}$-$\lim x=L$. If we set $y_k:=f(|x_k|)$, then we obtain from Theorem 2.1 that x is f-$D_{p,q}$-uniformly integrable.

$(ii)\Rightarrow(i)$. Assume that $S_{p,q}$-$\lim x=L$ and x is f-$D_{p,q}$-uniformly integrable. Let $\varepsilon>0$. $|x_k-L|\geq\varepsilon$ implies that $f(|x_k-L|)\geq f(\varepsilon)$. On the other hand, $\lim_{\varepsilon\to 0^+} f(\varepsilon)=0$ since f is continuous at zero. This implies that any deferred statistically convergent sequence satisfies the condition

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}|\{k: p_n<k\leq q_n, f(|x_k-L|)\geq f(\varepsilon)\}|=0. \quad (1)$$

Thus, f-$D_{p,q}$-uniformly integrability and (1) imply by Theorem 2.1 that x is strongly deferred Cesàro convergent to L with respect to f. This completes the proof.

Next, we define the class of f-strongly deferred Cesàro convergent sequences and display its relation with strongly deferred Cesàro convergent sequences.

Definition 3.2 Let f be a modulus function and $x=(x_k)$ be a sequence of real numbers. The sequence x is said to be f-strongly deferred Cesàro convergent to the number L if

$$\lim_{n\to\infty}\frac{1}{f(q_n-p_n)} f\left(\sum_{k=p_n+1}^{q_n}|x_k-L|\right)=0.$$

The set of all f-strong deferred Cesàro convergent sequences will be denoted by f-$w_{p,q}$.

Theorem 3.2 Let f be any unbounded modulus function and $x=(x_k)$ be a sequence of real numbers. If x is f-strongly deferred Cesàro convergent to L, then x is strongly deferred Cesàro convergent to L. That is f-$w_{p,q}\subset w_{p,q}$.

Proof. Assume that (x_k) is f-strongly deferred Cesàro convergent to L. Then for each $p\in\mathbb{N}$, there exists an $n_0\in\mathbb{N}$ such that for $n\geq n_0$ we have

$$f\left(\sum_{k=p_n+1}^{q_n}|x_k-L|\right)<\frac{1}{p}f(q_n-p_n)\leq f\left(\frac{q_n-p_n}{p}\right).$$

Since f is increasing, we have

$$\sum_{k=p_n+1}^{q_n}|x_k-L|\leq\frac{1}{n}(q_n-p_n) \quad (2)$$

for all $n\geq n_0$. From this, we obtain that (x_k) is strongly deferred Cesàro convergent to L. This completes the proof.

Now recall the concept of compatible modulus function used in [11] and also in [24, 6].

Definition 3.3 *[11]*Let f be a modulus function. We say that f is compatible provided for any $\varepsilon>0$ there exist $\tilde{\varepsilon}>0$ and $n_0=n_0(\varepsilon)$ such that $\frac{f(n\tilde{\varepsilon})}{f(n)}<\varepsilon$ for all $n\geq n_0$.

For example, $f(x)=x+\log(x+1)$, $g(x)=\frac{x}{\sqrt{1+x}}$ and $h(x)=\frac{x}{\log x+e^2}$ are unbounded compatible modulus functions, where logarithm is to the natural base e. On the other hand the $f(x)=\log(x+1)$ and $f(x)=\log(\log(x+e))$ are examples of modulus functions which are not compatible (For the details, see [24] and [11]).

Remark 3.1 We know from [14] that $S_{p,q}^f\subset S_{p,q}$ for any modulus function f. Now let $f(x)=\log(x+1)$, $q_n=n^2$, $p_n=n$ and

$$x_k=\begin{cases}1, & \text{if } k \text{ is square}\\ 0, & \text{otherwise}\end{cases}.$$

Then $S_{p,q}$-$\lim x=0$ but $S_{p,q}^f$-$\lim x\neq 0$ (see Example 2.6 of [14]). On the other hand if we replace the modulus function with $f(x)=x+\log(x+1)$, then we obtain that $S_{p,q}$-$\lim x=S_{p,q}^f-\lim x=0$. The following result shows that this case is always valid when we use compatible modulus functions.

Theorem 3.3 Let f be a compatible modulus function. Then $S_{p,q}^{f} = S_{p,q}$.

Proof. When f is a compatible modulus function, it is sufficient to prove that $S_{p,q} \subset S_{p,q}^{f}$.

Since $S_{p,q}^{f} \subset S_{p,q}$ for any modulus function, it is enough to prove that $S_{p,q} \subset S_{p,q}^{f}$ when f is a compatible modulus function. Let f be a compatible modulus function, $x = (x_k)$ be a real sequence and $S_{p,q}$-lim$x = L$. Since f is compatible, for any given $\varepsilon > 0$, there exist $\tilde{\varepsilon} > 0$ and $n_0 = n_0(\varepsilon)$ such that $\frac{f(n\tilde{\varepsilon})}{f(n)} < \varepsilon$ for all $n \geq n_0$. Also the assumption $q_n - p_n \to \infty$ $(n \to \infty)$ implies that there exists $N_0 = N_0(n_0)$ (thus $N_0 = N_0(\varepsilon)$) such that for all $n \geq N_0$ we have $q_n - p_n > n_0$. Hence, we obtain that $\frac{f((q_n - p_n)\tilde{\varepsilon})}{f(q_n - p_n)} < \varepsilon$ for all $n \geq N_0$. Now, let $\lambda > 0$ and fix $\tilde{\varepsilon}$. Since $S_{p,q}$-lim$x = L$, there exists N_1 such that

$$|\{k: p_n < k \leq q_n, |x_k - L| \geq \lambda\}| < (q_n - p_n)\tilde{\varepsilon}$$

for all $n \geq N_1$. Since f is increasing, we get

$$\frac{1}{f(q_n - p_n)} f(|\{k: p_n < k \leq q_n, |x_k - L| \geq \lambda\}|) < \frac{f\left((q_n - p_n)\tilde{\varepsilon}\right)}{f(q_n - p_n)} < \varepsilon$$

for all $n \geq \max\{N_0, N_1\}$. Thus, $S_{p,q}^{f}$-lim$x = L$ and this completes the proof.

By using the same technic we can prove the following.

Theorem 3.4 Let f be a compatible modulus function. Then f-$w_{p,q} = w_{p,q}$.

Using the same method with Proposition 1.1 of [25], one can also obtain the following result.

Theorem 3.5 Let f be a modulus function.

(i) If all deferred statistically convergent sequences are deferred f-statistically convergent, then f must be compatible.

(ii) If all strongly deferred Cesàro convergent sequences are f-strongly deferred Cesàro convergent, then f must be compatible.

Theorem 3.6 Let $x = (x_k)$ be a real sequence and f be a compatible modulus function. Then the following are equivalent.

(i) x is f-strongly deferred Cesàro convergent to L.

(ii) x is deferred f-statistically convergent to L and $D_{p,q}$-uniformly integrable.

Proof. $(ii) \Rightarrow (i)$. Let x be deferred f-statistically convergent to L and $D_{p,q}$-uniformly integrable. Since, $S_{p,q}^{f} \subset S_{p,q}$, Theorem 2.1, x is strongly deferred Cesàro convergent to L. Finally, since f is a compatible modulus, x is f-strongly deferred Cesàro convergent to L by Theorem 3.4.

$(i) \Rightarrow (ii)$. Assume that x is f-strongly deferred Cesàro convergent to L. Then applying Theorem 3.2 and Theorem 2.1 we obtain that x is $D_{p,q}$-uniformly integrable. Now prove that x is deferred f-statistically convergent to L. Let $\varepsilon > 0$ and choose any $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Since $E_{\varepsilon,p,q} \subset E_{\frac{1}{m},p,q}$ we have

$$\frac{1}{f(q_n - p_n)} f(|\{k: p_n < k \leq q_n, |x_k - L| \geq \varepsilon\}|) \leq \frac{1}{f(q_n - p_n)} f\left(\left|\left\{k: p_n < k \leq q_n, |x_k - L| \geq \frac{1}{m}\right\}\right|\right)$$

and so it is enough to prove that

$$\lim_{n\to\infty} \frac{1}{f(q_n - p_n)} f\left(\left|\left\{k: p_n < k \leq q_n, |x_k - L| \geq \frac{1}{m}\right\}\right|\right) = 0 \quad (3)$$

for any $n \in \mathbb{N}$. Hence for any $n \in \mathbb{N}$, we can write

$$f\left(\sum_{k=p_n+1}^{q_n} |x_k - L|\right) \geq f\left(\sum_{k\in E_{\frac{1}{m},p,q}} |x_k - L|\right) \geq f\left(\frac{1}{n} \sum_{k\in E_{\frac{1}{m},p,q}} 1\right) \geq \frac{1}{n} f\left(\sum_{k\in E_{\frac{1}{m},p,q}} 1\right)$$

$$= \frac{1}{m} f\left(\left|\left\{k: p_n < k \leq q_n, |x_k - L| \geq \frac{1}{n}\right\}\right|\right)$$

From this, we have

$$\frac{1}{f(q_n - p_n)} f\left(\left|\left\{k: p_n < k \leq q_n, |x_k - L| \geq \frac{1}{m}\right\}\right|\right) \leq \frac{m}{f(q_n - p_n)} f\left(\sum_{k=p_n+1}^{q_n} |x_k - L|\right).$$

Thus, by the assumption we obtain (3) and this completes the proof.

Conclusion

In this paper we have studied on deferred f-statistically convergent, strongly deferred Cesàro convergent and f-strongly deferred Cesàro convergent sequences. Some results are obtained through deferred uniformly integrable sequences and compatible modulus functions. Our results in this paper generalizes the results of [11]. For further study, similar ideas can be reformulated for double sequences.

Conflict of interests

There are no conflicts of interest in this work.

References

[1] Fast H., Sur la convergence statistique. *Colloq. Math.,* 2 (3/4) (1951) 241–244.

[2] Buck R. C., Generalized asymptotic density, *Amer. J. Math.,* 75 (1953) 335–346.

[3] Schoenberg I. J., The integrability of certain functions and related summability methods, *Amer. Math. Monthly*, 66 (1959) 361–375.

[4] Šalát T., On statistically convergent sequences of real numbers. *Math. Slovaca*, 30(2) (1980) 139-150.

[5] Fridy J. A., On statistical convergence, *Analysis,* 5 (1985) 301–313.

[6] Maddox I.J., Sequence spaces defined by a modulus, *Math. Proc., Cambridge Philos. Soc.,* 100 (1986) 161-166..

[7] Connor J., On strong matrix summability with respect to a modulus and statistical convergence, *Canad. Math. Bull*, 32 (1989) 194-198.

[8] Connor J., The statistical and strong p-Cesaro convergence of sequences, *Analysis,* 8 (1988) 47-63.

[9] Khan M. K. and Orhan C., Matrix characterization of A-statistical convergence, *J. Math. Anal. Appl.* 335 (2007) 406–417.

[10] Aizpuru A., Listàn-Garc1ɜ̆053′fa, M. C., Rambla-Barreno, F., Density by moduli and statistical convergence, *Quaest. Math.*, 37 (4) (2014) 525-530.

[11] León-Saavedra F., del Carmen Listán-Garc1ɜ̆053′fa M., Fernández F. J. P., de la Rosa M. P. R. On statistical convergence and strong Cesàro convergence by moduli. *J. Inequal. Appl.*, (2019) 2019:298.

[12] Agnew R. P., On deferred Cesàro means. *Annals of Mathematics*, (1932) 413-421.

[13] Küçükaslan M. and Yılmaztürk M., On deferred statistical convergence of sequences, *Kyungpook Math. J*, 56(2) (2016) 357-366.

[14] Gupta S. and Bhardwaj V. K., On deferred f-statistical convergence, *Kyungpook Math.,* J. 58 (2018) 91-103.

[15] Cinar M., Yilmaz E. and Et M. Deferred statistical convergence on time scales. *Proceedings of the Romanian Academy, Series A:,* 22 (4) (2021) 301–306.

[16] Dagadur I. and Sezgek S., Deferred Cesàro mean and deferred statistical convergence of double sequences. *J. Inequal. Spec. Funct*, 4 (2016) 118-136.

[17] Et M. and Yilmazer M. C., On deferred statistical convergence of sequences of sets. *AIMS Mathematics*, 5 (3) (2020) 2143-2152.

[18] Et M., Baliarsingh P., Kandemir H. Ş. and Küçükaslan M., On μ-deferred statistical convergence and strongly deferred summable functions. *RACSAM,* 115 (1) (2021) 34 1-14.

[19] Kişi Ö., Gürdal M. and Savaş E., On Deferred Statistical Convergence of Fuzzy Variables. *Applications and Applied Mathematics: An International Journal (AAM)*, 17 (2) (2022) 5.

[20] Kişi Ö. and Gürdal M., On deferred Cesàro summability and statistical convergence for the sets of triple sequences, *Annals of Fuzzy Mathematics and Informatics*, 24(2) (2022) 115–127.

[21] Kişi Ö. and Gürdal M., Certain aspects of deferred statistical convergence of fuzzy variables in credibility space, *The Journal of Analysis*, (2023) 1-19.

[22] Ulusu U., and Gülle E., Deferred Cesàro summability and statistical convergence for double sequences of sets, *Journal of Intelligent & Fuzzy Systems*, 42(4) (2022) 4095-4103.

[23] Nakano H., Concave modulars, *J. Math Soc. Japan*, 5(1) (1953) 29-49.

[24] Belen C., Yıldırım M. and Sümbül C., On statistical and strong convergence with respect to a modulus function and a power series method. *Filomat*, 34 (12) (2020) 3981-3993.

[25] León-Saavedra F., del Carmen Listán-Garc1ɜ̆053′fa M., Fernández F. J. P., de la Rosa M. P. R. Correction to: On statistical convergence and strong Cesàro convergence by moduli. *J. Inequal. Appl.*, J Inequal Appl 2023, 110 (2023). https://doi.org/10.1186/s13660-023-02988-0.