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Legendre Computational Algorithm for Linear Integro-Differential Equations

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ABSTRACT

This work presents a collocation computational algorithm for solving linear Integro-Differential Equations (IDEs) of the Fredholm and Volterra types. The proposed method utilizes shifted Legendre polynomials and breaks down the problem into a series of linear algebraic equations. The matrix inversion technique is then employed to solve these equations. To validate the effectiveness of the suggested approach, the authors examined three numerical examples. The results obtained from the proposed method were compared with those reported in the existing literature. The findings demonstrate that the proposed algorithm is not only accurate but also efficient in solving linear IDEs. In order to present the results, the study employs tables and figures. These graphical representations aid in displaying the numerical outcomes obtained from the algorithm. All calculations were performed using Maple 18 software.

Keywords: Legendre polynomials, Fredholm and Volterra integro-differential equations, Approximate solution, Matrix inversion.

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Introduction

Integro-Differential Equations (IDEs) are types of mathematical equations that involve both derivatives and integrals. They arise in various fields of science and engineering, including physics, biology, economics, and finance, where systems exhibit memory or history-dependent behavior. Unlike Ordinary Differential Equations (ODEs) that involve only derivatives, integro-differential equations incorporate the influence of past values of the unknown function through the integration term. Integro-differential equations often appear in problems involving diffusion, propagation of waves, population dynamics, and control theory, among others. They provide a more realistic description of phenomena that exhibit memory effects or spatial interactions. Since most IDEs cannot be solved analytically, researchers have focused on developing numerical methods to obtain approximate solutions. Several authors have contributed to this area. For example, [1] employed the differential transform method, [2] used the Bernstein operational matrix approach, [3] applied the Chebyshev collocation method, [4] employed Lucas collocation method, and [5] introduced the reliable iterative method for Volterra-Fredholm IDEs. In [6], Euler polynomials with the least squares method are used to solve IDEs. In [7], the Adomian decomposition method was used to solve Boundary Value Problems (BVPs) associated with fourth-order IDEs. The Trapezoidal rule and the Variational Iteration Method (VIM) were investigated for linear IDEs in [8], and VIM was also employed in [9] to solve fourth-order IDEs. For Fredholm-Volterra IDEs, various methods were utilized. [10] employed the projection method based on a Bernstein collocation approach; [11] used the Bernstein collocation method; [12] applied a fixed-point iterative algorithm; [13] utilized the Chebyshev polynomial approach; and [14] employed a collocation method based on Bernstein polynomials. In [15], a new numerical method was developed specifically for solving systems of Volterra IDEs. In [16], the Lucas polynomial is employed to solve nonlinear differential equations with variable delays. The use of third-kind Chebyshev polynomials for solving IDEs was examined in [17] and [18]. In [19], Chebyshev Computational Approach is used to find the numerical solution Volterra-Fredholm integro-differential equations. Other methods mentioned in this study include the Hermite collocation method [20], the extended minimal residual method [21], the quadrature-difference method [22], and Adomian's decomposition approach [23], which were used to solve Fredholm IDEs. Based on the works mentioned above, this study propose a computational algorithm that utilizes shifted Legendre polynomials. This technique is inspired by previous research and aims to enhance the outcomes achieved by [18].

The general form of the class of problem considered in this work is given as:

$$\sum_{i=0}^{n} \rho_i(w)\xi^i(w) = f(w) + \int_0^w K(w,v)\xi(v)dv \qquad (1)$$

$$\sum_{i=0}^{n} \rho_i(w)\xi^i(w) = f(w) + \int_0^1 K(w,v)\xi(v)dv, \qquad (2)$$

with the initial conditions

$$\xi^r(0) = \xi_r \qquad r = 0,1,2,\cdots n-1.$$

Where r^{th} represent derivatives, K and $\rho_i(w)$, $i = 0,1,2,\cdots n$ with $\rho_i(w) \neq 0$ are known functions, $f(w)$ is a known function and $\xi^i(w)$ is the i^{th} derivatives of the unknown function $\xi(w)$ to be determined, and (1) and (2) are referred to as Volterra and Fredholm IDEs respectively.

Materials and Method

Definition 1: An integral equation is an equation that has an unknown function, $\xi(w)$, that appears under the integral sign. Standard integral equation has the following form:

$$\xi(w) = f(w) + \lambda \int_{g(w)}^{h(w)} K(w,t)\xi(t)dt \ ,$$

where K(w,t) is a function of two variables w and t known as the kernel or the nucleus of the integral equation, $g(w)$ and $h(w)$ are the limits of integration, λ is a constant parameter.

Definition 2: Legendre's polynomial of degree n is denoted and defined by

$$\tau_n(w) = \sum_{r=0}^{[\frac{n}{2}]}(-1)^r \frac{(2n-2r)!}{2^n.r!(n-r)!(n-2r)!} w^{n-2r} \ ,$$

where

$$\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2} & if \ \ n \ is \ even \\ \frac{n-1}{2}, & if \ n \ is \ odd. \end{cases}$$

$n\tau_n(w) = (2n-1)w\tau_{n-1}(w) - (n-1)\tau_{n-2}(x); n \geq 2$, starting with

$$\tau_0(w) = 1, \tau_1(w) = w$$

Hence, the first few of Legendre Polynomials on the interval [-1,1] is given below:

$$\left.\begin{aligned} \tau_0(w) &= 1 \\ \tau_1(w) &= w \\ \tau_2(w) &= \frac{1}{2}(3w^2-1) \\ \tau_3(w) &= \frac{1}{2}(5w^3-3w) \end{aligned}\right\} \qquad (3)$$

The shifted equivalent of (3) that valid in [0, 1] are given as

$$\left.\begin{aligned} \tau^*_0(w) &= 1 \\ \tau^*_1(x) &= 2w-1 \\ \tau^*_2(w) &= 6w^2-6w+1 \\ \tau^*_3(x) &= 20w^3-30w^2+20w-1 \end{aligned}\right\} \qquad (4)$$

Definition 3: Absolute Error: We defined absolute error as follows in this study: Absolute Error=$|\boldsymbol{\tau}(w) - \tau(w)|$; $-1 \leq w \leq 1$, where $\boldsymbol{\tau}(w)$ is the exact solution and $\tau(w)$ is the approximate solution.

Proposed Method

In order to find the numerical approximation to the general class of problem considered in this study, we assumed an approximate solution by means of the shifted Legendre polynomials in the form:

$$(w) = \sum_{r=0}^{n} \tau^*_r(w)\, c_r, \qquad (5)$$

where, $c_r, r = 0(1)n$ are to be found. Thus, substituting Eq. (5) into Eq. (1) gives

$$\sum_{i=0}^{n} \rho_i(w) \sum_{r=0}^{n} \tau^{*i}{}_r(w)\, c_r = f(w) + \int_a^w k(w,v) \sum_{r=0}^{n} \tau^*_r(v)\, c_r dv\,, \qquad (6)$$

where $\tau^{*i}(w)$ is the i^{th} derivative of $\tau^*(w)$.

Let $p(w) = \sum_{i=0}^{n} \rho_i(w) \sum_{r=0}^{n} \tau^{*i}{}_r(w)\, c_r$ and $q(w) = \int_a^w k(w,v) \sum_{r=0}^{n} \tau^*_r(v)\, c_r dv$.

Thus, equation (6) becomes

$$p(w) - q(w) = f(w) \qquad (7)$$

The linear algebraic system of equations in $(n+1)$ unknown constants $e'_i s$ is obtained by collocating Eq. (7) at the evenly spaced point $w_i = a + \frac{(b-a)i}{n}$, $(i = 0(1)n)$.

Additional equations are obtained from initial conditions Eq. (3), which are represented in matrix form:

$$\begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & \cdots & Z_{1n} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} & \cdots & Z_{2n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ Z_{m1} & Z_{m2} & Z_{m3} & Z_{m4} & \cdots & Z_{mn} \\ Z_{11}{}^0 & Z_{12}{}^0 & Z_{13}{}^0 & Z_{14}{}^0 & \cdots & Z_{1n}{}^0 \\ Z_{21}{}^1 & Z_{22}{}^1 & Z_{23}{}^1 & Z_{24}{}^1 & \vdots & Z_{2n}{}^0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ Z_{m1}{}^{n-1} & Z_{m2}{}^{n-1} & Z_{m3}{}^{n-1} & Z_{m4}{}^{n-1} & \cdots & Z_{mn}{}^{n-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} E_{11} \\ E_{21} \\ \vdots \\ E_{mn} \\ E_{11}{}^0 \\ E_{22}{}^1 \\ \vdots \\ E_{mn}{}^{n-1} \end{pmatrix} \qquad (8)$$

where $Z_i's$ and $Z_i{}^*{}'s$ are the coefficients of c_{is} and $E_i's$ are values of $f(w_i)$. The matrix inversion approach is then used to solve the system of equations in order to obtain the unknown constants.

$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & \cdots & Z_{1n} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} & \cdots & Z_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{m1} & Z_{m2} & Z_{m3} & Z_{m4} & \vdots & Z_{mn} \\ Z_{11}{}^{0} & Z_{12}{}^{0} & Z_{13}{}^{0} & Z_{14}{}^{0} & \vdots & Z_{1n}{}^{0} \\ Z_{21}{}^{1} & Z_{22}{}^{1} & Z_{23}{}^{1} & Z_{24}{}^{1} & \vdots & Z_{2n}{}^{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{m1}{}^{n-1} & Z_{m2}{}^{n-1} & Z_{m3}{}^{n-1} & Z_{m4}{}^{n-1} & \cdots & Z_{mn}{}^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} E_{11} \\ E_{21} \\ \vdots \\ \vdots \\ E_{mn} \\ E_{11}{}^{0} \\ E_{22}{}^{1} \\ \vdots \\ E_{mn}{}^{n-1} \end{pmatrix} \quad (9)$$

The required approximate solution is obtained by solving Eq. (9) and then substituting the unknown constant values into the assumed approximate solution.

Numerical Examples

Example 1 **[18]:** Consider the second-order Fredholm integro-differential equation

$$\xi^{(ii)}(w) = e^{w} - \frac{4}{3}w + \int_0^1 wv\xi(v)dv$$

Subject to the initial conditions

$$\xi(0) = 1, \xi'(0) = 2.$$

The exact solution is $\xi(w) = w + e^{w}$

By applying the aforementioned technique to example 1, which is solved at n = 9, we obtained the following constants and the necessary approximation:

$c_0 = 2.21828182923867$, $c_1 = 1.34515451545867$,
$c_2 = 0.139863996066064$, $c_3 = 0.0139312558593461$,
$c_4 = 0.000992587579099529$, $c_5 = 0.0000550476656947328$,
$c_6 = 0.00000249897543908013$, $c_7 = 9.59643311863878 \times 10^{-8}$,
$c_8 = 3.20747900718829 \times 10^{-9}$, $c_9 = 1.18693253142732 \times 10^{-10}$

$$\xi(w) = 0.9999999999 + 1.999999998w + 0.5000000002w^2 + 0.1666668768w^3 + 0.04166493183w^4 + 0.008339701772w^5 + 0.001376126408w^6 + 0.0002131111947w^7 + 0.00001531135797w^8 + 0.000005770865966w^9$$

Table 1. Shows comparison of the absolute errors for Example 1.

w_i	Absolute Error of our Method n=9	Absolute Error of our Method n=10	Absolute Error of our Method n=11	Absolute Error of our Method n=15	Absolute Error of our Method n=16	Absolute Error of our Method n=17	Absolute Error of [5] n=10
0.0	1.000E-10	9.000E-10	9.000E-10	1.000E-10	9.000E-10	1.000E-10	4.79E-06
0.2	0.000E+00	7.100E-10	7.200E-10	4.000 E+00	3.000E-10	1.000E-10	5.03E-06
0.4	1.000E-10	1.220E-09	1.230E-09	1.100E-09	2.900E-09	1.100E-09	6.74E-06
0.6	0.000E+00	3.000E-10	1.300E-09	1.000E-09	2.200E-09	3.100E-09	7.91E-06
0.8	0.000E+00	1.000E-10	1.200E-09	1.000E-09	1.800E-09	1.100E-09	7.58E-06
1.0	0.000E+00	1.000E-10	1.000E-09	1.100E-09	2.000E-10	2.100E-09	1.11E-05

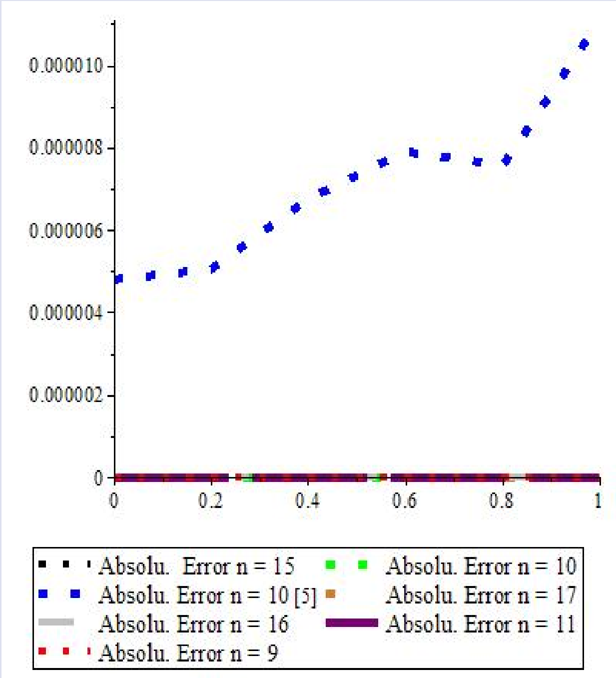

Figure 1. Shows the graphical representation of the exact and approximate solutions to the Example 1.

Example 2. [18]: Consider fourth-order Volterra integro- differential equation

$$\xi^{(iv)}(w) = -1 + \xi(w) + \int_0^w (w - v)\xi(v)dv$$

Subject to the initial conditions
$\xi(0) = -1, \xi'(0) = 1, \xi''(0) = 1, \xi'''(0) = -1.$

The exact solution is $\xi(w) = \sin w - \cos w$

By applying the aforementioned technique to example 2, which is solved at n = 9, we obtained the following constants and the necessary approximation:

$c_0 = -0.381773289713882$, $c_1 = 0.661692203358418$,
$c_2 = 0.0325913563567896$, $c_3 = -0.0111522290510834$,
$c_4 = -0.000234317964781481$, $c_5 = 0.0000444449128009975$,
$c_6 = 5.93499121704253 \times 10^{-7}$, $c_7 = -7.78764893501739 \times 10^{-8}$,
$c_8 = -7.57650814696993 \times 10^{-10}$ $c_9 = 7.61473616036433 \times 10^{-11}$

$$\xi(w) = -1.000000001 + w + 0.5w^2 - 0.166666666w^3 - 0.04166666964w^4 + 0.00833351048w^5 + 0.001388125394w^6 - 0.0001969077181w^7 - 0.00002641124722\tau^8 + 0.000003702284721w^9$$

Table 2. Shows comparison of the absolute errors for Example 2.

w_i	Absolute Error of our Method n=9	Absolute Error of our Method n=10	Absolute Error of our Method n=11	Absolute Error of our Method n=15	Absolute Error of our Method n=16	Absolute Error of our Method n=17	Absolute Error of [5] n=10
0.0	1.000E-09	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	6.00E-09
0.2	1.030E-09	1.416E-10	2.000E-10	1.870E-10	0.000E+00	3.000E-10	2.10E-09
0.4	7.400E-10	4.070E-10	3.700E-10	3.018E-10	0.000E+00	0.000E+00	6.20E-09
0.6	2.000E-10	5.993E-10	6.000E-10	5.548E-10	1.000E-09	9.000E-10	6.80E-09
0.8	9.000E-10	9.746E-10	8.000E-10	7.561E-10	1.000E-09	2.000E-10	4.77E-09
1.0	3.100E-09	1.228E-09	1.200E-09	1.244E-09	1.000E-09	1.000E-09	9.55E-07

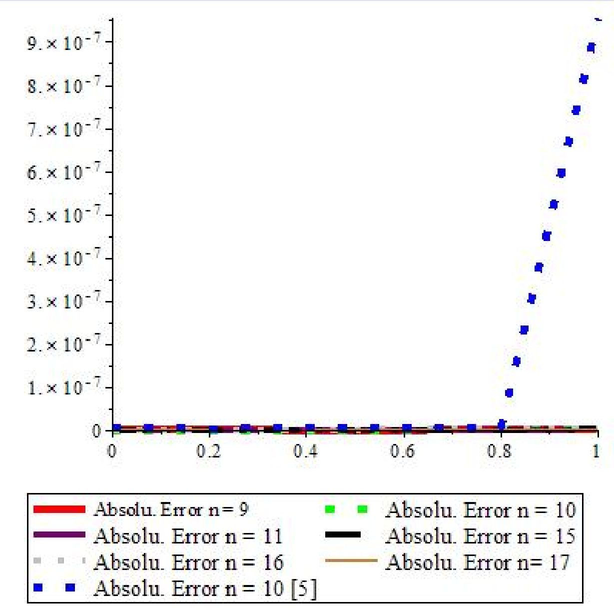

Figure 2. Shows the graphical representation of the exact and approximate solutions to the Example 2.

Example 3. [18]: Consider the following second-order Volterra integro- differential equation

$$\xi^{(ii)}(w) = 2 - 2w\sin w - \int_0^w (w-v)\xi(v)dv$$

Subject to the initial conditions

$$\xi(0) = 0, \xi'(0) = 0.$$

The exact solution is $\xi(w) = w\sin w$

By applying the aforementioned technique to example 3, which is solved at n = 9, we obtained following constants and the necessary approximation:

$c_0 = 0.301168679731766$, $c_1 = 0.435959616897170$,
$c_2 = 0.121472377673670$, $c_3 = -0.0153164866040020$,
$c_4 = -0.00191302106873919$, $c_5 = 0.0000925800901348493$,
$c_6 = 0.00000746980787237939$, $c_7 = -2.17442225378117 \times 10^{-7}$
$c_8 = -1.28999931101810 \times 10^{-8}$

$$\xi(w) = -1.24991699 \times 10^{-11} - 9.910471 \times 10^{-11}w + w^2 + 3.37729139 \times 10^{-9}w^3 - 0.1666699038w^4 + 0.00001365628761w^5 + 0.008301749153w^6 + 0.0000428124374w^7 - 0.0002324198696w^8 + 0.00001475487962w^9$$

Table 3. Shows comparison of the absolute errors for Example 3.

w_i	Absolute Error of our Method n=9	Absolute Error of our Method n=10	Absolute Error of our Method n=11	Absolute Error of our Method n=15	Absolute Error of our Method n=16	Absolute Error of our Method n=17	Absolute Error of [5] n=10
0.0	1.250E-11	2.297E-11	4.294E-11	4.761E-11	4.803E-11	6.272E-11	1.13E-10
0.2	3.100E-10	2.000E-11	7.706E-12	3.309E-11	3.522E-11	9.268E-11	2.56E-07
0.4	7.000E-10	1.000E-10	2.571E-10	1.003E-10	1.944E-13	1.162E-10	2.22E-07
0.6	1.000E-09	4.000E-10	6.571E-10	0.000E+00	1.150E-10	2.513E-10	1.68E-07
0.8	1.300E-09	5.000E-10	6.571E-10	9.654E-11	1.533E-10	2.206E-10	5.38E-07
1.0	1.700E-09	6.000E-10	9.571E-10	2.143E-10	3.412E-10	1.952E-10	9.55E-07

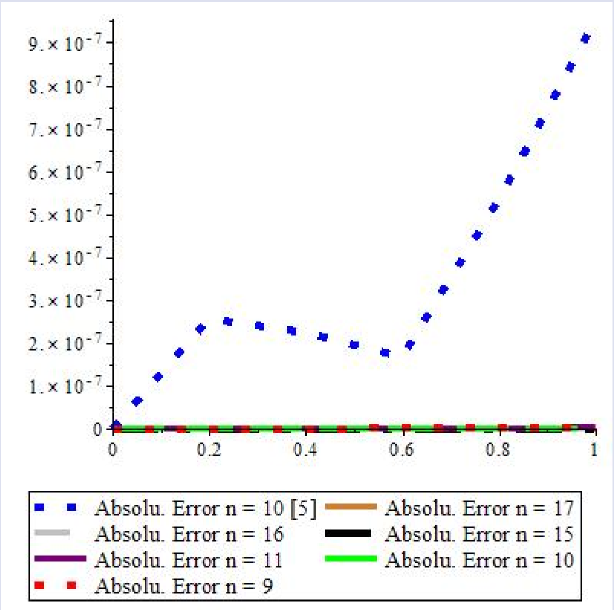

Figure 3: Shows the graphical representation of the exact and approximate solutions to the Example 3.

Conclusion

This work introduced a numerical approach for solving linear integro-differential equations by combining shifted Legendre polynomials with the collocation method. The method was applied to three specific numerical examples, and the results were compared to a previous study [18] that used the collocation method with Chebyshev third-kind polynomials at n = 10. The table of results clearly indicates that the proposed technique outperformed the method employed in [18] in terms of performance. The errors obtained using the suggested method were consistently smaller than those reported in [18]. This demonstrates the superiority of the recommended approach for solving IDEs. Based on these findings, we strongly recommend adopting the provided approach when dealing with linear integro-differential equations.

Conflicts of interest

There are no conflicts of interest in this work.

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