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New Circular Distribution with an Application to Biology

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ABSTRACT

In this paper, we introduce a new circular distribution known as the wrapped new weighted exponential distribution (WNWE) is introduced. We derive an explicit expression for its probability density function and establish closed-form expressions for the distribution function, characteristic function, and trigonometric moments. Furthermore, we discuss the properties of the proposed model. We employ the method of maximum likelihood estimation to estimate the parameters. To demonstrate the applicability of the proposed distribution, we analyze a real dataset consisting of 50 starhead top minnows.

Keywords: Circular data, Weighted exponential distribution, Characteristic function, Moments, Wrapped distributions.

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Introduction

Levy [1] put forward an innovative mechanism for introducing a wrapped variable to the symmetric and asymmetric distributions, which is acknowledged in the literature as wrapped distributions. The wrapping and inverse stereographic projection methods are established approaches to obtain circular models with the help of linear distributions [2]. Srinivasa Rao Jammalamadaka et al. [3-5] developed circular models obtained by wrapping the classical exponential and Laplace distributions on the real line around the unit circle and present explicit forms for their densities and distributions. Rao et al. [6] discuss wrapped versions of some models, such as wrapped Weibull, lognormal, extreme-value and logistic distributions and their population characteristics. Roy et al. [7] developed a new circular model called wrapped weighted exponential distribution. Roy et al. [8] derived a class of wrapped generalized Gompertz distribution and is used to analyze data on the headings of the orientation of the nests of noisy scrub birds. Girija et al. [9-10] derived some circular and semicircular models with the application of inverse stereographic projection and derived some statistical properties. Rao et al. [11] proposed a circular version of logistic distribution by inducing inverse stereographic projection. Phani et al. [12-15] constructed some semicircular distributions by using inverse stereographic projection on linear models and derived their trigonometric moments. Savitri Joshi et al. [16] explored wrapped Lindley distribution. Abdullah Yilmaz et al. [17] introduced a new wrapped exponential distribution by using the transmuted rank quadratic map method. Ahmad et al. [18-20] derived some wrapped circular distributions and studied their properties. Ayat T.R. Al-Meanazel et al. [21] constructed a wrapped Shankar distribution. Phani Yedlapalli et al. [22] derived a family of semicircular and circular Arc tan-exponential type distributions and presented some distributional properties. Ayesha Iftikhar et al. [23] developed half circular modified burr-III distribution and presented different estimation methods. In this study, we draw upon prior research contributions for contextual grounding, notably referencing Karakaya and Karakaya et al.'s [24] investigation into various estimation techniques for the one-parameter Akash distribution, Tanış et al.'s [25] examination of the Transmuted Complementary Exponential Power distribution, and Korkmaz et al.'s [26] study on parameter estimation for the Unit log-log Distribution.

In the present study, we introduce wrapped new weighted exponential distribution by wrapping the density function of the new weighted exponential distribution (Oguntunde et al. [27] along a circle of radius of unity and delve into its characteristics. In the second section of our paper, we delineate a methodology for developing a circular variant of the new weighted exponential distribution by leveraging the principle of wrapping a univariate density function. We are able to derive explicit equations for the probability density function (pdf) and cumulative distribution function (cdf) of the proposed model. In the third section, we derive the characteristic function, trigonometric moments and other properties are discussed. The maximum likelihood estimation method is discussed to estimate the proposed model parameters in the fourth section. In the fifth section, we utilize the proposed model to analyze a real dataset comprising the orientation of fifty starhead top minnows. To evaluate its effectiveness, we compare its performance with that of the wrapped Lindley distribution (Joshi et al., [16] and the wrapped exponential distribution (S. Rao et al., [2] using model selection criteria such as Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), and log-likelihood.

Definition and Derivation

In this section, we describe a method of synthesis for the circular version of a new weighted exponential distribution following the methodology of wrapping of univariate density (Jammalamadaka et al. [11]).

Let X follows new weighted exponential distribution (Oguntunde et al. [27], then the probability density function of X is given by

$f(x) = (\alpha + \lambda\alpha)\, exp(-(\alpha + \lambda\alpha)x);$

where

$x > 0, \alpha > 0$ is the scale parameter and $\lambda > 0$ the shape parameter.

Now, the circular (wrapped) new weighted exponential random variable is defined as

$\theta = X(mod\, 2\pi)$, such that for $\theta \in 0{,}2\pi)$, the probability density function of the proposed model can be expressed as follows:

$$g_{nwe}(\theta) = \sum_{j=0}^{\infty} f_X(\theta + \pi 2j)$$

$$= (\alpha + \lambda\alpha)\sum_{j=0}^{\infty} e^{-(\alpha+\lambda\alpha)(\theta+\pi 2j)}$$

$$= (\alpha + \lambda\alpha) e^{-(\alpha+\lambda\alpha)\theta}\sum_{j=0}^{\infty} e^{-2\pi(\alpha+\lambda\alpha)j} \quad (1)$$

$$g_{nwe}(\theta) = \frac{(\alpha + \lambda\alpha)\exp(-(\alpha + \lambda\alpha)(\theta - \pi))}{2\sinh(\pi(\alpha + \lambda\alpha))},$$

$\theta \in [0, 2\pi)$ Where $\lambda > 0, \alpha > 0$.

The random variable θ having wrapped new weighted exponential distribution is denoted by θ ~WNWE(α, λ).

Special case:

When$\lambda \to 0$, the wrapped new weighted exponential distribution approaches to the wrapped exponential distribution.

Figures 1 and 2 depict plots of the probability density function of the wrapped new weighted exponential distribution and its circular representation for various values of the shape parameters.

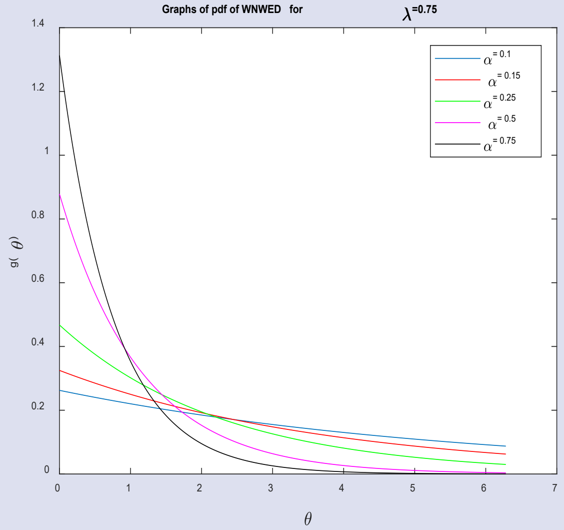

Figure 1. Pdf plots of wrapped new weighted exponential distribution.

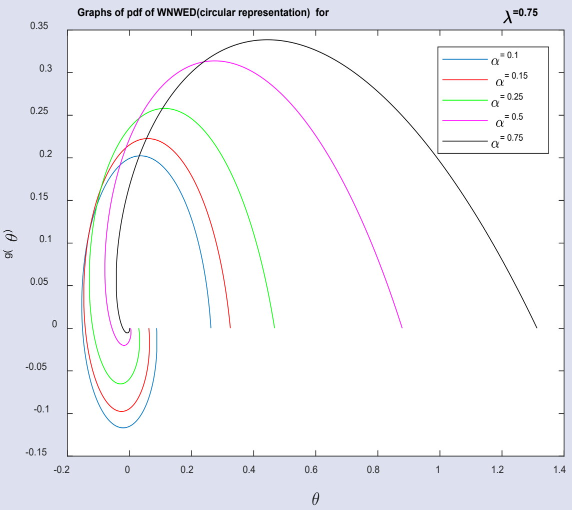

Figure 2. Pdf plots of wrapped new weighted exponential distribution (circular representation).

The cumulative distribution of WNWE(α, λ) is obtained as follows (Jammalamadaka and Sengupta (2001))

$$G_{nwe}(\theta) = \sum_{j=0}^{\infty}[F_X(\theta + \pi 2j) - F_X(2\pi j)],$$

$$G_{nwe}(\theta) = \frac{exp(\pi\alpha(1+\lambda))\,(1 - exp(-\alpha(1+\lambda)\theta))}{2\,sinh(\pi\alpha(1+\lambda))}, \quad (2)$$

$\theta \in 0{,}2\pi)$.

Figures 3. and 4. depict plots of the cumulative distribution function for both the wrapped new weighted exponential distribution and its circular representation.

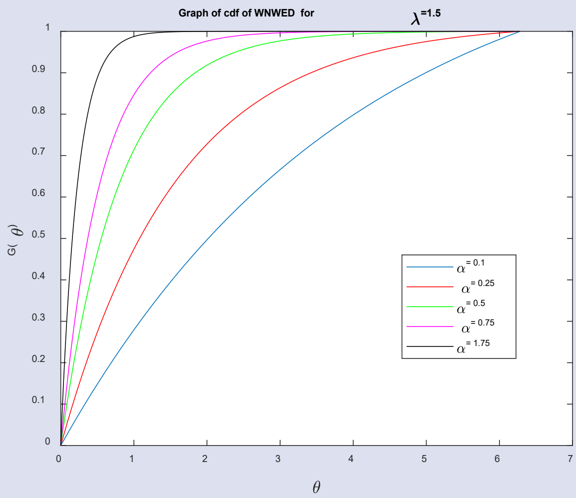

Figure 3. Plots of cdf of wrapped new weighted exponential distribution.

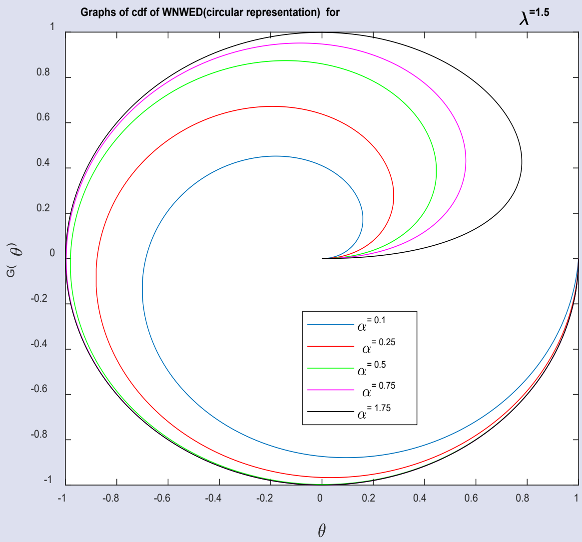

Figure 4. Plots of cdf of wrapped new weighted exponential distribution (circular representation)

Properties

In this section, we derive the closed-form expressions for various statistical properties of the proposed model. These include the characteristic function, trigonometric moments, and several other important characteristics such as the resultant length, circular mean, circular variance, standard deviation, coefficient of skewness, kurtosis, and entropy. By obtaining these analytical expressions, we gain valuable insights into the behavior and descriptive measure of the proposed model, allowing for a comprehensive understanding of its characteristics.

Characteristic Function and Trigonometric Moments

The characteristic function of a circular random variable θ is the doubly-infinite sequence of complex numbers $\{\varphi_p : p \in Z\}$ given by $\varphi_p = E\left(e^{ip\theta}\right)$

$$\varphi_p = \int_0^{2\pi} e^{ip\theta} d\left(G(\theta)\right)$$

$$= \frac{\alpha(1+\lambda)}{\left(1-e^{-2\pi\alpha(1+\lambda)}\right)} \int_0^{2\pi} e^{ip\theta} e^{-\alpha(1+\lambda)\theta}\, d\theta$$

$$= \frac{\alpha(1+\lambda)}{\left(1-e^{-2\pi\alpha(1+\lambda)}\right)} \int_0^{2\pi} e^{-(\alpha(1+\lambda)-ip)\theta}\, d\theta$$

$$\varphi_p = \frac{\alpha^2(1+\lambda)^2}{\alpha^2(1+\lambda)^2+p^2} + i\,\frac{\alpha(1+\lambda)p}{\alpha^2(1+\lambda)^2+p^2} = \alpha_p + i\beta_p. \tag{3}$$

Where $\alpha_p = \frac{\alpha^2(1+\lambda)^2}{\alpha^2(1+\lambda)^2+p^2}$, $\beta_p = \frac{\alpha(1+\lambda)p}{\alpha^2(1+\lambda)^2+p^2}$, $p \in \mathbb{Z}$.

are non-central trigonometric moments of wrapped new weighted exponential distribution.

$\varphi_p = \rho_p e^{i\mu_p^0}, p \geq 0$, so that

$$\rho_p = \sqrt{\alpha_p^2+\beta_p^2} = \frac{(1+\lambda)\alpha}{\sqrt{(1+\lambda)^2\alpha^2+p^2}}. \tag{4}$$

$$\mu_p^0 = tan^{-1}\left(\frac{p}{\alpha(1+\lambda)}\right). \tag{5}$$

Quantile Function

The quantile function can be obtained from the root of the equation $G(\theta) - u = 0$with respect to θ as

$$Q(u) = \frac{1}{(\alpha\lambda+\alpha)} ln\left(\frac{1}{\left(1-u\left(1-e^{-2\pi\alpha(\lambda+1)}\right)\right)}\right), \tag{6}$$

where $u \in (0,1)$.

Median

$$Q(0.5) = \frac{1}{\alpha(1+\lambda)} ln\left(\frac{2}{\left(1+e^{-2\pi\alpha(1+\lambda)}\right)}\right). \tag{7}$$

Resultant Length

$$\rho = \rho_1 = \frac{(1+\lambda)\alpha}{\sqrt{(1+\lambda)^2\alpha^2+1}}. \tag{8}$$

Mean Direction

$$\mu = \mu_1^0 = tan^{-1}\left(\frac{1}{\alpha(1+\lambda)}\right). \tag{9}$$

Circular Variance

$$V_0 = 1-\rho = 1-\frac{(1+\lambda)\alpha}{\sqrt{(1+\lambda)^2\alpha^2+1}}. \tag{10}$$

Circular Standard Deviation

$$\sigma_0 = \sqrt{-2\log_e(1-V_0)} = \sqrt{\log_e\left(1+\frac{1}{(1+\lambda)^2\alpha^2}\right)}. \tag{11}$$

Central Trigonometric Moments

$$\overline{\alpha_p} = \rho_p \cos(\mu_p - p\mu_1) = \frac{(1+\lambda)\alpha}{\sqrt{(1+\lambda)^2\alpha^2 + p^2}} \cos\left(\tan^{-1}\left(\frac{p}{\alpha(1+\lambda)}\right) - p\tan^{-1}\left(\frac{1}{\alpha(1+\lambda)}\right)\right). \quad (12)$$

$$\overline{\beta_p} = \rho_p \sin(\mu_p - p\mu_1) = \frac{(1+\lambda)\alpha}{\sqrt{(1+\lambda)^2\alpha^2 + p^2}} \sin\left(\tan^{-1}\left(\frac{p}{\alpha(1+\lambda)}\right) - p\tan^{-1}\left(\frac{1}{\alpha(1+\lambda)}\right)\right). \quad (13)$$

Skewness

$$\varsigma_1^0 = \frac{\overline{\beta_2}}{V_0^{\frac{3}{2}}} = \frac{\frac{(1+\lambda)\alpha}{\sqrt{(1+\lambda)^2\alpha^2 + 2^2}} \sin\left(\pi - \tan^{-1}\left(\frac{2}{\alpha^3(1+\lambda)(\alpha^2(1+\lambda)^2+3)}\right)\right)}{\left(\frac{(1+\lambda)\alpha}{\sqrt{(1+\lambda)^2\alpha^2+1}}\right)^{\frac{3}{2}}} \quad (14)$$

Kurtosis

$$\varsigma_2^0 = \frac{\overline{\alpha_2} - (1-V_0)^4}{V_0^2} = \frac{\frac{(1+\lambda)\alpha}{\sqrt{(1+\lambda)^2\alpha^2 + 2^2}} \cos\left(\pi - \tan^{-1}\left(\frac{2}{\alpha^3(1+\lambda)(\alpha^2(1+\lambda)^2+3)}\right)\right) - \left(\frac{(1+\lambda)\alpha}{\sqrt{(1+\lambda)^2\alpha^2+1}}\right)^4}{\left(\frac{(1+\lambda)\alpha}{\sqrt{(1+\lambda)^2\alpha^2+1}}\right)^2}. \quad (15)$$

Alternative representation of density function of wrapped new weighted exponential distribution as Fourier series

$$g_{nwe}(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{p=1}^{\infty}\left(\frac{\alpha^2(1+\lambda)^2}{\alpha^2(1+\lambda)^2+p^2}\cos(p\theta) + \frac{\alpha(1+\lambda)p}{\alpha^2(1+\lambda)^2+p^2}\sin(p\theta)\right)\right]. \quad (16)$$

Renyi and Shannon Entropy

Entropy serves as a metric to quantify the level of uncertainty associated with a random variable. There are two widely recognized types of entropy: Renyi entropy and Shannon entropy, with the latter being the limiting case of the former. In this section, we delve into the discussion of Renyi and Shannon entropy within the context of the wrapped new weighted distribution. The entropy of a circular random variable with a probability density function is given by $RE_\theta(\gamma) = \frac{1}{1-\gamma} ln \int_0^{2\pi} g^\gamma(\theta) d\theta$, where $\gamma \geq 0$ and $\gamma \neq 1$. Thus, Renyi entropy of WNWE (α, λ)distribution is obtained as

$$RE_\theta(\gamma) = \frac{2\pi\gamma}{1-\gamma}\left(ln\left(\frac{\alpha(1+\lambda)}{1-e^{-2\pi\alpha(1+\lambda)}}\right) - \pi\alpha(1+\lambda)\right). \quad (17)$$

Shannon entropy is obtained as

$$SE_\theta(1) = \pi\alpha(1+\lambda) - ln\left(\frac{\alpha(1+\lambda)}{1-e^{-2\pi\alpha(1+\lambda)}}\right). \quad (18)$$

Table 1: Population characteristics of wrapped new weighted exponential distribution

$\alpha = 1.5$ Trigonometric moments	$\lambda = 0.1$	$\lambda = 0.15$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$
α_1	07314	0.7485	0.7785	0.8351	0.8733
α_2	0.4050	0.4266	0.4678	0.5586	0.6327
β_1	0.4433	0.4339	0.4152	0.3711	0.3327
β_2	0.2454	0.2473	0.2495	0.2483	0.2410
Resultant length ρ	0.8552	0.8651	0.8824	0.9138	0.9345
Mean direction μ	0.5449	0.5254	0.4900	0.4282	0.3640
Circular variance V_0	0.3774	0.3670	0.3478	0.3077	0.2759
Circular standard deviation σ_0	0.5593	0.5383	0.5003	0.4246	0.3681
Central trigonometric moments					
$\overline{\alpha_1}$	0.7314	0.7485	0.7785	0.8351	0.8733
$\overline{\alpha_2}$	0.3962	0.4188	0.4616	0.553	0.6308
$\overline{\beta_1}$	0	0	0	0	0
$\overline{\beta_2}$	-0.0839	-0.0813	-0.0756	-0.0612	-0.0486
Coefficient of skewness ς_1^0	-1.5230	-1.6406	-1.8728	-2.4193	-2.8482
Coefficient of kurtosis ς_2^0	1.7267	1.9171	2.3205	3.4384	4.6759

Estimation of Parameters

In this section, we delve into the intricacies of the maximum likelihood estimation method for estimating the model parameters of the WNWE(α,λ) distribution. Consider a random sample denoted by $\theta_1,\theta_2,\theta_3,\ldots,\theta_m$ comprising observations drawn from the WNWE (α,λ)distribution. Then the likelihood function is given by

$$L(\theta;\alpha,\lambda)=\prod_{i=1}^{m} g_{wnw}(\theta_i;\alpha,\lambda).$$

The log-likelihood function is given by

$$l=log_e(L)=m\,log_e(\lambda+1)+m\,log_e(\alpha)-\alpha(\lambda+1)\sum_{i=1}^{m}(\theta_i)-m\,log_e\left(1-e^{-\pi\alpha 2(\lambda+1)}\right). \quad (19)$$

Taking first order partial differentiation of the log-likelihood function with respect to α and λ then equating them to zero, we get the following normal equations.

$$\frac{\partial l}{\partial\alpha}=\frac{m}{\alpha}-(\lambda+1)\sum_{i=1}^{m}(\theta_i)+\pi(\lambda+1)m2\left[\frac{e^{-\pi\alpha 2(\lambda+1)}}{1-e^{-\pi\alpha 2(\lambda+1)}}\right]=0. \quad (20)$$

$$\frac{\partial l}{\partial\lambda}=\frac{m}{(\lambda+1)}-(\alpha)\sum_{i=1}^{m}(\theta_i)+\pi\alpha m2\left[\frac{e^{-\pi\alpha 2(\lambda+1)}}{1-e^{-\pi\alpha 2(\lambda+1)}}\right]=0. \quad (21)$$

We are not able to get the closed form expression for maximum likelihood estimators. So we opted alternative way, therefore we employ a numerical approach to get the values for the parameters α and λ.

Application

In the following section, we aim to showcase the modeling behavior of the wrapped new weighted exponential distribution by applying it to a real-life dataset. Specifically, we consider the well-known dataset on sun compass orientations of 50-star head topminnows (Fundulus dispar), which is recognized as an endangered aquatic species. To analyze this dataset, we employ the wrapped new weighted exponential distribution and estimate its parameters using Mathematica's *NMaximize* command. Additionally, we calculate various statistics such as log-likelihood, AIC, and BIC for the derived distribution. The outcomes of these computations, along with the corresponding results for other comparable distributions (wrapped Lindley and wrapped exponential distributions), are summarized in Table 2.Directional preferences of starhead topminnows (Fisher B.4, [28], [29]

Sun compass orientations of 50 starhead topminnows, measured under heavily overcast conditions.

2	9	18	24	30	35	35	39	39	44	44	49	56
70	76	76	81	86	91	112	121	127	133	134	138	147
152	157	166	171	177	187	206	210	211	215	238	246	269
270	285	292	305	315	325	328	329	343	354	359		

Table2: Summary of statistics

Model	MLE	Log-likelihood	AIC	BIC
WE(λ)	(0.1149)	-95.8932	193.5157	195.1783
WL(λ)	(0.5138)	-94.7979	191.5958	193.5079
WNWE(α,λ)	(0.03014,0.11158)	-90.8212	185.4420	189.2660

The higher value of the log-likelihood statistic, along with the smaller values of AIC and BIC, unequivocally indicates that the wrapped new weighted exponential distribution provides a superior fit to the dataset compare to the wrapped exponential and wrapped Lindley distributions.

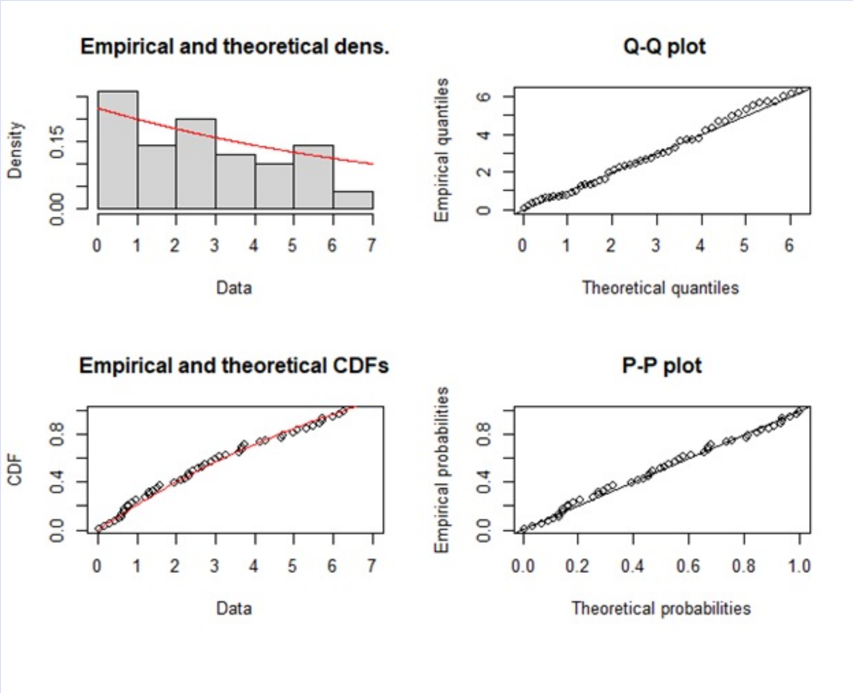

Figure 5: Depicts fitted pdf, cdf, Q-Q, and P-P plots of the wrapped new weighted exponential distribution for star head topminnows.

Closing Remarks

In the present study, we have introduced a new circular distribution referred to as the wrapped new weighted exponential (WNWE) distribution, designed specifically for modeling circular data. The probability density function (pdf), cumulative distribution function (cdf), and some characteristics have been derived in explicit form, allowing for a comprehensive understanding of the distribution's properties. To estimate the model parameters, the maximum likelihood estimation method has been employed. Furthermore, we have applied the proposed distribution to a real-life dataset for empirical validation. The performance of the proposed distribution has been rigorously examined, comparing it with that of the wrapped exponential and wrapped Lindley distributions, utilizing various statistical criteria such as log-likelihood, Akaike Information Criteria (AIC), and Bayesian Information Criteria (BIC). Based on the obtained results, it is evident that the proposed distribution exhibits a superior fit to the dataset of 50 star head top minnows compared to the wrapped exponential and wrapped Lindley distribution.

Scope for Future Work: Future research can focus on validating the WNWE distribution on diverse datasets, exploring its application in statistical and machine learning models, extending or modifying it for specific data patterns, and conducting comparative studies with other circular distributions to enhance its utility and understanding.

Conflicts of interest

The authors declare that, there are no conflicts of interest in this work.

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