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A Note on the Trace of Generalized Permuting Tri-Derivations

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ABSTRACT

Many researchers have studied permuting tri-derivation and generalized derivation in prime or semi-prime rings, BCK-algebras, lattices, d-algebras, MV-algebras and many algebraic structures. Later, they introduced the concept of generalized permuting tri-derivation by combining the concepts of generalized derivation and permuting tri-derivation. In this article, we have examined the properties of generalized permuting triderivation by adding conditions on their traces in prime or semi-prime rings. In addition, we examined the properties of two permuting triderivation by giving a relation between their traces.

Keywords: Prime ring, Semiprime ring, Permuting tri-derivation, Generalized permuting tri-derivation.

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Introduction and Preliminaries

The derivation is one of the important topics of many areas of mathematics. It has also been carried to the ring in algebra. In 1957, Posner defined the derivation in the ring and examined the commutativity of the ring [1]. In 1991, the generalized derivation in rings was introduced by Bresar [2]. Later inspired by partial derivation in analysis, in 1980, the symmetric bi-derivation was defined by Maksa and the permuting tri-derivation by Öztürk in 1999 [3-4]. They investigated the commutativity of prime or semi-prime rings with the help of conditions on these derivations and their traces. After then, many authors studied symmetric bi-derivation and permuting tri-derivation on many kind of algebraic structures as lattice, BCK-alebras, MV-algebras, hyperring, etc. [5-10]. In 2017, Yazarli defined generalized permuting tri-derivation, combining the concepts of generalized derivation and permuting tri-derivation in the ring [11].

Assume that $\mathfrak{A}$ is a ring and $d\colon \mathfrak{A} \to \mathfrak{A}$ is an additive map. If there exists a derivation α of $\mathfrak{A}$ such that $d(ab) = d(a)b + a\alpha(b)$ in $\mathfrak{A}$, then d is called generalized derivation.

Let $\mathfrak{A}$ is a ring and $F(.,.,.)\colon \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is a map. If $F(a,b,c) = F(a,c,b) = F(c,a,b) = F(c,b,a) = F(b,c,a) = F(b,a,c)$ is provided in $\mathfrak{A}$, then F is called permuting and a map $f\colon \mathfrak{A} \to \mathfrak{A}$ defined by $f(a) = F(a,a,a)$ is called trace of $F(.,.,.)$. Suppose that $F(.,.,.)\colon \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is permuting tri-additive, in this case F satisfies the relation $f(a+b) = f(a) + f(b) + 3F(a,a,b) + 3F(a,b,b)$ in $\mathfrak{A}$. A permuting tri-additive map $F(.,.,.)\colon \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is called permuting tri-derivation if $F(ad,b,c) = F(a,b,c)d + aF(d,b,c)$ in $\mathfrak{A}$. The trace f is an odd function.

Let $\mathfrak{A}$ be a ring and $\Gamma\colon \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ be a permuting tri-additive map. Then, $\Gamma\colon \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is called generalized permuting tri-derivation of $\mathfrak{A}$ associated with F if

$$\Gamma(ad,b,c) = \Gamma(a,b,c)d + aF(d,b,c)$$
$$\Gamma(a,bd,c) = \Gamma(a,b,c)d + bF(a,d,c)$$
$$\Gamma(a,b,cd) = \Gamma(a,b,c)d + cF(a,b,d),$$

in $\mathfrak{A}$ where $F(.,.,.)\colon \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is a permuting tri-derivation.

Example 1 *[11] Let $\mathfrak{A}$ be a ring, F be a tri-derivation of $\mathfrak{A}$ and $\alpha\colon \mathfrak{A} \times \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ be a tri-additive map. If $\alpha(a,b,cd) = \alpha(a,b,c)d$, $\alpha(a,bd,c) = \alpha(a,b,c)d$ and $\alpha(ad,b,c) = \alpha(a,b,c)d$ in $\mathfrak{A}$, then $F + \alpha$ is a generalized F tri-derivation of $\mathfrak{A}$.*

Lemma 1 *[13] Let $\mathfrak{A}$ be a prime ring. If there exists a right ideal of $\mathfrak{A}$ which is contained in $\mathfrak{Z}(\mathfrak{A})$, $\mathfrak{A}$ must be commutative where $\mathfrak{Z}(\mathfrak{A})$ be the center of $\mathfrak{A}$*

In this article, we will apply the problems examined for generalized bi-derivations in the article of Ali, Shujat, Khan in 2015 to generalized permuting tri-derivations [12].

Results

In the Theorem 1, Theorem 2 and Theorem 3, we will assume that $\mathfrak{A}$ be a prime ring with 2,3-torsion free, $\mathfrak{B}$ be a non-zero ideal of $\mathfrak{A}$, $\mathfrak{Z}(\mathfrak{A})$ be the center of $\mathfrak{A}$, Γ is a generalized permuting tri-derivation with associated permuting tri-derivation F, γ is the trace of Γ and f is the trace of F.

Theorem 1 If $[\gamma(\mathrm{a}), \mathrm{a}] = 0$ in $\mathfrak{B}$ and $\mathfrak{A}$ is non-commutative then Γ is a left tri-multiplier on $\mathfrak{B}$.
Proof. Assume that

$$[\gamma(a), a] = 0 \text{ in } \mathfrak{B} \qquad (1)$$

Writting $\mathrm{a} + \mathrm{b}$ instead of a, for $\mathrm{b} \in \mathfrak{B}$ in (1), we get

$$[\Gamma(a,a,a), b] + 3[\Gamma(a,a,b), a] + 3[\Gamma(a,a,b), b] \qquad (2)$$
$$+3[\Gamma(a,b,b), a] + 3[\Gamma(a,b,b), b] + [\Gamma(b,b,b), a] = 0.$$

Substituting $-\mathrm{b}$ instead of b in (2) and subtracting from (2), we get

$$[\Gamma(a,a,b), b] + [\Gamma(a,b,b), a] = 0. \quad (3)$$

Writting $\mathrm{a} + \mathrm{b}$ instead of b, for $\mathrm{b} \in \mathfrak{B}$ in (3) and using (1) and (3), we get

$$[\Gamma(a,a,a), b] + 3[\Gamma(a,a,b), a] = 0 \quad (4)$$

Substituting br for b, for $\mathrm{r} \in \mathfrak{A}$ in (4) and using (4), we get

$$b[\Gamma(a,a,a), r] + 3\Gamma(a,a,b)[r,a] + 3[b,a]F(a,a,r) \qquad (5)$$
$$+3b[F(a,a,r), a] = 0.$$

Writting a instead of r, for $\mathrm{a} \in \mathfrak{B}$ in (5), we obtain

$$3[b,a]F(a,a,a) + 3b[F(a,a,a), a] = 0. \qquad (6)$$

Substituting rb instead of b, for $\mathrm{r} \in \mathfrak{A}$ in (6) and using (6), we get

$$[r,a]bF(a,a,a) = 0. \qquad (7)$$

Writting rs instead of r, $\mathrm{s} \in \mathfrak{A}$ in (7), we get

$$[r,a]sbF(a,a,a) = 0, \quad a,b \in \mathfrak{B} \text{ and } r,s \in \mathfrak{A}.$$

From here, we obtain $[\mathrm{r}, \mathrm{a}] = 0$ or $\mathrm{bF}(\mathrm{a},\mathrm{a},\mathrm{a}) = 0$ for all $\mathrm{a}, \mathrm{b} \in \mathfrak{B}$ and $\mathrm{r} \in \mathfrak{A}$, since $\mathfrak{A}$ is prime ring. If $[\mathrm{r}, \mathrm{a}] = 0$, then $\mathfrak{B} \subseteq Z(\mathfrak{A})$. In this case, $\mathfrak{A}$ is commutative ring. This is a contradiction. Assume that $\mathrm{bF}(\mathrm{a},\mathrm{a},\mathrm{a}) = 0$ for all $\mathrm{a}, \mathrm{b} \in \mathfrak{B}$ If we write $\mathrm{a} + \mathrm{c}$, $\mathrm{c} \in \mathfrak{B}$ instead of a, we have

$$bF(a,a,a) + 3bF(a,a,c) + 3bF(a,c,c) + bF(c,c,c) = 0$$

for all a, b, $\mathrm{c} \in \mathfrak{B}$. Writting $-\mathrm{c}$ instead of c and comparing last two expressions we get $\mathrm{bF}(\mathrm{a},\mathrm{c},\mathrm{c}) = 0$. Writting $\mathrm{c} + \mathrm{d}$ instead of c, $\mathrm{d} \in \mathfrak{B}$, we get $\mathrm{bF}(\mathrm{a},\mathrm{c},\mathrm{c}) + 2\mathrm{bF}(\mathrm{a},\mathrm{c},\mathrm{d}) + \mathrm{bF}(\mathrm{a},\mathrm{d},\mathrm{d}) = 0$. That is, $\mathrm{bF}(\mathrm{a},\mathrm{c},\mathrm{d}) = 0$ in $\mathfrak{B}$. Thus, we obtain that $\Gamma(\mathrm{ad},\mathrm{b},\mathrm{c}) = \Gamma(\mathrm{a},\mathrm{b},\mathrm{c})\mathrm{d}$ on $\mathfrak{B}$.

Theorem 2 If $\gamma(\mathrm{a}) \circ \mathrm{a} = 0$ in $\mathfrak{B}$ and $\mathfrak{A}$ is non-commutative ring, then Γ is a left tri-multiplier on $\mathfrak{B}$.
Proof. Assume that

$$\gamma(a) \circ a = 0 \text{ in } \mathfrak{B} \qquad (8)$$

Writting $\mathrm{a} + \mathrm{b}$ instead of a, for $\mathrm{b} \in \mathfrak{B}$ in (8) and using (8), we get

$$3\Gamma(a,a,b)a + 3\Gamma(a,b,b)a + \Gamma(b,b,b)a + \Gamma(a,a,a)b$$
$$+3\Gamma(a,a,b)b + 3\Gamma(a,b,b)b + 3a\Gamma(a,a,b) + 3a\Gamma(a,b,b)$$
$$+a\Gamma(b,b,b) + b\Gamma(a,a,a) + 3b\Gamma(a,a,b) + 3b\Gamma(a,b,b) = 0. \quad (9)$$

Substituting $-\mathrm{b}$ for b in (9) and subtracting from (9), we get

$$\Gamma(a,b,b)a + \Gamma(a,a,b)b + a\Gamma(a,b,b) + b\Gamma(a,a,b) = 0. \qquad (10)$$

Writting $\mathrm{a} + \mathrm{b}$ instead of b, for $\mathrm{b} \in \mathfrak{B}$ in (10) and using (8) and (10), we obtain

$$0 = \Gamma(a,a,a)b + 3\Gamma(a,a,b)a + 3a\Gamma(a,a,b) + b\Gamma(a,a,a). \qquad (11)$$

Substituting bc instead of b, c ∈ $\mathfrak{B}$ in (11), we get

$$0 = \Gamma(a,a,a)bc + 3\Gamma(a,a,b)ca + 3bF(a,a,c)a$$
$$+3a\Gamma(a,a,b)c + 3abF(a,a,c) + bc\Gamma(a,a,a) \quad (12)$$

If we multiply (11) by c on the right side and compare with (12), then we have

$$0 = 3\Gamma(a,a,b)[c,a] + 3bF(a,a,c)a + 3abF(a,a,c) + b[c,\Gamma(a,a,a)]. \quad (13)$$

In (13), writting a instead of c, we get

$$3bf(a)a + 3abf(a) + b[a,\gamma(a)] = 0. \quad (14)$$

Substituting rb instead of b, for r ∈ $\mathfrak{A}$ in (14), we get

$$3rbf(a)a + 3arbf(a) + rb[a,\gamma(a)] = 0.$$

Multiplying (14) by r on the right side and subtracting from last equation, we have

$$[a,r]bf(a) = 0,$$

since $\mathfrak{A}$ is 3-torsion free.
Since $\mathfrak{A}$ is non-commutative ring, we get $bf(a) = 0$ for all a, b ∈ $\mathfrak{B}$. Writting $a + c$ instead of a, we get

$$0 = bF(a,a,c) + bF(a,a,c),$$

since $\mathfrak{A}$ is 3-torsion free. Substituting $-c$ instead of c, we get $bF(a,a,c) = 0$ for a,b, c ∈ $\mathfrak{B}$. From here, $F = 0$ on $\mathfrak{B}$ Thus, we obtain that $\Gamma(ad,b,c) = \Gamma(a,b,c)d$ on $\mathfrak{B}$

Theorem 3 Let $\mathfrak{A}$ be a semiprime ring with 2,3-torsion free and $\mathfrak{B}$ be a non-zero ideal of $\mathfrak{A}$. If $[\gamma(a),a]$, $[f(a),a] \in \mathfrak{Z}(\mathfrak{A})$ in $\mathfrak{B}$, then $[\gamma(a),a] = 0$ in $\mathfrak{B}$.

Proof. Assume that γ is centralizing on $\mathfrak{B}$. That is,

$$[\gamma(a),a] \in \mathfrak{Z}(\mathfrak{A}) \text{ in } \mathfrak{B} \quad (15)$$

Writting $a + b$ instead of a, for b ∈ $\mathfrak{B}$ in (15) and using (15), we get

$$[\gamma(a),b] + [\gamma(b),a] + 3[\Gamma(a,a,b),a] + [\Gamma(a,a,b),b]$$
$$+3[\Gamma(a,b,b),a] + 3[\Gamma(a,b,b),b] \in \mathfrak{Z}(\mathfrak{A}) \quad (16)$$

Taking $-b$ instead of b in (16) and subtracting from (16), we get

$$[\gamma(a),b] + [\gamma(b),a] + 3[\Gamma(a,a,b),a]$$
$$+3[\Gamma(a,b,b),b] \in \mathfrak{Z}(\mathfrak{A}), \quad (17)$$

since $\mathfrak{A}$ is 2,3-torsion free.
Writting $b + c$ instead of b, for c ∈ $\mathfrak{B}$ in (17) and using (17), we get

$$[\Gamma(b,b,c),a] + [\Gamma(b,c,c),a] + [\Gamma(a,b,b),c] + 2[\Gamma(a,b,c),b]$$
$$+2[\Gamma(a,b,c),c] + [\Gamma(a,c,c),b] \in \mathfrak{Z}(\mathfrak{A}), \quad (18)$$

since $\mathfrak{A}$ is 3-torsion free.
Replacing $-b$ by b in (18) and subtracting from (18), we get

$$[\Gamma(b,b,c),a] + [\Gamma(a,b,b),c] + 2[\Gamma(a,b,c),b] \in \mathfrak{Z}(\mathfrak{A}) \quad (19)$$

Taking b instead of c in (19), we get

$$[\gamma(b),a] + 3[\Gamma(a,b,b),b] \in \mathfrak{Z}(\mathfrak{A}) \quad (20)$$

Writting b^2 instead of a and using γ is commuting on $\mathfrak{B}$,

$$5[\gamma(b), b]b + 3b[f(b), b] \in \mathfrak{Z}(\mathfrak{A}).$$

Setting $[\gamma(b), b] = s$, $[f(b), b] = t$ and $5[\gamma(b), b]b + 3b[f(b), b] = z$, we get $3bt = z - 5sb$, $b \in \mathfrak{B}$
Now, we calculate

$$\begin{aligned}
&[\gamma(a^2), a^2] = [\Gamma(a^2, a^2, a^2), a^2] \\
&= [\gamma(a), a]a^4 + a[\gamma(a), a]a^3 + 3a^2[F(a), a]a^2 \\
&+3a[F(a), a]a^3 + 3a^3[F(a), a]a + 3a^2[F(a), a]a^2 \\
&+a^4[F(a), a] + a^3[F(a), a]a. \qquad (21)
\end{aligned}$$

Since $3at = z - 5sa$, we get from (21),

$$-18a^4s + 4a^3z + 2a^4t \in \mathfrak{Z}(\mathfrak{A}). \qquad (22)$$

Commuting with $\gamma(a)$, we have

$$\begin{aligned}
&[\gamma(a), -18a^4s + 4a^3z + 2a^4t] \\
&= -18s[\gamma(a), a^4] + 4z[\gamma(a), a^3] + 2t[\gamma(a), a^4] \\
&= -72a^3s^2 + 12a^2zs + 8a^3st = 0.
\end{aligned}$$

Again commuting with $\gamma(a)$, we obtain

$$\begin{aligned}
&[\gamma(a), -72a^3s^2 + 12a^2zs + 8a^3st] \\
&= -216a^2s^3 + 24s^2az + 24s^2a^2t = 0.
\end{aligned}$$

From here, we have

$$\begin{aligned}
&[\gamma(a), -216a^2s^3 + 24s^2az + 24s^2a^2t = 0] \\
&= -512s^4a + 40s^3z = 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
&[\gamma(a), -512s^4a + 40s^3z] \\
&= -512s^5 = 0.
\end{aligned}$$

That is, $s = 0$ from $\mathfrak{A}$ is semiprime ring. Thus, γ is commuting on $\mathfrak{B}$

Theorem 4 Let Δ and Γ be permuting tri-derivations with δ, γ the traces of Δ and Γ, respectively and $\mathfrak{A}$ is non-commutative prime ring. If $\delta(a)a + a\gamma(a) = 0$ in $\mathfrak{B}$, then $\Gamma = 0$ and $\Delta = 0$.

Proof. Assume that $\delta(a)a + a\gamma(a) = 0$ for $b \in \mathfrak{B}$. Writting $a + b$ instead of a, for $b \in \mathfrak{B}$, we get

$$\begin{aligned}
&0 = \delta(a + b)(a + b) + (a + b)\gamma(a + b) \\
&= \delta(b)a + 3\Delta(a, a, b)a + 3\Delta(a, b, b)a \\
&+\delta(a)b + 3\Delta(a, a, b)b + 3\Delta(a, b, b)b \\
&+a\gamma(b) + 3a\Gamma(a, a, b) + 3a\Gamma(a, b, b) \\
&+b\gamma(a) + 3b\Gamma(a, a, b) + 3b\Gamma(a, b, b). \qquad (23)
\end{aligned}$$

Taking $-a$ instead of a in (23) and subtracting from (23), we have

$$\Delta(a, a, b)b + \Delta(a, b, b)a + b\Gamma(a, a, b) + a\Gamma(a, b, b) = 0. \qquad (24)$$

Substituting $b + c$ for b, for $c \in \mathfrak{B}$ in (24) and using (24), we get

$$\begin{aligned}
&\Delta(a, a, b)c + \Delta(a, a, c)b + 2\Delta(a, b, c)a \qquad (25) \\
&+b\Gamma(a, a, c) + c\Gamma(a, a, b) + 2a\Gamma(a, b, c) = 0.
\end{aligned}$$

Writing cd instead of c, $d \in \mathfrak{B}$ in (25), we get

$$0 = \Delta(a, a, b)cd + \Delta(a, a, c)db + c\Delta(a, a, d)b$$

$$+2\Delta(a,b,c)da + 2c\Delta(a,b,d)a + b\Gamma(a,a,c)d$$
$$+bc\Gamma(a,a,d) + cd\Gamma(a,a,b) + 2ac\Gamma(a,b,d) + 2a\Gamma(a,b,c)d. \quad (26)$$

Multiplying (25) by d on the right side and comparing with (26), we get

$$0 = \Delta(a,a,c)[d,b] + c\Delta(a,a,d)b + 2\Delta(a,b,c)[d,a]$$
$$+2c\Delta(a,b,d)a + bc\Gamma(a,a,d) + c[d,\Gamma(a,a,b)]$$
$$+2ac\Gamma(a,b,d). \quad (27)$$

In (27), writting rc instead of c, for $r \in \mathfrak{A}$, we get

$$0 = r\Delta(a,a,c)[d,b] + \Delta(a,a,r)c[d,b] + rc\Delta(a,a,d)b$$
$$+2r\Delta(a,b,c)[d,a] + 2\Delta(a,b,r)c[d,a] + 2rc\Delta(a,b,d)a$$
$$+brc\Gamma(a,a,d) + rc[d,\Gamma(a,a,b)] + 2arc\Gamma(a,b,d). \quad (28)$$

Multiplying (27) by r on the left side and comparing with (28), we get

$$\Delta(a,a,r)c[d,b] + 2\Delta(a,b,r)c[d,a] + [b,r]c\Gamma(a,a,d)$$
$$+2[a,r]c\Gamma(a,b,d) = 0. \quad (29)$$

Writting a instead of b and d in (29), we obtain

$$[a,r]c\gamma(a) = 0.$$

From here, $\mathfrak{B} \subseteq \mathfrak{Z}(\mathfrak{A})$ or $\gamma(a) = 0$ in $\mathfrak{B}$. Since $\mathfrak{A}$ is non-commutative, we obtain $\gamma(a) = 0$. And so, $\Gamma = 0$ and $\Delta = 0$.

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Conflicts of interest

There are no conflicts of interest in this work.

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