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An Analytical Approach to Contaminant Transport with Spatially and Temporally Dependent Dispersion in a Heterogeneous Porous Medium

Raja Ram Yadav[1,a], Sujata Kushwaha[1,b,*], Lav Kush Kumar[2,c], Joy Roy[3,d]

[1] *Department of Mathematics and Astronomy, University of Lucknow, Lucknow-226007, U.P., India*
[2] *Department of Applied Science, BabuBanarasi Das Institute of Technology and Management, Lucknow-226028, U.P., India*
[3] *Department of Mathematics, Aryavart Institute of Higher Education, Lucknow-226005, U.P., India*
**Corresponding author*

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ABSTRACT

This study presents an analytical solution to the one-dimensional advection-dispersion equation (ADE) for a semi-infinite heterogeneous aquifer system with space and time-dependent groundwater velocity and dispersion coefficient. The dispersion coefficient is assumed to be proportional to the groundwater flow velocity. In addition, retardation factor, first-order decay and zero-order production terms are also considered. Contaminants and porous media are assumed to be chemically inert. Initially, it is assumed that some uniformly distributed solutes are already present in the aquifer domain. The input point source is considered uniformly continuous and increasing nature in a semi-infinite porous medium. The solutions are obtained analytically using the Laplace Integral Transform Technique (LITT). The nature of the concentration profile of the resulting solution for different parameters in different time domains is illustrated graphically.

Keywords: Groundwater, Pollutant, Advection, Dispersion, Retardation factor.

[a] *yadav_rr2@yahoo.co.in* *https://orcid.org/0000-0002-1311-5435*
[b] *sujatakushwaha14@gmail.com* *https://orcid.org/0000-0003-0467-3862*
[c] *lk.verma13@gmail.com* *https://orcid.org/0000-0001-9039-7205*
[d] *joyroy2904@gmail.com* *https://orcid.org/0000-0003-0403-3048*

Introduction

In recent years, excessive use of pesticides in agriculture, unplanned urbanization, illegal industrial activities, and seepage of chemicals into groundwater have contributed to the deterioration of water quality. Hazardous substances can sometimes cause serious health problems even at low concentrations, so treatment is often required to achieve sustainability. Several factors, including source geometry, solute shape and orientation, play an important role in influencing the concentration profile throughout transport process. The complex structure of the aquifer systems makes forecasting the degree of contamination and remedial measures extremely challenging. In order to deal with groundwater pollution, it is imperative to study solute transport in groundwater systems. To efficiently estimate and predict such problems, mathematical models are very useful. The advection-dispersion equation (ADE) mathematically describes the physical, chemical and biological processes that are primarily responsible for the movement of pollutants in porous media. Analytical or semi-analytical solutions of the advection-dispersion equation are often used for preliminary risk assessment of groundwater pollution and accurate prediction of pollution intensity in aquifer systems.

Many studies have been published in the literature to solve the advection-dispersion equation (ADE) assuming constant pore velocity and dispersion coefficient (Shi et al., 2016, [1] Guerrero et al., 2013 [2]). However, groundwater flow fluctuates both in temporally and spatially due to the different recharge rates of aquifer systems at different locations and time. Especially in summer and rainy season, the change of groundwater flow pattern is more obvious. The dispersion coefficient measured in the laboratory is quite different from the dispersion coefficient obtained in the field. Bear (1972) [3] and Mishra et al., (1991) [4] observed these discrepancies in dispersion as scale dependence of the dispersivity. Elder (1959) [5] derived an expression for the longitudinal dispersion coefficient for an infinitely wide open channel assuming a logarithmic velocity distribution using Taylor's theory. After establishing that the transverse velocity distribution is the dominant mechanism for longitudinal diffusion, Fischer (1967) [6] proposed an alternative formulation for the longitudinal diffusion coefficient. According to stochastic analysis, the dispersion transport is time dependent and rises until it reaches an asymptotic value (Gelhar et al., 1979) [7]. Valocchi (1989) [8] used spatial moment analysis to investigate how adsorption kinetics affects the spatial moments of depth-averaged pollution plumes. Shan and Javendel (1997) [9] developed an analytical solution for solute transport in a vertical section of a homogeneous aquifer with steady and uniform groundwater flow. In a semi-infinite homogeneous river, Wadi et al., (2014) [10] obtained an analytical solution to the one-dimensional solute transport problem using the Laplace transform technique. Kumar and Yadav (2014) [11] developed a 1D analytical solution for conservative solute transport in heterogeneous porous media for uniform and varying pulse type input point sources. Natarajan (2016) [12] studied the effect of time- and distance-dependent dispersion coefficients on the transport of multi-species pollutants in porous media. Rubol et al., (2016) [13] simulated solute transport in the soil layer of plant roots using the advection-diffusion equation (ADE) with spatially variable coefficients. Kumar et al., (2019) [14] investigated the effect of the source/sink term on the transport of solutes in porous media. Younes et al., (2020) [15] studied the effect of travel distance on the dispersion value in porous media and found that the difference in travel distance is mainly determined by the transverse dispersion. Yadav et al., (2023) [16] studied analytical solutions for scale and time dependent solute transport in heterogeneous porous medium.

As this work could have a big impact on other research areas that uses forms of the ADE, such as the study of fish survival in rivers, radionuclide release from an area source, and radioactive substance dispersion released from nuclear power plants among others. For determining conditions and suitable locations for fish survival by using the solution of the two coupled pollution and aeration equations by Raafat, P. B., et.al (2023)[17]. A novel analytical approach for advection diffusion equation for radionuclide release from an area source by Esmail et. al (2020)[18].

In the current model, the Laplace Integral Transform approach is employed to get analytical solutions for the convection-diffusion equation to explain a conservative solute transport process in a heterogeneous porous formation. The variable coefficients of the ADE are reduced to constant coefficients by employing appropriate transformations that take into account the space and time-dependent expressions. Dispersion and velocity both depend on space and time and are considered separately for each time period. Such studies may be updated daily, monthly, fortnightly or annually based on information obtained from different positions and times within the contaminated area. This work may also apply to other physical conditions that are affected by an increase in source concentration. The developed model is very useful for understanding the concentration levels at any position and time and in the aquifer systems.

Mathematical Formulation and Analytical Solutions

Here, we focus on groundwater flow that transports a neutral and inert solute through a heterogeneous porous medium. The solute is conservative and the porous structure is heterogeneous, adsorbing and semi-infinitely long. The one-dimensional contaminant transport equation can be expressed in Cartesian form as follows (Bear, 1972) [3]:

$$R\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left\{D(x,t)\frac{\partial c}{\partial x}-U(x,t)c\right\}-\gamma c+\mu \quad (1)$$

Where, $c[ML^{-3}]$ is the contaminant concentration at position $x[L]$ and time $t[T]$. $D(x,t)[L^2T^{-1}]$ and $U(x,t)\ [LT^{-1}]$ are longitudinal dispersion coefficient and groundwater velocity, respectively. $\gamma[T^{-1}]$ is the first order decay constant and $\mu[ML^{-3}T^{-1}]$ is the zero order production rate coefficient for solute which represents internal/external production of the solute in the medium. All solute transport through porous media is affected by a dimensionless quantity called the retardation factor (R). Adsorption causes a significant delay, which is considered as a retardation factor in the equation. The zero gradient boundary condition is enforced at the outlet, which can not only meet the mass balance requirements, but also ensure the consistency of the concentration at $x\to\infty$.

In the process of finding the analytical solution of the convection-diffusion equation by using the Laplace Integral Transformation, it is necessary to convert the variable coefficients into constant coefficients. The time- and position-dependent dispersion and groundwater velocity along the longitudinal direction is taken in degenerated form as follows:

$$\left.\begin{array}{l} U(x,t)=U_0(1+ax)F(t),\\ D=D_0(1+ax)^2F(t),\\ \gamma=\gamma_0F(t),\\ \mu=\mu_0F(t),\\ R=R_0 \end{array}\right\} \quad (2)$$

where, D_0 and U_0 denote the initial dispersion coefficient and longitudinal groundwater velocity, respectively. γ_0, μ_0 and R_0 are the respective initial first-order decay constant, zero-order production coefficient and retardation factor, respectively.

where,

$$F(t)=\begin{Bmatrix} f_1\ (m_1t)\ ;0\le t<t_1\\ f_2\ (m_2t)\ ;t_1\le t<t_2\\ f_3\ (m_3t)\ ;t_2\le t<\infty \end{Bmatrix} \quad (3)$$

The expression $f_i(m_it); i=1,2,3$ is dimensionless. m_i are unsteady parameters having dimension, inverse of time variable. Here $f_1\ (m_1t)$ is chosen such that $f_1\ (m_1t)=1$ for $m_1=0$ or $t=0$. It is obvious that $F(t)$ is dimensionless function. a is the heterogeneity parameter having dimension, inverse of position variable.

Analytical Solution of the Problem for Uniform Input Point Source

The first condition Eq. (4) is an initial condition that says, at time $t=0$, there is some uniformly concentration c_i everywhere in the semi-infinite flow domain where, $x\ge 0$.The second condition Eq. (5) specifies continuous injection of a uniform input from the inlet boundary at $x=0$ (Yadav and Kumar, 2018)[16]. At other end as $x\to\infty$, it is assumed that a gradient of zero concentration exist forever Eq. (6).

So, the consideration of initial and boundary conditions for the above governing Eq. (1) are as follows:

Initial and Boundary Conditions

$$c(x,t) = c_i;\ \ t = 0, x \geq 0 \quad (4)$$

$$c(x,t) = c_0 G(t);\quad x = 0, t > 0 \quad (5)$$

$$\frac{\partial c(x,t)}{\partial x} = 0;\ x \to \infty, t \geq 0 \quad (6)$$

where, $G(t) = m^* \int_0^t F(t)dt$ (7)

where, m^* is unsteady parameters having dimension, inverse of time variable and $G(t)$ is also dimensionless function.

Incorporating of Eq.(2) into Eqs. (1) and (4-6) yields, we have as follows:

$$R_0 \frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left\{D_0(1+ax)^2 F(t)\frac{\partial c}{\partial x} - U_0(1+ax)F(t)c\right\} - \gamma_0 F(t)c + \mu_0 F(t) \quad (8)$$

The initial and boundary conditions are reduced as follows:

$$c(x,t) = c_i\ ; t = 0, x \geq 0 \quad (9)$$

$$c(x,t) = c_0 G(t)\ ; x = 0, t > 0 \quad (10)$$

$$\frac{\partial c(x,t)}{\partial x} = 0;\quad x \to \infty, t \geq 0 \quad (11)$$

Mathematical substitution can simplify the structure of the equation and provide greater degree of freedom for the solution. So, we introduce the following transformation Eq. (12) to keep the variable coefficient independence of space in Eq. (8) [19]:

$$X = \int_0^x \frac{1}{(1+ax)} dx \text{or} X = \frac{1}{a}log(1+ax) \quad (12)$$

The value of the variable X is 0 for $x = 0$ and $X > 0$ for $x > 0$.

In order to determine the analytical solution to the convection-diffusion equation using the Laplace integral transform, the variable coefficients must be changed to constant coefficients. Incorporating Eq. (12) into Eqs. (8-11) yields:

$$\frac{R_0}{F(t)}\frac{\partial c}{\partial t} = D_0 \frac{\partial^2 c}{\partial X^2} - U_1 \frac{\partial c}{\partial X} - \gamma_1 c + \mu_0 \quad (13)$$

where,

$$U_1 = U_0 - D_0 a\ ,\ \ \gamma_1 = \gamma_0 + U_0 a\ ,$$
$$c(X,t) = c_i\ ;\qquad X \geq 0, t = 0 \quad (14)$$

$$c(X,t) = c_0 G(t);\quad X = 0, t > 0 \quad (15)$$

$$\frac{\partial c(X,t)}{\partial X} = 0; X \to \infty, t \geq 0 \quad (16)$$

We introduce the new time variable T with the help of the following transformation Eq.(17) as follows (Crank, 1975) [20]:

$$T = \int_0^t F(t)dt = \int_0^{t_1} f_1(m_1 t)dt + \int_{t_1}^{t_2} f_2(m_2 t)dt + \int_{t_2}^{t} f_3(m_3 t)dt \quad (17)$$

It is obvious that $T = 0$ for $t = 0$ and $T > 0$ for $t > 0$.

Use the transformation given in Eq. (17), Eqs. (13-16) simplifies to the following form:

$$R_0 \frac{\partial c}{\partial T} = D_0 \frac{\partial^2 c}{\partial X^2} - U_1 \frac{\partial c}{\partial X} - \gamma_1 c + \mu_0 \quad (18)$$

$$c(X,T) = c_i\ ; T = 0, X \geq 0 \quad (19)$$

$$c(X,T) = c_0 m^* T;\quad X = 0, T > 0 \quad (20)$$

$$\frac{\partial c(X,T)}{\partial X} = 0; X \to \infty, T \geq 0 \quad (21)$$

We employ another transformation as follows to remove the convective term from the Eq. (18):

$$c(X,T) = k(X,T)\exp\left[\frac{U_1}{2D_0}X - \frac{1}{R_0}\left(\frac{U_1^2}{4D_0} + \gamma_1\right)T\right] + \frac{\mu_0}{\gamma_1} \quad (22)$$

Where, $k(X,T)$ is the dependent variable that depends on the position and time variables.

Eqs.(18-21) are converted to the following form by applying the transformation in Eq. (22), as :

$$R_0 \frac{\partial k}{\partial T} = D_0 \frac{\partial^2 k}{\partial X^2} \quad (23)$$

$$k(X,T) = \left(c_i - \frac{\mu_0}{\gamma_1}\right) exp(-\beta X); X \geq 0, T = 0 \quad (24)$$

$$k(X,T) = \left(c_0 m^* T - \frac{\mu_0}{\gamma_1}\right) exp(\eta^2 T)\ ;\ X = 0, T > 0 \quad (25)$$

$$\frac{\partial k(X,T)}{\partial X} + \beta k(X,T) = 0; X \to \infty, T > 0 \quad (26)$$

where, $\eta^2 = \frac{1}{R_0}\left(\frac{U_1^2}{4D_0} + \gamma_1\right)$, $\beta = \frac{U_1}{2D_0}$.

We solve this boundary value problem in the Laplace domain by applying the Laplace Integral Transformation Technique (LITT) to the Eqs. (23-26) as follows:

$$\bar{k}(X,p) = \frac{c_0 m^*}{(p-\eta^2)^2} exp\left(-X\sqrt{\frac{pR_0}{D_0}}\right) - \frac{\mu_0}{\gamma_1}\frac{1}{(p-\eta^2)} exp\left(-X\sqrt{\frac{pR_0}{D_0}}\right) - \left(c_i - \frac{\mu_0}{\gamma_1}\right)\frac{1}{(p-\rho^2)} exp\left(-X\sqrt{\frac{pR_0}{D_0}}\right) + \left(c_i - \frac{\mu_0}{\gamma_1}\right)\frac{exp(-\beta X)}{(p-\rho^2)} \quad (27)$$

Where

$$\rho = \frac{U_1}{2\sqrt{R_0 D_0}}.$$

The analytic solution to the advection dispersion equation, applying an Inverse Laplace Integral to the Eq. (27) and apply the transformation Eq. (22) in reverse order, we have:

$$c(X,T) = \left\{\begin{matrix} m^* c_0 J_1(X,T) - \frac{\mu_0}{\gamma_1} H_1(X,T) - \left(c_i - \frac{\mu_0}{\gamma_1}\right) I_1(X,T) + \left(c_i - \frac{\mu_0}{\gamma_1}\right) \\ exp(-\beta X + \rho^2 T) \end{matrix}\right\} exp\left\{\frac{U_1}{2D_0} - \frac{1}{R_0}\left(\frac{U_1^2}{4D_0} + \gamma_1\right)\right\} + \frac{\mu_0}{\gamma_1} \quad (28)$$

where,

$$J_1(X,T) = \frac{1}{4\eta}\left(2\eta T - \sqrt{\frac{R_0}{D_0}}X\right) exp\left(\eta^2 T - \eta\sqrt{\frac{R_0}{D_0}}X\right) erfc\left(\frac{\sqrt{R_0}}{2\sqrt{D_0 T}}X - \eta\sqrt{T}\right) + \frac{1}{4\eta}\left(2\eta T + \sqrt{\frac{R_0}{D_0}}X\right) exp\left(\eta^2 T + \eta\sqrt{\frac{R_0}{D_0}}X\right) erfc\left(\frac{\sqrt{R_0}}{2\sqrt{D_0 T}}X + \eta\sqrt{T}\right)$$

$$H_1(X,T) = \frac{1}{2}\left\{\begin{matrix} \exp\left(\eta^2 T - \eta\sqrt{\frac{R_0}{D_0}}X\right) erfc\left(\frac{\sqrt{R_0}}{2\sqrt{D_0 T}}X - \eta\sqrt{T}\right) + \\ \exp\left(\eta^2 T + \eta\sqrt{\frac{R_0}{D_0}}X\right) erfc\left(\frac{\sqrt{R_0}}{2\sqrt{D_0 T}}X + \eta\sqrt{T}\right) \end{matrix}\right\}$$

$I_1(X,T)$ is obtained by replacing η with ρ in $H_1(X,T)$.

Analytical solution of the problem for varying input point source

In this case, we also consider a one-dimensional transport problem with decay and source terms in a semi-infinite heterogeneous porous domain. The varying nature of the input point source is described by Eq. (29), applied along the flow to the analytical solution of the advection dispersion equation.

In the first scenario, the concentration of the input at the origin $(x = 0)$ of the medium always remains the same, which may not be possible in the real world. Illegal human activities on the surface of the earth may lead to an increase/decrease in pollution levels at the source. Such condition is mathematically expressed as mixed-type condition Eq. (29).

$$-D(x,t)\frac{\partial c(x,t)}{\partial x} + U(x,t)c = U(x,t)c_0 G(t)\ ; \quad x = 0, t > 0 \quad (29)$$

Substituting the expression of Eq. (2) into Eq. (29), we have:

$$-D_0\frac{\partial c(x,t)}{\partial x} + U_0 c = U_0 c_0 G(t); \quad x = 0,\ t > 0 \quad (30)$$

Incorporating the transformation of Eq. (12), Eq. (30) reduced as follows:

$$-D_0\frac{\partial c(X,t)}{\partial X} + U_0 c = U_0 c_0 G(t); \quad X = 0,\ t > 0 \quad (31)$$

Incorporating Eq. (17), Eq. (31) can be simplified as follows:

$$-D_0\frac{\partial c(X,T)}{\partial X} + U_0 c = U_0 c_0 m^* T\ ;\ X = 0, T > 0 \quad (32)$$

Using transformation given in Eq. (22), the Eq. (32) reduces into following form;

$$-D_0\frac{\partial k(X,T)}{\partial X} + \left(U_0 - \frac{U_1}{2}\right)k = U_0\left(c_0 m^* T - \frac{\mu_0}{\gamma_1}\right) exp(\eta^2 T)\ ;\ X = 0, T > 0 \quad (33)$$

where, $\eta^2 = \frac{1}{R_0}\left(\frac{U_1{}^2}{4D_0} + \gamma_1\right)$

Apply the Laplace Integral Transformation technique in Eq.(33) using Eqs.(23,24, 26) as in previous case, obtained following result:

$$\bar{k}(X,p) = \frac{U_0 c_0 m^*}{\sqrt{R_0 D_0}} \frac{1}{(p-\eta^2)^2(\sqrt{p}+\sigma)} exp\left(-X\sqrt{\frac{pR_0}{D_0}}\right) - \frac{U_0\mu_0}{\gamma_1\sqrt{R_0 D_0}} \frac{1}{(p-\eta^2)(\sqrt{p}+\sigma)} exp\left(-X\sqrt{\frac{pR_0}{D_0}}\right)$$
$$- \frac{U_0}{\sqrt{R_0 D_0}}\left(c_i - \frac{\mu_0}{\gamma_1}\right)\frac{1}{(p-\rho^2)(\sqrt{p}+\sigma)} exp\left(-X\sqrt{\frac{pR_0}{D_0}}\right) + \left(c_i - \frac{\mu_0}{\gamma_1}\right)\frac{exp(-\beta X)}{(p-\rho^2)} \quad (34)$$

where, $\rho = \frac{U_0 - aD_0}{2\sqrt{R_0 D_0}}, \sigma = \frac{U_0 + aD_0}{2\sqrt{R_0 D_0}}$.

Taking the Inverse Laplace Integral Transform into Eq. (34) and use the transformation Eq. (22), we obtain the desired solution of the advection-diffusion equation as follows:

$$C(x,T) = \left\{\begin{matrix} \frac{m^* c_0 U_0}{\sqrt{R_0 D_0}} I(X,T) + \frac{U_0\mu_0}{\gamma_1\sqrt{R_0 D_0}} J(X,T) - \frac{U_0}{\sqrt{R_0 D_0}}\left(c_i - \frac{\mu_0}{\gamma_1}\right) H(X,T) + \left(c_i - \frac{\mu_0}{\gamma_1}\right) \\ exp(\beta X - \rho^2 T) \end{matrix}\right\}$$
$$exp\left(\frac{U_1}{2D_0}X - \frac{1}{R_0}\left(\frac{U_1^2}{4D_0}\right)T\right) + \frac{\mu_0}{\gamma_1} \quad (35)$$

where

$$I(X,T) = c_1\left\{\frac{1}{\sqrt{\pi T}} exp\left(-\frac{R_0 X^2}{4D_0 T}\right) + \eta\, exp\left(\eta^2 T - \eta\sqrt{\frac{R_0}{D_0}}X\right) erfc\left(\frac{1}{2}\sqrt{\frac{R_0}{D_0}}X - \eta\sqrt{T}\right)\right\}$$
$$+ c_2\left\{\begin{matrix}\left(1 - \eta\sqrt{\frac{R_0}{D_0}}X + 2\eta^2 T\right) exp\left(\eta^2 T - \eta\sqrt{\frac{R_0}{D_0}}X\right) \\ erfc\left(\frac{1}{2}\sqrt{\frac{R_0}{D_0}}X - \eta\sqrt{T}\right) + 2\eta T\left\{\frac{1}{\sqrt{\pi T}} exp\left(-\frac{R_0 X^2}{4D_0 T}\right)\right\}\end{matrix}\right\}$$
$$+ c_3\left\{\frac{1}{\sqrt{\pi T}} exp\left(-\frac{R_0 X^2}{4D_0 T}\right) - \eta\, exp\left(\eta^2 T + \eta\sqrt{\frac{R_0}{D_0}}X\right) erfc\left(\frac{1}{2}\sqrt{\frac{R_0}{D_0}}X + \eta\sqrt{T}\right)\right\}$$

$$c_4\left\{\begin{matrix}\left(1 + \eta\sqrt{\frac{R_0}{D_0}}X + 2\eta^2 T\right) exp\left(\eta^2 T + \eta\sqrt{\frac{R_0}{D_0}}X\right) erfc\left(\frac{1}{2}\sqrt{\frac{R_0}{D_0}}X + \eta\sqrt{T}\right) \\ -2\eta T\left\{\frac{1}{\sqrt{\pi T}} exp\left(-\frac{R_0 X^2}{4D_0 T}\right)\right\}\end{matrix}\right\}$$
$$+ c_5\left\{\frac{1}{\sqrt{\pi T}} exp\left(-\frac{R_0 X^2}{4D_0 T}\right) - \rho\, exp\left(\rho^2 T + \rho\sqrt{\frac{R_0}{D_0}}X\right) erfc\left(\frac{1}{2}\sqrt{\frac{R_0}{D_0}}X + \rho\sqrt{T}\right)\right\}$$

$$J(X,T) = \frac{1}{2(\eta+\sigma)} \exp\left(\eta^2 T - \eta\sqrt{\frac{R_0}{D_0}}X\right) erfc\left(\frac{\sqrt{R_0}}{2\sqrt{D_0 T}}X - \eta\sqrt{T}\right) - \frac{1}{2(\eta-\sigma)} \exp\left(\eta^2 T + \eta\sqrt{\frac{R_0}{D_0}}X\right)$$
$$erfc\left(\frac{\sqrt{R_0}}{2\sqrt{D_0 T}}X + \eta\sqrt{T}\right) + \frac{\eta}{\eta^2 - \sigma^2} \exp\left(\sigma^2 T + \sigma\sqrt{\frac{R_0}{D_0}}X\right) \times erfc\left\{\frac{\sqrt{R_0}}{2\sqrt{D_0 T}}X + \sigma\sqrt{T}\right\},$$

Denklemi buraya yazın.

$c_1 = -\frac{2\eta+\sigma}{4\eta^3(\eta+\sigma)^2}, c_2 = \frac{1}{4\eta^2(\eta+\sigma)},$ $c_3 = -\frac{2\eta-\sigma}{4\eta^3(\eta-\sigma)^2}, c_4 = \frac{1}{4\eta^2(\sigma-\eta)},$

$c_5 = \frac{1}{(\sigma-\eta)^2(\sigma+\eta)^2}$,

$$X = \frac{1}{a}\log(1+ax), T = \int_0^t F(t)dt,$$

The value of $H(X,T)$ is obtained by replacing η with ρ in $J(X,T)$

Result and Discussion

Analytical solutions are developed for the advection-diffusion equation (ADE) under the hypothetical scenario of time-dependent groundwater flow in a 1D heterogeneous porous formulation. Point source is considered to enter the domain from the left end $x = 0$. Separate graphs are drawn to show the effect of each parameter on the variation in solute concentration. For a set of input hydrological data, the obtained analytical solutions Eq. (28) and Eq. (35) for uniform and varying input point sources are shown with various graphs. The values of the parameters and empirical constants involved in the boundary conditions and the governing equations are taken from published papers, e.g. (Kumar et al., 2010)[21]. The parameters affecting the concentration distribution are illustrated with the help of various graphs.Contaminant concentration values are evaluated for longitudinal porous domains $0 \le x(km) \le 25$. The values of common input parameters and constants are assumed as follows:

$c_0(\text{kg/km}^3) = 1.0, c_i(\text{kg/km}^3) = 0.1, U_0(\text{km/year}) = 0.65, \quad \mu_0(\text{kg/km}^3 year^{-1}) = 0.11, \gamma_0(year^{-1}) = 0.11.$

The mathematical expression for a time-dependent function $F(t)$ that varies in each time domain is assumed to be as follows:

$$F(t) = \begin{cases} exp\ \ (m_1 t)\ ; 0 \le t < t_1 \\ s_1\ exp(m_2(t-t_1)) + s_2\ ; t_1 \le t < t_2 \\ s_3 \dfrac{m_3(t-t_2)}{m_3(t-t_2)+1} + s_4 ; t_2 \le t < \infty \end{cases}$$

The values of the unsteady parameters m^*, m_1, m_2 and m_3 are $0.4(year^{-1})$, $0.1(year^{-1})$, $0.3\ (year^{-1})$ and $0.4\ (year^{-1})$, respectively. The values of time t_1 and t_2 are taken $1(year)$ and $5(year)$, respectively and the value of $s_1 = 0.36839, s_2 = 0.736781\ s_3 = 0.917324$ and $s_4 = 1.9598795012568466$, respectively.

Uniform input solution

Figs. (1-4) demonstrate the concentration pattern of a uniform input point source for a heterogeneous porous medium described by the analytical solution Eq. (28).

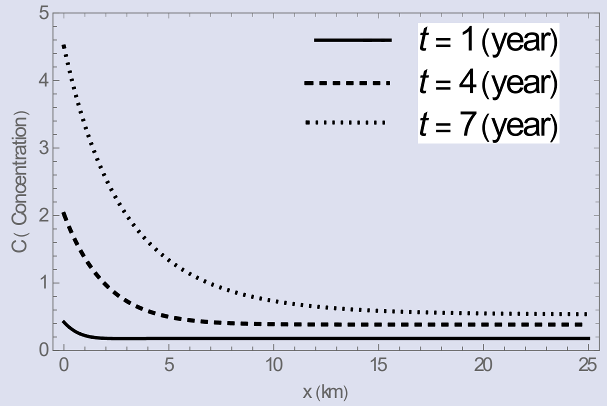

Figure 1. Contaminant concentration profile obtained in Eq. (28) for different time t(year)=1,4,7

Fig.(1) demonstrates dimensionless concentration distributions for different time $t(year) = 1,4,7$ and given groundwater velocity $U_0(kmyear^{-1}) = 0.65$, dispersion coefficients $D_0(km^2 year^{-1}) = 0.67$ and retardation factor $R = 1.15$ while other parameters are fixed. It exemplifies the pattern of concentration dispersion caused by a constant uniform point source. Concentration levels were found to be higher for longer time, lower for shorter time, and then stabilize as moved away from the source boundary. The concentration value appears different at different times at the origin, decreases rapidly as the position increases, and tends to be stable at the other end.

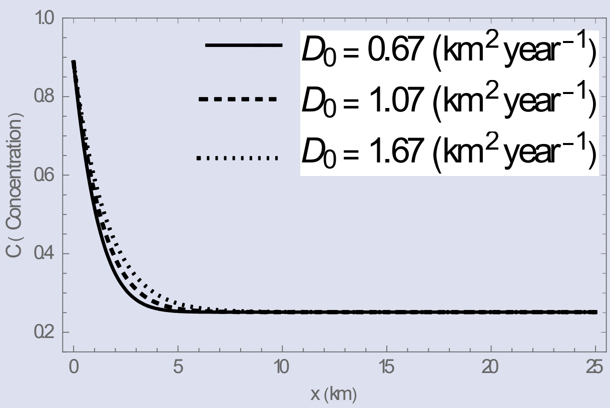

Figure 2. Contaminant concentration profile obtained in Eq. (28) for different dispersion $D_0(km^2 year^{-1}) = 0.67,1.07,1.67$

Fig.(2) shows the effect of various dispersion coefficients $D_0(km^2 year^{-1}) = 0.67,1.07,1.67$ on the concentration distribution for a specific period of time $t(year) = 2$ and retardation factor $R = 1.15$ while other parameters remain unchanged. It can be seen that the concentration pattern near the entry point becomes more pronounced as the dispersion coefficient increases. Although the concentration values for all dispersion coefficients at the entry point $(x = 0)$ are the same. The concentration level decreases with space from a starting point ($x = 0$) and becomes constant towards the outlet boundary.

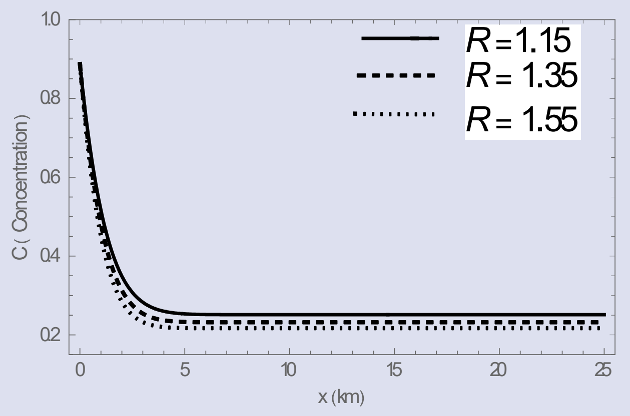

Figure 3. Contaminant concentration profile obtained in Eq. (28) for different retardation $R = 1.15, 1.35, 1.55$

Fig.(3) depicts the effect of several retardation factor $R = 1.15, 1.35, 1.55$for dispersion coefficient $D_0(km^2 year^{-1}) = 0.67$, on pollutant concentrations at time $t(year) = 2$.For different retardation factors, the concentration values at the entry point are same, but fluctuations in the concentration distribution can be observed near the boundary point and towards the other boundary in the steady state.

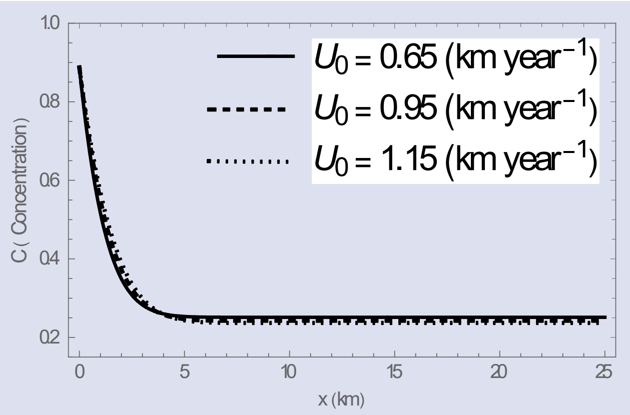

Figure 4. Contaminant concentration profile obtained in Eq. (28) for different velocity $U_0(kmyear^{-1}) = 0.65, 0.95, 1.15$

Fig. (4) shows the effect of variation of different ground water velocity $U_0(kmyear^{-1}) = 0.65, 0.95, 1.05$at common values of time $t(year) = 2$, retardation factor $R = 1.15$ and dispersion coefficient $D_0(km^2 year^{-1}) = 0.67$, keeping the other parameters constant. It can be observed that the concentration pattern is higher for higher ground water velocity, the variation in the concentration distribution profile is quite significant the near inlet boundary. Near the middle position from $x = 1.5$ to $x = 2.4$, the greatest change can be seen.

Varying input solution

Figs. (5–8) display the solute concentration patterns for varying input point sources in the heterogeneous porous medium described by Eq. (35).

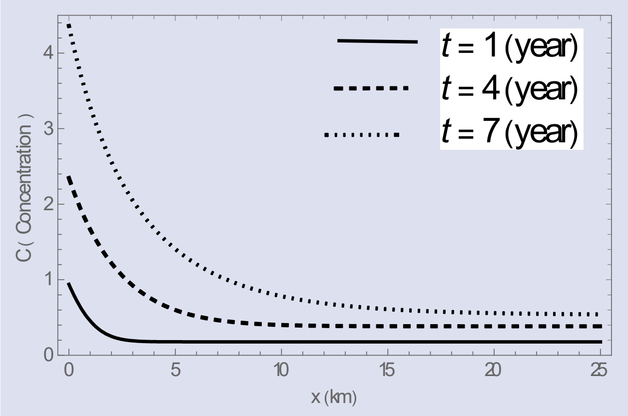

Figure 5. Contaminant concentration profile obtained in Eq. (35) for different time $t(year) = 1, 4, 7$

Fig.(5) illustrates the dimensionless concentration distribution for different time $t(year) = 1, 4, 7$and for fix retardation factor $R = 1.15$, ground water velocity $U_0(km\ year^{-1}) = 0.65$ and dispersion coefficient $D_0(km^2 year^{-1}) = 0.67$. It reveals the concentration distribution pattern produced by a uniform continuous point source. Concentration levels near to the source boundary are observed to be higher for higher time and lower for lower time, and stabilize after a distance away from the origin.

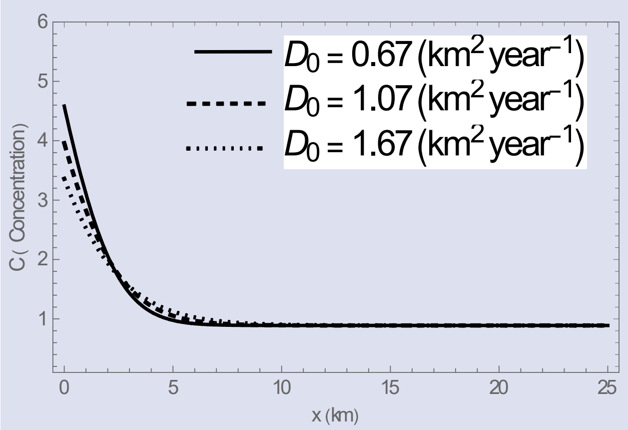

Figure 6. Contaminant concentration profile obtained in Eq. (35) for different dispersion $D_0(km^2 year^{-1}) = 0.67, 1.07, 1.67$

Fig.6. shows the pollutant concentration with various dispersion coefficients$D_0(km^2 year^{-1}) = 0.67, 1.07, 1.67$ at time $t(year) = 2$, retardation factor $R = 1.15$and ground water velocity $U_0(kmyear^{-1}) = 0.65$, keeping the other parameters fixed. The variation in solute concentration caused by various diffusion coefficients in both space and time are shown in this figure. It can be observed that at particular position the concentration level is higher for lower dispersion coefficient near the source boundary but after some distance traveled from the origin order gets reversed. This is due to the fact that dispersion phenomenon causes the spreading of the plume in the medium.

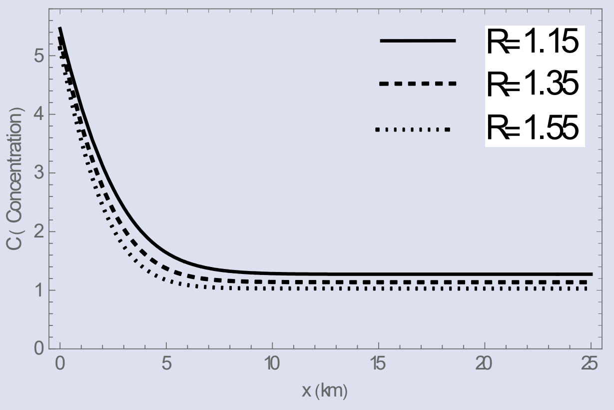

Figure 7. Contaminant concentration profile obtained in Eq. (35) for different retardation $R = 1.15, 1.35, 1.55$

Fig. (7) shows the pollutant concentration with various retardation factors $R = 1.15, 1.35, 1.55$ on the concentration distribution for dispersion coefficient at $D_0(km^2 year^{-1}) = 0.67$, time $t(year) = 2$, ground water velocity $U_0(kmyear^{-1}) = 0.65$, and keeping all other parameters are fixed. For each location on a certain domain, the pollution concentration decreases as the retardation factor increases. The solute concentration is directly affected by the retardation factor (R) during solute transport. It is clear that as the retardation factor increases, the concentration level decreases, and near the inlet boundary, the concentration profile changes significantly. At particular point, the lower the retardation factor, the higher the concentration level.

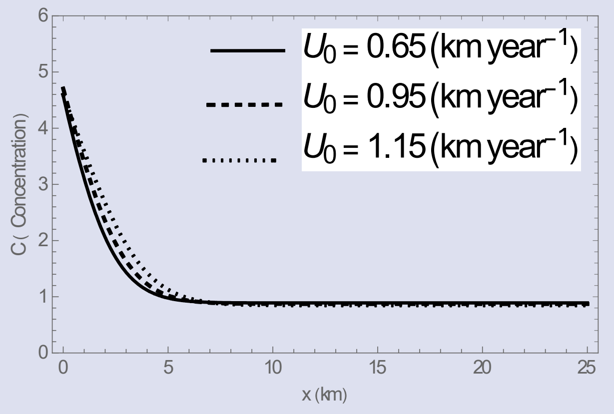

Figure 8. Contaminant concentration profile obtained in Eq. (35) for different velocity $U_0(kmyear^{-1}) = 0.65, 0.95, 1.15$

Fig.(8) shows the effect of groundwater flow velocity $U_0(year^{-1}) = 0.65, 0.95, 1.05$on the concentration profile for a fixed retardation factor$R = 1.15$ and dispersion coefficient$D_0(km^2 year^{-1}) = 0.67$ at time $t(year) = 2$ and keeping all other parameters fixed. The concentration distribution fluctuates more near the inlet boundary, and it can also be observed that the higher the groundwater velocity, the steeper the concentration distribution.

Special Case I. If we take $c_i = 0, \gamma = 0, \mu = 0, R = 1, F(t) = 1, G(t) = 1$ in our model for uniform input point source then obtained analytical solution of the problem for uniform input point source is similar to one obtained by Kumar et al. (2010) [21]and given by;

$$c(X,t) = \begin{Bmatrix} c_0 Q_1(X,t) \\ exp(-\beta X + \rho^2 t) \end{Bmatrix} exp\left\{\frac{U_1}{2D_0}X - \left(\frac{U_1^2}{4D_0}\right)t\right\} \quad (36)$$

where,

$$Q_1(X,t) = \frac{1}{2}\begin{Bmatrix} exp\left(\eta^2 t - \eta\sqrt{\frac{R_0}{D_0}}X\right) erfc\left(\frac{\sqrt{R_0}}{2\sqrt{D_0 T}}X - \eta\sqrt{t}\right) + \\ exp\left(\eta^2 t + \eta\sqrt{\frac{R_0}{D_0}}X\right) erfc\left(\frac{\sqrt{R_0}}{2\sqrt{D_0 T}}X + \eta\sqrt{t}\right) \end{Bmatrix}$$

Special Case II. If we take $c_i = 0, \gamma = 0, \mu = 0, R = 1, F(t) = 1, G(t) = 1$ in our model for varying input point source then obtained analytical solution of the problem for varying input point source is similar to one obtained by Jaiswal et al. (2011) [22]and given by;

$$C(x,T) = \left\{\frac{c_0 U_0}{\sqrt{D_0}} N(X,t)\right\} \exp\left(\frac{U_1}{2D_0}X - \left(\frac{U_1^2}{4D_0}\right)t\right) \quad (37)$$

where,

$$N(X,t) = \frac{1}{2(\eta+\sigma)} exp\left(\eta^2 t - \eta\sqrt{\frac{1}{D_0}}X\right) erfc\left(\frac{\sqrt{1}}{2\sqrt{D_0 T}}X - \eta\sqrt{t}\right) - \frac{1}{2(\eta-\sigma)} exp\left(\eta^2 t + \eta\sqrt{\frac{1}{D_0}}X\right) erfc\left(\frac{\sqrt{1}}{2\sqrt{D_0 t}}X + \eta\sqrt{t}\right) + \frac{\eta}{\eta^2 - \sigma^2} exp\left(\sigma^2 t + \sigma\sqrt{\frac{1}{D_0}}X\right) \times erfc\left\{\frac{\sqrt{1}}{2\sqrt{D_0 t}}X + \sigma\sqrt{t}\right\},$$

Conclusions

With solute degrading at varying rate, an analytical solution of this kind vividly shows concentration details at various time domains. Such solutions are important for assessing contamination levels in a variety of downstream conditions away from the source of ingestion. From the obtained solution, we can estimate the variation of the concentration with time and space. The fact that the values of groundwater velocity and diffusivity vary over time intervals makes the results more realistic. The obtained results show that temporally dependent input sources have a significant impact on pollutant transport. The two key parameters, diffusion coefficient and retardation factor, have a great influence on the change of solute concentration. Aquifers are generally heterogeneous in nature. Current mathematical models can predict the concentration of pollutants in the medium, which can help to take remedial measures. The developed mathematical model can be considered as an effective tool for understanding the transport behavior of pollutants in surface water and groundwater phenomena. Changes in dispersion and velocity have an impact on the levels of pollutant concentration in the aquifer system. From the obtained solution we can estimate the variation of the concentration with time and space. The results of this study can be used to enhance the planning and management of water pollution problems, as well as the many physical conditions defined by traffic phenomena.

Conflicts of interest

There are no conflicts of interest in this work.

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