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Bayesian Analysis for the Modified Frechet–Exponential Distribution with Covid-19 Application

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ABSTRACT

In this manuscript, the maximum likelihood estimators and Bayes estimators for the parameters of the modified Frechet–exponential distribution. Because the Bayes estimators cannot be obtained in closed forms, the approximate Bayes estimators are computed using the idea of Lindley's approximation method under squared-error loss function. Then, the approximate Bayes estimates are compared with the maximum likelihood estimates in terms of mean square error and bias values using Monte Carlo simulation. Finally, real data sets belonging to COVID-19 death cases in Europe and China to are used to demonstrate the emprical results belonging to the approximate Bayes estimates, the maximum likelihood estimates.

Keywords: Modified Frechet–Exponential distribution, Bayes estimator, Lindley's approximation, Monte Carlo simulation, Squared-error loss function.

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Introduction

The exponential distribution is one of the popular lifetime distributions with wide application in reliability analysis, medical studies, and applied statistics. In addition, the Frechet distribution is one of the important distributions. Recently, some generalizations and extensions of the Frechet distribution are derived for modeling data: such as exponentiated Frechet distribution [1], gamma extended Frechet distribution [2], Marshall–Olkin Frechet distribution [3], weibull Frechet distribution [4], modified Frechet–Rayleigh distribution [5], novel Kumaraswamy power Frechet distribution [6], modified Frechet–exponential distribution [7].

The modified Frechet–exponential (MFE) distribution is introduced by Farhat et al. [7]. The MFE distributions with parameter (α,λ) is shown with $MFE(\alpha,\lambda)$ where $\alpha>0$ and $\lambda>0$. The probability density function (pdf), cumulative distribution function (cdf), hazard function and survival function of X random variable has the modified frechet-exponential distribution with α and λ parameters are as follows;

$$f(x;\alpha,\lambda)=\frac{\alpha\lambda\, exp(-\lambda x-(1-exp(-\lambda x))^{\alpha})\,(1-exp(-\lambda x))^{\alpha-1}}{1-exp(-1)},x>0,\alpha>0,\lambda>0 \quad (1)$$

$$s(x;\alpha,\lambda)=\frac{\exp(-1)-\exp\left(-(1-\exp(-\lambda x))^{\alpha}\right)}{\exp(-1)-1} \quad (2)$$

$$h(x;\alpha,\lambda)=\frac{\alpha\lambda\, exp(-\lambda x-(1-exp(-\lambda x))^{\alpha})\,(1-exp(-\lambda x))^{\alpha-1}}{exp(-(1-exp(-\lambda x))^{\alpha})-exp(-1)} \quad (3)$$

$$s(x;\alpha,\lambda)=\frac{\exp(-1)-\exp\left(-(1-\exp(-\lambda x))^{\alpha}\right)}{\exp(-1)-1} \quad (4)$$

where x > 0, $\alpha>0$ and $\lambda>0$.

In this study, Bayesian estimators of the modified Frechet–exponential distribution is investigated. There are a lot of studies that refer to the bayes estimation of different distributions under the complete samples [8-17].

The main objective of this manuscript is to develop the approximate Bayes estimators under square error loss functions and compare them with maximum likelihood estimators (MLEs) in terms of MSE and bias values of estimates. The rest of the manuscript is organized as follows. In Section 2, the MLEs for parameters are derived. In section 3, the approximate Bayes estimators under squared error loss function are achieved by using Lindley's approximation. Using Monte Carlo simulation, The approximate Bayes estimation are compared with the maximum likelihood estimation in terms of MSE and bias values. Then, results are tabulated in section 4. In section 5, real data sets belonging to COVID-19 death cases in Europe and China to are used to demonstrate the emprical results belonging to the approximate Bayes estimates, the maximum likelihood estimates. Finally, conclusions are given in Section *6*.

Maximum Likelihood Estimation

Let $(X_1, X_2, \dots, X_n)$ denote a random sample taken from$MFE(\alpha,\lambda)$ distribution with observed values $(x_1, x_2, \dots, x_n)$. Then, the likelihood function $\ell(\alpha, \lambda)$ and the log-likelihood function$(\alpha, \lambda) = log\, \ell\, (\alpha, \lambda)$ can be written as

$$\ell(\alpha,\lambda) = \frac{(\alpha\lambda)^n}{(1-exp(-1))^n} exp\left[-\sum_{i=1}^{n} \lambda x_i\right] exp\left[-\sum_{i=1}^{n}(1-exp(-\lambda x_i))^{\alpha}\right] \prod_{i=1}^{n}(1-exp(-\lambda x_i))^{\alpha-1} \tag{5}$$

and

$$L(\alpha,\lambda) = n\, log(\alpha\lambda) - n\, log(1-exp(-1)) - \sum_{i=1}^{n}\lambda x_i - \sum_{i=1}^{n}(1-exp(-\lambda x_i))^{\alpha} + (\alpha-1)\sum_{i=1}^{n} log(1-exp(-\lambda x_i))^{\alpha-1} \tag{6}$$

respectively. Taking the partial derivatives of $L(\alpha, \lambda)$ according to α and λ parameters and as a result of equalizing them to zero, the following equations are obtained;

$$L_1 = \frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}(1-exp(-\lambda x_i))^{\alpha}\, log(1-exp(-\lambda x_i)) + \sum_{i=1}^{n} log(1-exp(-\lambda x_i)) = 0 \tag{7}$$

$$L_2 = \frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i - \alpha \sum_{i=1}^{n} x_i\, exp(-\lambda x_i)\,(1-exp(-\lambda x_i))^{\alpha-1} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i\, exp(-\lambda x_i)}{(1-exp(-\lambda x_i))} = 0 \tag{8}$$

The nonlinear equations (7) and (8) can be solved by using Newton-Raphson method, which is one of the numerical methods in MATLAB program.

Bayes Estimation

Assuming that $(X_1, X_2, \dots, X_n)$ a sample having $MFE(\alpha,\lambda)$ distribution and α and λ parameters are independent random variables with prior Gamma distributions. Then the density functions for α and λ parameters are given by,

$$\pi_1(\alpha) = \frac{\alpha^{a_1-1}\, exp(-b_1\alpha)\, b_1^{a_1}}{\Gamma(a_1)} a_1, b_1, \alpha > 0 \tag{9}$$

$$\pi_2(\lambda) = \frac{\lambda^{a_2-1}\, exp(-b_2\lambda)\, b_2^{a_2}}{\Gamma(a_2)} a_2, b_2, \lambda > 0 \tag{10}$$

respectively. In this case, the joint prior and the log of joint prior density functions can be written as follows

$$\pi(\alpha,\beta) = \frac{\alpha^{a_1-1} b_1^{a_1} \lambda^{a_2-1} b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)} exp(-b_1\alpha)\, exp(-b_2\lambda) \tag{11}$$

And

$$\rho(\alpha,\lambda) = (a_1-1)\, log\, \alpha + (a_2-1)\, log\, \lambda - b_1\alpha - b_2\lambda + +a_1\, log(b_1) + a_2\, log(b_2) - log[\Gamma(a_1)] - log[\Gamma(a_2)] \tag{12}$$

respectively. Then, the joint posterior density function of α and λ parameters is obtained by

$$P(\alpha,\lambda|X) = \frac{k(\alpha,\lambda)\, exp[-\sum_{i=1}^{n}\lambda x_i]\, exp[-\sum_{i=1}^{n}(1-exp(-\lambda x_i))^{\alpha}] \prod_{i=1}^{n}(1-exp(-\lambda x_i))^{\alpha-1}}{\int_0^{\infty}\int_0^{\infty} k(\alpha,\lambda)\, exp[-\sum_{i=1}^{n}\lambda x_i]\, exp[-\sum_{i=1}^{n}(1-exp(-\lambda x_i))^{\alpha}] \prod_{i=1}^{n}(1-exp(-\lambda x_i))^{\alpha-1}\, d\alpha d\lambda} \tag{13}$$

where $k(\alpha,\lambda) = \alpha^{n+a_1-1}\lambda^{n+a_2-1}\, exp(-b_1\alpha)\, exp(-b_2\lambda)$. Thus, the Bayes estimate of $u(\alpha, \lambda)$ that is any function of α and λ under a squared error loss function can be written as follows,

$$\hat{u}_B(\alpha,\lambda) = E[u(\alpha,\lambda)] = \frac{\int_0^\infty \int_0^\infty u(\alpha,\lambda) e^{[L(\alpha,\lambda)+\rho(\alpha,\lambda)]} d\alpha d\lambda}{\int_0^\infty \int_0^\infty e^{[L(\alpha,\lambda)+\rho(\alpha,\lambda)]} d\alpha d\lambda} \quad (14)$$

Due to the fact that the equation given in (14), which consists of the ratio of two integrals, can not be obtained in closed-form, the Bayes Estimators of parameters using Lindley's approximation under the squared error loss (quadratic loss) function are computed.

Lindley's Approximation

Lindley's approximation which is an approximation of the Bayes estimate was suggested by Lindley [18]. Lindley's approximation used to approximate the ratio of two integrals such as (14) that can not be solved analytically. Lindley's approximation has been used by many authors (Ahmad et al. [19], Kundu et al. [20], Preda et al. [21], Singh et al. [22], Akdam et al. [23], Çiftci et al. [24], Akdam [25]) to compute the approximate Bayes estimators of different lifetime distributions. The formulas as regards Lindley's approximation are given by,

$$u_{BL}(\hat{\alpha},\hat{\lambda}) = E[u(\alpha,\lambda)/X] \approx\approx \left[u(\hat{\alpha},\hat{\lambda}) + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}(u_{ij}+2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} L_{ijk}\sigma_{ij}\sigma_{kl}u_l \right] \quad (15)$$

where $\hat{\alpha}$ and $\hat{\lambda}$are the MLE of α and λ, respectively, and
$u_i, i=1,2$are the unary partial derivatives and $u_{ij}, i,j=1,2$ are the binary partial derivatives of $u(\alpha,\lambda)$ with respect to α and λ parameters, respectively.$L_{ij}, i,j=1,2$ are the binary partial derivatives and$L_{ijk}, i,j,k=1,2$ are the trinary partial derivatives of log-likelihood function $L(\alpha,\lambda)$ with respect to α and λ parameters, respectively, and

$\left[-L_{ij}\right]^{-1} = \left[\sigma_{ij}\right],\ i,j=1,2$. σ_{ij} is the (*i, j*)-th element of the matrix$[\sigma_{ij}]$. From(12), we get

$$\rho_1 = \frac{\partial\rho(\alpha,\lambda)}{\partial\alpha} = \frac{a_1-1}{\alpha} - b_1\ ,\quad \rho_2 = \frac{\partial\rho(\alpha,\lambda)}{\partial\lambda} = \frac{a_2-1}{\lambda} - b_2$$

and then, we have the following values of L_{ij} for *i, j* = 1,2 and L_{ijk} for *i, j,k* = 1,2

$$L_{11} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n}\left(1-\exp(-\lambda x_i)\right)^{\alpha}\log\left(1-\exp(-\lambda x_i)\right)^2$$

$$L_{12} = -\sum_{i=1}^{n}\frac{\left(1-\exp(-\lambda x_i)\right)^{\alpha}\alpha x_i\exp(-\lambda x_i)\log\left(1-\exp(-\lambda x_i)\right)^2}{\left(1-\exp(-\lambda x_i)\right)} + \frac{\left(1-\exp(-\lambda x_i)\right)^{\alpha}x_i\exp(-\lambda x_i)}{\left(1-\exp(-\lambda x_i)\right)} + \sum_{i=1}^{n}\frac{x_i\exp(-\lambda x_i)}{\left(1-\exp(-\lambda x_i)\right)} = L_{21}$$

$$L_{22} = -\frac{n}{\alpha^2} - \left[\sum_{i=1}^{n}\frac{\left(1-\exp(-\lambda x_i)\right)^{\alpha}\alpha^2x_i^2\left(\exp(-\lambda x_i)\right)^2}{\left(1-\exp(-\lambda x_i)\right)^2} - \frac{\left(1-\exp(-\lambda x_i)\right)^{\alpha}\alpha x_i^2\exp(-\lambda x_i)}{\left(1-\exp(-\lambda x_i)\right)} - \frac{\left(1-\exp(-\lambda x_i)\right)^{\alpha}\alpha x_i^2\left(\exp(-\lambda x_i)\right)^2}{\left(1-\exp(-\lambda x_i)\right)^2}\right] + (\alpha-1)\sum_{i=1}^{n}\left[-\frac{x_i^2\exp(-\lambda x_i)}{\left(1-\exp(-\lambda x_i)\right)} - \frac{x_i^2\left(\exp(-\lambda x_i)\right)^2}{\left(1-\exp(-\lambda x_i)\right)^2}\right]$$

$$L_{111} = -\frac{2n}{\alpha^3} - \sum_{i=1}^{n}\left(1-\exp(-\lambda x_i)\right)^{\alpha}\log\left(1-\exp(-\lambda x_i)\right)^3$$

$$L_{112}=-\left[\sum_{i=1}^{n}\frac{\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\alpha x_i\left(\exp\left(-\lambda x_i\right)\right)\log\left(1-\exp\left(-\lambda x_i\right)\right)^{2}}{\left(1-\exp\left(-\lambda x_i\right)\right)}+\frac{2\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\log\left(1-\exp\left(-\lambda x_i\right)\right)x_i\exp\left(-\lambda x_i\right)}{\left(1-\exp\left(-\lambda x_i\right)\right)}\right]$$

$$L_{112}=L_{121}=L_{211}$$

$$L_{122}=-\left[\sum_{i=1}^{n}\frac{\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\alpha^2x_i^2\left(\exp\left(-\lambda x_i\right)\right)^2\log\left(1-\exp\left(-\lambda x_i\right)\right)}{\left(1-\exp\left(-\lambda x_i\right)\right)^2}-\frac{\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\alpha x_i^2\left(\exp\left(-\lambda x_i\right)\right)\log\left(1-\exp\left(-\lambda x_i\right)\right)}{\left(1-\exp\left(-\lambda x_i\right)\right)}-\frac{\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\alpha x_i^2\left(\exp\left(-\lambda x_i\right)\right)^2\log\left(1-\exp\left(-\lambda x_i\right)\right)}{\left(1-\exp\left(-\lambda x_i\right)\right)^2}+\frac{2\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\alpha x_i^2\left(\exp\left(-\lambda x_i\right)\right)^2}{\left(1-\exp\left(-\lambda x_i\right)\right)^2}-\frac{\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}x_i^2\left(\exp\left(-\lambda x_i\right)\right)}{\left(1-\exp\left(-\lambda x_i\right)\right)}-\frac{\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}x_i^2\left(\exp\left(-\lambda x_i\right)\right)^2}{\left(1-\exp\left(-\lambda x_i\right)\right)^2}\right]+\sum_{i=1}^{n}\left(\frac{x_i^2\left(\exp\left(-\lambda x_i\right)\right)}{\left(1-\exp\left(-\lambda x_i\right)\right)}-\frac{x_i^2\left(\exp\left(-\lambda x_i\right)\right)^2}{\left(1-\exp\left(-\lambda x_i\right)\right)^2}\right)$$

$$L_{122}=L_{221}=L_{212}$$

$$L_{222}=\frac{2n}{\alpha^3}-\left[\sum_{i=1}^{n}\frac{\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\alpha^3x_i^3\left(\exp\left(-\lambda x_i\right)\right)^3}{\left(1-\exp\left(-\lambda x_i\right)\right)^3}-\frac{3\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\alpha^2x_i^3\left(\exp\left(-\lambda x_i\right)\right)^2}{\left(1-\exp\left(-\lambda x_i\right)\right)^2}-\frac{3\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\alpha^2x_i^3\left(\exp\left(-\lambda x_i\right)\right)^3}{\left(1-\exp\left(-\lambda x_i\right)\right)^3}+\frac{\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\alpha x_i^3\left(\exp\left(-\lambda x_i\right)\right)}{\left(1-\exp\left(-\lambda x_i\right)\right)}+\frac{3\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\alpha x_i^3\left(\exp\left(-\lambda x_i\right)\right)^2}{\left(1-\exp\left(-\lambda x_i\right)\right)^2}+\frac{2\left(1-\exp\left(-\lambda x_i\right)\right)^{\alpha}\alpha x_i^3\left(\exp\left(-\lambda x_i\right)\right)^3}{\left(1-\exp\left(-\lambda x_i\right)\right)^3}\right]+\left(\alpha-1\right)\sum_{i=1}^{n}\left(\frac{x_i^3\left(\exp\left(-\lambda x_i\right)\right)}{\left(1-\exp\left(-\lambda x_i\right)\right)}+\frac{3x_i^3\left(\exp\left(-\lambda x_i\right)\right)^2}{\left(1-\exp\left(-\lambda x_i\right)\right)^2}+\frac{2x_i^3\left(\exp\left(-\lambda x_i\right)\right)^3}{\left(1-\exp\left(-\lambda x_i\right)\right)^3}\right)$$

Finally, the approximate Bayes estimators for α and λ parameter of $MFE(\alpha,\lambda)$distribution under the squared error loss function are obtained as follows;

$$\hat{\alpha}_{LINDLEY}=\hat{\alpha}_{MLE}+\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\left(u_{ij}+2u_i\rho_j\right)\sigma_{ij}+\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}L_{ijk}\sigma_{ij}\sigma_{kl}u_l\,,$$

$$\hat{\lambda}_{LINDLEY}=\hat{\lambda}_{MLE}+\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\left(u_{ij}+2u_i\rho_j\right)\sigma_{ij}+\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}L_{ijk}\sigma_{ij}\sigma_{kl}u_l\,,$$

respectively.

Simulation Study

In this section, By means of Monte Carlo simulation study for different sample sizes (*n*) in terms of the MSE and bias values the performances of the approximate Bayes estimates (computed with Lindley under the squared error loss function) for α and λ parameters of $MFE(\alpha,\lambda)$ are compared with those of the MLE. Informative priors for $a_1=1, b_1=2, a_2=2, b_2=1$ are used while computing the approximate Bayes estimates. MSE for the estimate of α and λ parameters can be computed with $MSE=\frac{1}{10000}\sum_{i=1}^{10000}\left(\left(\hat{\alpha}_i,\hat{\lambda}_i\right)-\left(\alpha,\lambda\right)\right)^2$, where $\left(\hat{\alpha},\hat{\lambda}\right)$ is MLE or approximate Bayes estimation. α and λ is generated from Gamma distribution with parameter (a_1,b_1) and (a_2,b_2) respectively. All the computations are based on 10.000 replications in MATLAB program. Finally, the MSE and bias values are tabulated in Table 1-4.

Table 1. ML and approximate Bayes estimates, MSE and bias values, for $(\alpha=2,\lambda=2)$

n	MLE			BAYES$_{LINDLEY}$			MLE			BAYES$_{LINDLEY}$		
	$\hat{\alpha}$	bias	MSE	$\hat{\alpha}$	bias	MSE	$\hat{\lambda}$	bias	MSE	$\hat{\lambda}$	bias	MSE
20	2.3033	0.5851	0.6629	1.5434	0.4581	0.2718	2.2027	0.4794	0.4025	1.5084	0.4916	0.2789
30	2.2317	0.4754	0.4652	1.7874	0.2866	0.1224	2.1540	0.3971	0.2737	1.7261	0.3225	0.1485
40	2.1681	0.3950	0.3055	1.8763	0.2756	0.1112	2.1097	0.3338	0.1903	1.8103	0.2825	0.1159
50	2.1275	0.3471	0.2217	1.9122	0.2669	0.1072	2.0876	0.2930	0.1444	1.8570	0.2548	0.0960
70	2.0921	0.2808	0.1394	1.9503	0.2331	0.0854	2.0641	0.2468	0.1006	1.9058	0.2216	0.0742
100	2.0586	0.2277	0.0876	1.9659	0.2029	0.0644	2.0432	0.2016	0.0660	1.9361	0.1867	0.0537
150	2.0406	0.1820	0.0548	1.9814	0.1685	0.0447	2.0283	0.1622	0.0422	1.9586	0.1545	0.0369
200	2.0300	0.1605	0.0419	1.9865	0.1519	0.0362	2.0220	0.1437	0.0328	1.9703	0.1386	0.0295
350	2.0186	0.1187	0.0224	1.9944	0.1148	0.0206	2.0139	0.1060	0.0179	1.9847	0.1036	0.0168
500	2.0111	0.0981	0.0153	1.9945	0.0960	0.0145	2.0081	0.0886	0.0124	1.9879	0.0874	0.0119

Table 2. ML and approximate Bayes estimates, MSE and bias values, for $(\alpha=3,\lambda=2)$

n	MLE			BAYES$_{LINDLEY}$			MLE			BAYES$_{LINDLEY}$		
	$\hat{\alpha}$	bias	MSE	$\hat{\alpha}$	bias	MSE	$\hat{\lambda}$	bias	MSE	$\hat{\lambda}$	bias	MSE
20	3.3136	0.7989	1.0526	1.7531	1.2469	1.7122	2.1111	0.3876	0.2398	1.4005	0.5995	0.3823
30	3.3510	0.7637	1.1211	2.2675	0.7325	0.6274	2.1153	0.3506	0.2041	1.6376	0.3676	0.1727
40	3.2821	0.6571	0.8499	2.5387	0.4745	0.3239	2.0954	0.3030	0.1563	1.7532	0.2823	0.1122
50	3.2330	0.5795	0.6417	2.6802	0.4146	0.2493	2.0785	0.2706	0.1221	1.8145	0.2454	0.0876
70	3.1592	0.4714	0.3999	2.8027	0.3744	0.2065	2.0566	0.2254	0.0830	1.8768	0.2070	0.0638
100	3.1123	0.3820	0.2545	2.8795	0.3280	0.1629	2.0386	0.1842	0.0548	1.9173	0.1733	0.0454
150	3.0746	0.3109	0.1596	2.9272	02806	0.1203	2.0270	0.1505	0.0363	1.9478	0.1442	0.0317
200	3.0513	0.2624	0.1120	2.9441	0.2436	0.0918	2.0194	0.1300	0.0269	1.9611	0.1256	0.0244
350	3.0310	0.1975	0.0623	2.9716	0.1886	0.0556	2.0108	0.0971	0.0149	1.9780	0.0951	0.0141
500	3.0192	0.1624	0.0421	2.9783	0.0805	0.0391	2.0059	0.0814	0.0104	1.9832	0.0805	0.0101

Table 3. ML and approximate Bayes estimates, MSE and bias values, for $(\alpha=0.6,\lambda=0.8)$

n	MLE			BAYES$_{LINDLEY}$			MLE			BAYES$_{LINDLEY}$		
	$\hat{\alpha}$	bias	MSE	$\hat{\alpha}$	bias	MSE	$\hat{\lambda}$	bias	MSE	$\hat{\lambda}$	bias	MSE
20	0.6727	0.1403	0.0397	0.6733	0.1279	0.0307	0.9823	0.3234	0.2266	0.8447	0.2874	0.0557
30	0.6424	0.1043	0.0201	0.6485	0.1000	0.0182	0.9110	0.2406	0.1115	0.8488	0.1807	0.0554
40	0.6307	0.0880	0.0137	0.6370	0.0860	0.0130	0.8793	0.1969	0.0715	0.8415	0.1621	0.0450
50	0.6262	0.0105	0.0782	0.6317	0.0769	0.0102	0.8664	0.1782	0.0578	0.8390	0.1538	0.0407
70	0.6159	0.0637	0.0068	0.6204	0.0630	0.0067	0.8435	0.1441	0.0360	0.8271	0.1305	0.0286
100	0.6128	0.0523	0.0045	0.6161	0.0520	0.0045	0.8319	0.1185	0.0239	0.8216	0.1108	0.0205
150	0.6083	0.0436	0.0031	0.6106	0.0434	0.0031	0.8219	0.0973	0.0155	0.8157	0.0932	0.0141
200	0.6058	0.0372	0.0022	0.6076	0.0371	0.0022	0.8160	0.0825	0.0112	0.8116	0.0800	0.0104
350	0.6038	0.0278	0.0012	0.6048	0.0277	0.0012	0.8102	0.0611	0.0060	0.8078	0.0600	0.0058
500	0.6019	0.0232	0.0009	0.6026	0.0232	0.0009	0.8053	0.0508	0.0041	0.8037	0.0502	0.0040

Table 4 . ML and approximate Bayes estimates, MSE and bias values, for $(\alpha = 1.6, \lambda = 0.8)$

n	MLE			BAYES$_{LINDLEY}$			MLE			BAYES$_{LINDLEY}$		
	$\hat{\alpha}$	bias	MSE	$\hat{\alpha}$	bias	MSE	$\hat{\lambda}$	bias	MSE	$\hat{\lambda}$	bias	MSE
20	1.8675	0.4801	0.5056	1.6782	0.2930	0.1320	0.8993	0.2146	0.0855	0.8129	0.1441	0.0327
30	1.7571	0.3534	0.2532	1.6800	0.2788	0.1316	0.8637	0.1669	0.0495	0.8193	0.1333	0.0291
40	1.7185	0.2968	0.1655	1.6733	0.2529	0.1099	0.8474	0.1425	0.0349	0.8180	0.1220	0.0244
50	1.6921	0.2557	0.1192	1.6616	0.2270	0.0890	0.8389	0.1242	0.0262	0.8173	0.1102	0.0199
70	1.6651	0.2100	0.0765	1.6473	0.1940	0.0636	0.8268	0.1036	0.0177	0.8128	0.0955	0.0147
100	1.6483	0.1744	0.0514	1.6375	0.1654	0.0456	0.8193	0.0854	0.1190	0.8101	0.0807	0.0105
150	1.6299	0.1394	0.0324	1.6236	0.1348	0.0300	0.8125	0.0690	0.0077	0.8068	0.0666	0.0071
200	1.6203	0.1201	0.0233	1.6160	0.1172	0.0221	0.8088	0.0598	0.0057	0.8047	0.0583	0.0054
350	1.6119	0.0894	0.0127	1.6096	0.0882	0.0124	0.8056	0.0447	0.0032	0.8033	0.0440	0.0031
500	1.6092	0.0752	0.0090	1.6076	0.0745	0.0089	0.8041	0.0377	0.0022	0.8025	0.0373	0.0022

As shown in Tables 1-4, MSE and the bias values of all estimates are given for different n values. The performances of the Lindley approximate Bayes estimates outdo those of the ML estimates. For ML and Bayes estimator methods, it is observed that when the n values increase, the bias, MSEs of the estimates decrease to zero.

Data Analysis

In this section, the analaysis of real data set is presented for illustrative purposes.

Real Data-1: The real data-1 set represent daily deaths due to COVID-19 in Europe from 1st March to 30 March (https://covid19.who.int/) [26].

Real Data-1, n=31:

6	18	29	28	47	55	40	150	129	184	263	237	336	219	612	434
648	706	838	1129	1421	118	116	1393	1540	1941	2175	2278	2824	2803	2667	

First, it is checked whether MFE distribution can be used or not to analyze these data set. The Kolmogorov-Smirnov (KS) Z have been used to check the goodness-of-fit via MATLAB program. ML estimates, KS-Z and p- values based on above data is shown in Table 5.

Table 5 . Kolmogorov-Smirnov Z and the corresponding p-values for ML estimates

Methods	Estimates	Kolmogorov-Smirnov Z	p-values
MLE	$\hat{\alpha}$=0.7057$\hat{\lambda}$=0.00073	0.1076	0.8654

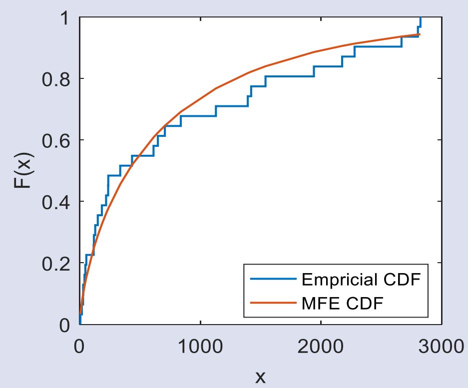

Figure 1: Plot Empricial CDF and MFE CDF for data-1

In this case, ML and approximate Bayes estimates for α and λ parameters are as follows.

	MLE		***BAYES$_{Lindley}$***	
n	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
31	0.7057	0.00073	0.7540	0.00081

Real Data-2: The real data-2 set represent Daily deaths due to COVID-19 in China from 23 January to 28 March(https://www.worldometers. info/coronavirus/country/china/)[19].

Real Data-2, n=66:

8	16	15	24	26	26	38	43	46	45	57	64	65	73	73	86
89	97	108	97	146	121	143	142	105	98	136	114	118	109	97	150
71	52	29	44	47	35	42	31	38	31	30	28	27	22	17	22
11	7	13	10	14	13	11	8	3	7	6	9	7	4	6	5
3	5														

First, it is checked whether MFE distribution can be used or not to analyze these data set. The Kolmogorov-Smirnov Z have been used to check the goodness-of-fit via MATLAB program. ML estimates, KS-Z and p- values based on above data is shown in Table 6.

Table 6 . Kolmogorov-Smirnov Z and the corresponding p-values for ML estimates

Methods	Estimates	Kolmogorov- Smirnov Z	p-values
MLE	$\hat{\alpha}$=1.2666 $\hat{\lambda}$=0.0184	0.0928	0.6208

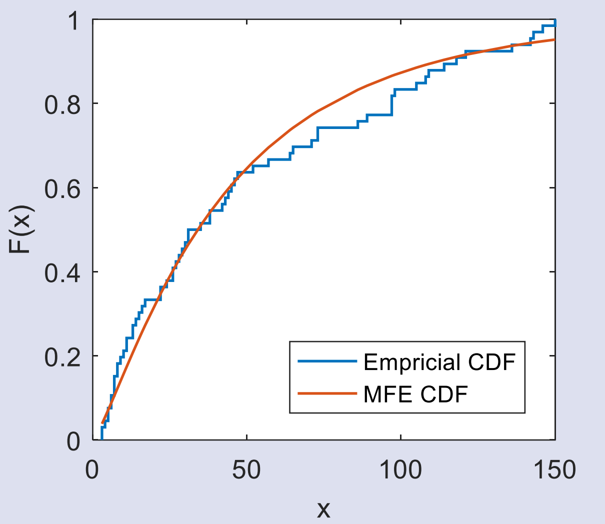

Figure 2: Plot Emprical CDF and MFE CDF for data-2

In this case, ML and approximate Bayes estimates for α and λ parameters are as follows.

	MLE		*BAYES*$_{Lindley}$	
n	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
66	1.2666	0.0184	1.3009	0.0190

Conclusions

In this article, I have considered the MLE and approximate Bayes estimators for parameters of MFE distribution based on complete sample. It has been observed that the maximum likelihood estimators of the parameters can be obtained by using Newton-Raphson method. Because the Bayes estimators of the parameters cannot be obtained in explicit forms, we have obtained the approximate Bayes estimators using Lindley approximation method under squared-error loss function. I have compared the performance of the approximate Bayes estimates with the ML estimates by means of Monte Carlo simulations, and it has been observed that the performances of approximate Bayes estimates are better than those of ML estimates. Further, MSE values of the estimates of α and λ parameters obtained by using Lindley's approximation method are lower than those obtained by using MLE.

Concluding Remarks

The MFE distribution introduced by [7] is studied in this work with relation to MLE and Bayes estimation methods. For different parameter values and different sample sizes, simulations are performed. When the sample size is increased, the MSE and bias values are observed to decrease and approach zero. In addition, the estimations and KS outcomes for MLE and Bayes estimators are examined for two real datasets. The simulation results show that Bayesian estimators perform well on the bias and MSE criteria, and that the MLE and Bayesian estimators are close to each other in a large sample.

Conflicts of interest

All authors declare that they have no conflict of interest.

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