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Investigation of Solution Behavior By Sumudu Methods of Random Complex Partial Differential Equations

Mehmet Merdan [1,a,*], Merve Merdan [1,b], Rıdvan Şahin [1,c]

[1] *Department of Mathematical Engineering, Gümüşhane University, Gümüşhane, Türkiye.*
*Corresponding author

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ABSTRACT

In this study, solutions of random complex partial differential equations were found using the two-dimensional Sumudu transformation method(STM). The initial conditions of a deterministic equation or the non-homogeneous part of the equation are transformed into random variables to obtain a random complex partial differential equation. With the help of the properties of two-dimensional Sumudu and inverse Sumudu transformation, an approximate analytical solution of a complex partial differential equation with random constant coefficients was obtained by selecting a random variable with an initial condition of Normal and Gamma distribution. The probability characteristics of the resulting solutions, such as expected value and variance, were obtained and graphically shown with the help of the Maple package program.

Keywords: Two-dimensional Sumudu Transform and Inverse Sumudu Transformation, Complex partial differential equation, Expected value, Variance and confidence interval.

[a] mmerdan@gumushane.edu.tr https://orcid.org/0000-0002-8509-3044
[b] mervemerdan94@hotmail.com https://orcid.org/0000-0002-6045-6531
[c] ridvansahin@gumushane.edu.tr https://orcid.org/0000-0001-7434-4269

Introduction

Complex partial differential equations first were discovered in the early 1900. D. Pompeiu is one of the important mathematicians who made substantial contributions to the field and left a name. He described the Pompeiu integral operator, which carries his name. It still serves as the foundation for the theory of complex differential equations today. Integral transforms, the Adomian decomposition approach, and the reduced differential transform were used to solve complex differential equations [1-4]. These approaches divide the equation into real and imaginer components before solving it. The Sumudu differential transformation method can be utilized in this article to resolve the problem without dividing it into real and imaginer components. Because of this, a solution can be developed with minimal effort. Watugala introduced the Sumudu transform with the article [5] in 1993. It is a transformation that is crucial in the solution of several ordinary differential equations in control engineering. The Sumudu transform's characteristics and uses are described[5-6]. The Laplace transform and its characteristics are also taken into account, and the theory that establishes the relationship between Sumudu and Laplace transformations is presented. Sumudu transformation is used in this study to solve partial differential equations more quickly. The fundamental details for integral transformations, as well as the fundamental properties of the Laplace and Sumudu transformations, are provided in the study's first sections. In the following chapters, the solution of the partial equation was obtained by applying the Sumudu transform to appropriate ordinary complex differential equations and partial differential equations.

Watugala's Sumudu transform method (STM) was used to address engineering issues [6]. Weerakoon used the technique to solve partial differential equations [7]. Subsequently, Weerakoon discovered this transformation's inverse formula [8]. The Sumudu transformation technique (STM) was employed by Demiray et al. [9] to discover precise answers to fractional differential equations. The Sumudu transform iterative method (STIM) was expanded by Kumar and Daftardar-Gejji [10] to handle various time and spatial FPDEs and FPDE systems. To solve linear fuzzy fractional differential equations (FFDEs) using Caputo fuzzy fractional derivative, Rahman and Ahmad used the fuzzy Sumudu transform (FST) [11]. Prakash et al. [12] solved nonlinear fractional Zakharov–Kuznetsov equations with the help of the Sumudu transform method, which is a new iterative technique.

Given the wealth of literature on stochastic and deterministic differential equation models, the application of random differential equations (RDE) in mathematical models is not as common. By including random effect terms in the model's parameters, deterministic models can be changed to random models. Because random parameters offer the chance to account for parameter changes, this method enables more realistic modeling of physical processes. The probability properties of the equations that are randomized in the deterministic model by choosing the coefficients on the second side of the equation or the initial conditions from continuous probability distributions will be investigated. Ordinary differential equations in mathematical modeling are not necessarily adequate for investigating natural processes. Using random and stochastic differential equations to investigate occurrences with random components yields superior findings. Using ordinary differential equations, random differential equations can be produced in three methods[13]. ii. Differential equations with random non-homogeneous sections iii. Equations using random coefficients. In this study, we considered complex partial differential equations with inhomogeneous parts and initial conditions with random values. Many physical and engineering problems can be modeled more reliably with random differential equations. In recent decades, there has been a significant amount of research in the disciplines of parameter uncertainty and randomness, as well as stochastic differential processes.

In this study, the two-dimensional Sumudu transform method is used to solve linear complex partial differential equations. The structure of the essay is as follows. Section 2 provides information on the one dimensional and two-dimensional Sumudu transformation and its attributes. Examples of the random complex differential equation are provided in Section 3. The solutions' probability features were computed and graphically displayed. Section 4 provides the conclusions.

Materials and Methods

Sumudu Transform Method

Definition 2.1. Let A be a function set defined[14] by

$$A = \{g\,(t)|\, \exists M, \tau_1, \tau_2 > 0, |g(t)| < Me^{\frac{t}{\tau_j}}, \quad (1)$$
$$if\ t \in (-1)^j x\ [0, \infty\,)\}$$

the Sumudu transform is defined over the set of functions by

$$G(u) = S[g(t)] = \int_0^\infty g(ut)e^{-t}dt,\ u \in (-\tau_1, \tau_2) \quad (2)$$

Table 1. Special Sumudu transforms[14].

	$g(t)$	$G(u) = S[g(t)]$
1	1	1
2	t	u
3	$\sin(at)$	$\frac{au}{1+(au)^2}$
4	$\cos(at)$	$\frac{1}{1+(au)^2}$
5	$t^n, n > 0$	$\Gamma(n+1)u^n$
6	e^{at}	$\frac{1}{1-au}$
7	$e^{bt}\cos(at)$	$\frac{1-bu}{(1-bu)^2+(au)^2}$
8	$e^{bt}\sin(at)$	$\frac{au}{(1-bu)^2+(au)^2}$
9	$t^n e^{at}$	$\frac{\Gamma(n+1)u^n}{(1-au)^n}$

The Double Sumudu Transform

The Sumudu transform is a simple and somewhat elegant approach to implement the double Sumudu transform, assuming the function has a power series transformation with regard to its variables. The double Laplace transform of a function defined in the positive quadrant of the xy plane is:

$$\mathcal{L}_2[g(x,y);(r,s)] = \int_0^\infty \int_0^\infty g(x,y)e^{-(rx+sy)}dxdy \quad (3)$$

where p and q are the transformation variables of x and y, respectively.

Definition 1. [15-17] Let $g(t,x)$: $t, x \in \mathbb{R}^+$, If a function is written as a convergent infinite series, its double Sumudu transform is as follows:

$$G(u,v) = \mathbb{S}_2[g(t,x);(u,v)]$$
$$= \mathbb{S}[\mathbb{S}\{g(t,x);t \to u\};x \to v]$$
$$= \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{t}{u}+\frac{x}{v}\right)}g(t,x)dtdx \quad (4)$$

We propose applications of the double Sumudu transform to several specific functions, which are similar to those obtained by solving population dynamics equations with age structure. However, it is a simple exercise to demonstrate that the double Sumudu and Laplace transforms are also theoretical dual. That is:

$$uvG(u,v)$$
$$= \mathcal{L}_2\left[g(x,y);\left(\frac{1}{u},\frac{1}{v}\right)\right] \quad (5)$$

The relationship between the Sumudu Transform and double Laplace Transform can be expressed as in (5).

Theorem 1. [15-17] Let $G\,(x,y)$ be a real-valued function of $x, y\ \in \mathbb{R}^+$.

Then,

$$\mathbb{S}_2[g(x+y);(u,v)] = \frac{1}{u-v}\{uG(u) - vG(v)\}. \quad (6)$$

g represents population density, age x, and time y, or vice versa. The $(x - y)$ example is even more interesting from the point of view of biology, which is often encountered with these works in Mathematical Biology. The proof of the condition $x \geq y$ is also simple and sufficient. Thus, geometrically, although the line dividing the first quarter into two equal parts represents the η axis (represented by the lower part Q_1 and the upper part Q_2), dividing both the second and fourth quarters is the η -axis (the row facing up) and the ζ —axis (the arrow from the starting point to the fourth quarter), then the test is: Let's assume that g is a dual function, then $g\ (0)$ is odd

$$\mathbb{S}_2[g(x-y);(u,v)]=\frac{1}{uv}\int\int_{Q_1} g(x-y)e^{-\left(\frac{x}{u}+\frac{y}{v}\right)}dxdy-\frac{1}{uv}\int\int_{Q_2} g(x-y)e^{-\left(\frac{x}{u}+\frac{y}{v}\right)}dxdy. \tag{7}$$

$x=\frac{1}{2}(\zeta+\eta); y=\frac{1}{2}(\zeta-\eta)$

variable transformation

$$\int\int_{Q_1} g(x-y)e^{-\left(\frac{x}{u}+\frac{y}{v}\right)}dxdy=\frac{1}{2}\int_0^\infty g(\zeta)d\zeta\int_\zeta^\infty e^{-\frac{1}{2}\left(\frac{1}{u}+\frac{1}{v}\right)\zeta-\frac{1}{2}\left(\frac{1}{u}-\frac{1}{v}\right)\eta}d\eta$$

$$=\frac{uv}{u-v}\int_0^\infty e^{-\frac{\zeta}{v}}g(\zeta)d\zeta$$

$$=\frac{uv^2}{u-v}G(u).$$

similarly,

$$\int\int_{Q_1} g(x-y)e^{-\left(\frac{x}{u}+\frac{y}{v}\right)}dxdy=\frac{v^2u}{u-v}G(v).$$

Therefore, for the odd function g

$$\mathbb{S}_2[g(x-y);(u,v)]=\frac{uG(u)-vG(v)}{u+v} \tag{8}$$

and if g is a even function

$$\mathbb{S}_2[g(x-y);(u,v)]=\frac{uG(u)+vG(v)}{u-v} \tag{9}$$

$G(u)=\mathbb{S}[g(t)]=\int_0^\infty g(ut)e^{-t}dt,\ \ u\epsilon(-\tau_1,\tau_2)$ Sumudu conversion

From the equations $G\ (u)$ and (8), if g is an even function, then it is obvious that

$$(u+v)\mathbb{S}_2[g(x-y)]=(u-v)\mathbb{S}_2[g(x+y)] \tag{10}$$

are obtained. If STM is applied to partial derivatives as follows: $g(0,a)=\ G_0(a)$,

$$\mathbb{S}_2\left[\frac{\partial g(t,a)}{\partial t};(u,v)\right]=\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{t}{u}+\frac{s}{v}\right)}\frac{\partial}{\partial t}g(t,a)dtda=\frac{1}{v}\int_0^\infty e^{-\frac{s}{v}}\left\{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\frac{\partial}{\partial t}g(t,a)dt\right\}da.$$

The inner integral given in the equation (5),

$$\frac{G(u,a)-g(0,a)}{u} \tag{11}$$

$$\mathbb{S}_2\left[\frac{\partial g(t,a)}{\partial t};(u,v)\right]=\frac{1}{u}\left\{\frac{1}{v}\int_0^\infty e^{-\frac{a}{v}}G(u,a)da-\frac{1}{v}\int_0^\infty e^{-\frac{a}{v}}g_0(a)da\right\}$$

$$=\frac{1}{u}\{G(u,v)-G_0(v)\} \tag{12}$$

Also,

$$\mathbb{S}_2\left[\frac{\partial g(t,a)}{\partial a};(u,v)\right]=\frac{1}{v}\int_0^\infty e^{-\frac{a}{v}}\left\{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\frac{\partial}{\partial a}g(t,a)dt\right\}da$$

$$=\frac{1}{v}\int_0^\infty e^{-\frac{a}{v}}\frac{\partial}{\partial a}G(u,a)da$$

$$=G_u(u,v). \tag{13}$$

Alternatively,

$$\mathbb{S}_2\left[\frac{\partial g(t,a)}{\partial t};(u,v)\right] = \frac{1}{u}\int_0^\infty e^{-\frac{a}{v}}\left\{\frac{1}{v}\int_0^\infty e^{-\frac{a}{u}}\frac{\partial g}{\partial a}da\right\}dt$$

$$= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\frac{1}{v}[G(t,v)-g(t,0)]dt$$

$$= \frac{1}{v}(G(u,v)-G_0(u)) \tag{14}$$

Where $G\,(u,0) = G_0(u)$ and $G(0,v) = G_0(v)$. From the equations (12) and (13)

$$G_v(u,v) = \frac{G(u,v)-G_0}{v}$$

it is expressed as.

Definition 2.[15-17]: Let $G(t,X)$ and $H(T,X)$ functions have a two-dimensional Sumudu transformation. Then the two-dimensional Sumudu transformation of $g(t,x)$ and $h(t,x)$ of two-dimensional convolution,

$$(g ** h)(t,x) = \int_0^x\int_0^t g(\zeta,\eta)h(t-\zeta,x-\eta)d\zeta d\eta$$

$$\mathbb{S}_2[(g ** h)(t,x);u,v] = uvG(u,v)H(u,v)$$

Also, the two-dimensional Sumudu transformation of the partial derivative of the two-dimensional convolution with respect to x e was obtained below,

$$\mathbb{S}_2\left[\frac{\partial}{\partial x}(g ** h)(t,x);u,v\right] = uv\mathbb{S}_2\left[\frac{\partial}{\partial x}(t,x);u,v\right] = \mathbb{S}_2[h(t,x);u,v]$$

or

$$uv\mathbb{S}_2[f(t,x);u,v]\mathbb{S}_2\left[\frac{\partial}{\partial x}h(t,x);u,v\right].$$

Thus, the relationship between the Sumudu and Laplace Transform of the two-dimensional function,

$$\mathbb{S}_2[(g ** h)(t,x);u,v] = \frac{1}{uv}\mathcal{L}_t\mathcal{L}_x[g ** h)(t,x)]$$

to solve PDEs using the Sumudu transformation, partial derivatives of this transformation are needed. Thus, by applying the two-dimensional Sumudu transformation to its second-order partial derivatives with respect to x

$$\mathbb{S}_2\left[\frac{\partial^2}{\partial x^2}g(t,x);u,v\right] = \frac{1}{v^2}G(u,v) - \frac{1}{v^2}G(u,0) - \frac{1}{v}\frac{\partial}{\partial x}G(u,0)$$

expression is obtained. Similarly, given in the second-order partial derivative with respect to t;

$$\mathbb{S}_2\left[\frac{\partial^2}{\partial t^2}g(t,x);u,v\right] = \frac{1}{u^2}G(u,v) - \frac{1}{u^2}G(0,v) - \frac{1}{u}\frac{\partial}{\partial t}G(0,v).$$

Numerical Examples

Sumudu transformation method (STM) is used in this section to solve random complex partial differential equations. Examples of variances, confidence intervals, and estimated values for various probability distributions of these solutions are also provided. Complex differential equations with random effect terms and various probability distributions are included in each case. Mean-square computation has been used in the past to solve a few first-order random differential models and equations [18–23]. A stochastic differential equation (SDE) is a differential equation in which one or more elements are stochastic processes[24], and the solution is also stochastic. SDEs have numerous applications in pure mathematics, including modeling the behaviors of stochastic models such as stock markets [25], random growth models [26], and physical systems exposed to thermal fluctuations.

SDEs have a random differential, which in the most basic example is random white noise computed as the derivative of a Brownian motion, or more broadly, a semimartingale. Other sorts of random behavior are possible, such as hopping processes like Lévy processes[27] or hopping semimartingales. Random differential equations are equivalent to stochastic differential equations. Stochastic differential equations can also be expanded to differential manifolds[28-30].

Example 3.1.

$$\frac{\partial w}{\partial z}-\frac{\partial w}{\partial \bar{z}}-5w=0, \qquad w(x,0)=Be^{3x} \tag{15}$$

obtain the probability characteristics by solving the approximate analytical solution of the given partial differential equation $B\sim N(\mu,\sigma^2)$, independent random variables with a normal distribution, using the Sumudu method.
In equation (15)

$$\frac{\partial w}{\partial z}=\frac{1}{2}\left(\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y}\right)$$
$$\frac{\partial w}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}\right)$$

if their equality is written instead

$$\frac{1}{2}\left(\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}\right)-5w=0$$

are obtained. If $w=u+iv$ is written in the given equation,

$$\frac{1}{2}\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}-i\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\right]-\frac{1}{2}\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+i\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\right]-5(u+iv)=0$$
$$\frac{\partial v}{\partial y}-5u=0$$
$$-\frac{\partial u}{\partial y}-5v=0$$

if Sumudu transformation is applied to its equations, $S[u(x,y)]=R_1(x,s)$
$S[v(x,y)]=R_2(x,s)$
when,

$$\frac{1}{s}[R_2(x,s)-v(x,0)]-5R_1(x,s)=0$$
$$-\frac{1}{s}[R_1(x,s)-u(x,0)]-5R_2(x,s)=0$$

if the Cramer rule applies to the resulting Sumudu transformations,

$$-5R_1(x,s)+\frac{1}{s}R_2(x,s)=\frac{v(x,0)}{s}$$
$$-\frac{1}{s}R_1(x,s)-5R_2(x,s)=-\frac{u(x,0)}{s}.$$
$$\begin{vmatrix}-5 & \frac{1}{s}\\ \frac{-1}{s} & -5\end{vmatrix}=25+\frac{1}{s^2}=\Delta$$
$$R_1(x,y)=\frac{\begin{vmatrix}\frac{v(x,0)}{s} & \frac{1}{s}\\ -\frac{u(x,0)}{s} & -5\end{vmatrix}}{\Delta}=\frac{Be^{3x}}{1+(5s)^2}$$
$$R_2(x,y)=\frac{\begin{vmatrix}-5 & \frac{v(x,0)}{s}\\ \frac{-1}{s} & -\frac{u(x,0)}{s}\end{vmatrix}}{\Delta}=\frac{Be^{3x}5s}{(5s)^2+1}$$

taking the inverse Sumudu transformation into his equations,$u(x,y)=S^{-1}[R_1(x,s)]$

$$=S^{-1}\left[\frac{Be^{3x}}{1+(5s)^2}\right], \qquad \left\{S[cos(ay)]=\frac{1}{1+(as)^2}, S^{-1}\left[\frac{1}{1+(as)^2}\right]=cos(ay)\ \right\}$$
$$=Be^{3x}\cos(5y)$$

$$v(x,y) = S^{-1}[R_2(x,y)]$$
$$= S^{-1}\left[\frac{Be^{3x}5s}{(5s)^2+1}\right], \left\{S[\sin(ay)] = \frac{as}{1+(as)^2}, S^{-1}\left[\frac{as}{1+(as)^2}\right] = \sin(ay)\ \right\}$$
$$= Be^{3x}\sin(5y)$$

are obtained. Then

$$w(z) = Be^{3x}[cos(5y) + isin(5y)]$$
$$= Be^{4z-\bar{z}}$$

can be found. Let's try to find the probability characteristics of the solution we found. A random variable X is normally distributed ($X \sim N(\mu,\sigma^2)$) if its probability distribution function is $f(x) = \frac{1}{\sigma\sqrt{2\pi}}exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$.

Let $B \sim N(\mu,\sigma^2)$ be a random variable with a Normal distribution[32]. Moment generating function of Normal distribution,

$$M_x(t) = E[e^{tx}] = e^{\mu t+\frac{1}{2}\sigma^2 t^2}.$$

Expected value and variance of the first, second moments of the random variable $x \sim N(\mu,\sigma^2)$

$$E[x] = \mu, E[x^2] = \mu^2 + \sigma^2 \quad ve\ Var[x] = \sigma^2.$$

Using these moments and if x and y are random arguments $E[xy] = E[x]E[y]$ since, approximate formulas of expected value and variance

$$E[w(k,h)] = \sum_{k=0}^{n}\sum_{h=0}^{n} E[w(k,h)]x^k y^h$$

The expected value and variance of the solution we found above are, respectively,

$$E[w(z)] = E[B]e^{4z-\bar{z}}$$
$$= \mu e^{4z-\bar{z}}.$$

Variance

$$Var[w(z)] = Var[B]e^{8z-2\bar{z}}$$
$$= \sigma^2 e^{8z-2\bar{z}}.$$

Specifically, the expected value if $\mu = 2, \sigma^2 = 1$ is selected,

$$E[w(z)] = 2e^{4z-\bar{z}}$$

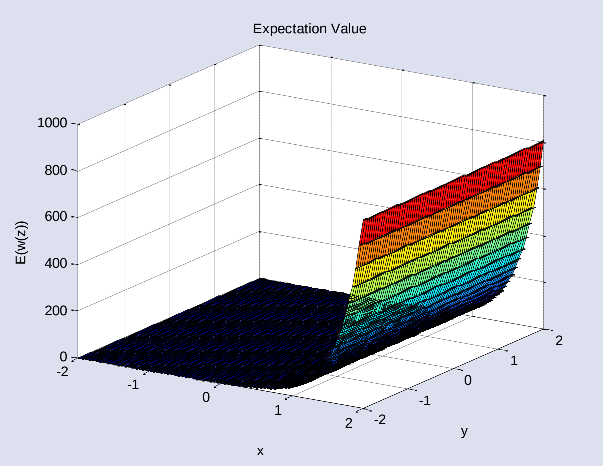

Figure 1 Expected value of equation (15) for values $\mu = 2, \sigma^2 = 1$

The expectations can be given in a single graph for a comparison with the deterministic results of equation (1) as above (Figure 1). Maximum and minimum values of expected values of the random variables are obtained as follows: $w(z)$ takes its maximum value $806{,}8576$ and its minimum value $0{,}0050$.

Variance

$$Var[w(z)] = e^{8z-2\bar{z}}.$$

If the variance found for the selected parameter values is plotted with MATLAB (2013a), the graph in Figure 2. is obtained.

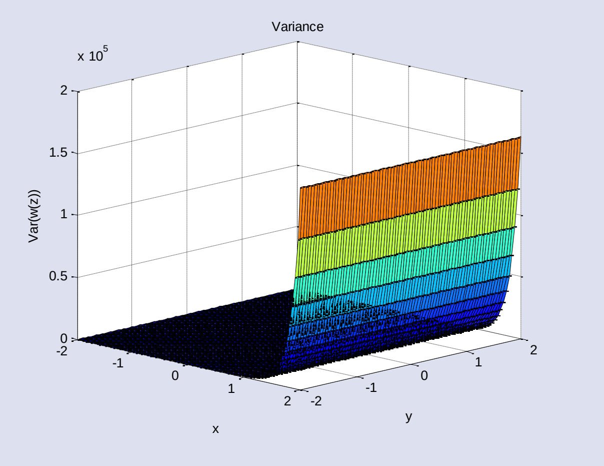

Figure 2 Variance of equation (13) for values $\mu = 2, \sigma^2 = 1$

These are the results for the confidence intervals for the expectations (Figure 3). The dashed line indicates the upper end of the confidence range while the dashed-dot lines show the lower ends of the interval in this case. Three standard deviations were utilized to produce the confidence intervals.

The variance of $w(z)$ is given above (Figure 2). Extremum values of the variances of the random variables are obtained as follows: $min[Var(w(z))] = 6.1442x10^{-6}$ and $max[Var(w(z))] = 1.6275x10^5$.

Confidence intervals for expected values of random variables,

$$(E[w(z)] - K.std(w(z)), E(w(z)) + K.std(w(z))$$

is equal to and this can be obtained through standard deviations. For $K = 3$, this formula gives approximately 99% confidence interval for the approximate expected value of the normally distributed random variable [21]. If the 99% confidence interval is plotted with MATLAB (2013a), the graph in Figure 3. is obtained. Known as the three-sigma rule, this popular rule indicates that about 99.73% of values for a normally distributed variable are within about three standard deviations of the mean. Therefore, using appropriate parameters, we will compare the variations of the results for two continuous distributions with limited and unlimited support, respectively. Appropriate parameters will ensure that almost all possible values for random effects are drawn from the same range for both distributions.

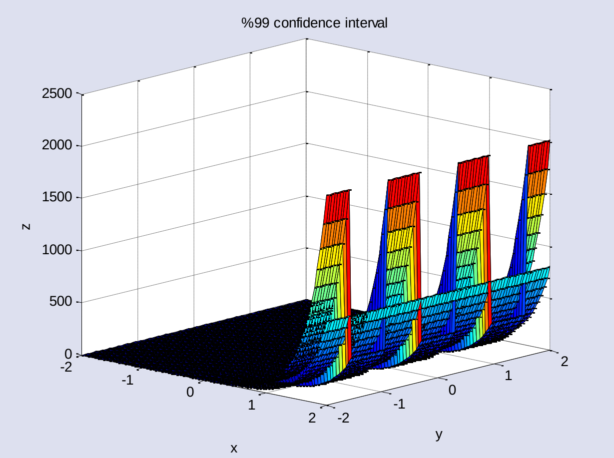

Figure 3. Confidence interval(%99) of equation (15) for values $\mu = 2, \sigma^2 = 1$

The confidence intervals of $w(z)$ are given in Figure 3. The extremum values of the confidence intervals are as follows: $min\big(E(w(z)) - 3std(w(z))\big)$ =0.0025 and $max(E(w(z)) + 3std(w(z)))$ = 2017.1 . Here, $K = 3$ gives an approximate 99% confidence interval.

Example 3.2.

$$w_z - w_{\bar{z}} = A - B,\ w(x,0) = (A+B)x \qquad (16)$$

obtain the probability characteristics by solving the approximate analytical solution of the given partial differential equation with $A, B \sim G(\alpha,\beta)$ gamma distribution, independent random variables with Sumudu method. In equation (16).

$$\frac{\partial w}{\partial z} = w_z = \frac{1}{2}\left[w_x - i w_y\right]$$
$$\frac{\partial w}{\partial \bar{z}} = w_{\bar{z}} = \frac{1}{2}\left[w_x + i w_y\right]$$

if $w = u + iv$ is written instead of equations,

$$\frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right] - \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right] = A - B$$

are obtained. After the necessary procedures are carried out,

$$\frac{\partial v}{\partial y} = A - B$$
$$-\frac{\partial u}{\partial y} i = 0$$

if Sumudu transformation is applied to its equations,

$$S[u(x,y)] = R_1(x,s)$$
$$S[v(x,y)] = R_2(x,s)$$

when,

$$\frac{1}{s}[R_2(x,s) - v(x,0)] = A - B$$
$$-\frac{1}{s}[R_1(x,s) - u(x,0)] = 0.$$

if the Cramer rule applies to the resulting Sumudu transformations,

$$R_1(x,s) = u(x,0) = (A+B)x,$$
$$R_2(x,s) = v(x,0) + s(A-B) = s(A-B)$$

taking the inverse Sumudu transformation into his equations,

$$u(x,y) = S^{-1}[(A+B)x]$$
$$= (A+B)x$$
$$v(x,y) = S^{-1}[s(A-B)]$$
$$= (A-B)y$$

are obtained. Then $w(z) = u(x,y) + iv(x,y) = (A+B)x + i(A-B)y = Az + B\bar{z}$ can be found. A random variable X is gamma distributed ($X \sim G(\alpha,\beta)$) if its probability distribution function is $f(x;\alpha,\beta) = \frac{x^{\alpha-1}e^{-\beta x}\beta^{\alpha}}{\Gamma(\alpha)}$. When $z \sim G(\alpha,\beta)$, the expected value and its variance are given below, respectively [32].

$E[z] = \alpha\beta$, $E[z^2] = (\alpha + \alpha^2)\beta^2$, $Var[z] = \alpha\beta^2$,

The expected value and variance of the solution we found above are, respectively,

$$E[w(z)] = E[Az + B\bar{z}]$$
$$= E[A]z + E[B]\bar{z}$$
$$= \alpha\beta(z + \bar{z}).$$
$$Var[w(z)] = Var[Az + B\bar{z}]$$
$$= Var[A]z^2 + Var[B]\bar{z}^2$$
$$= \alpha\beta^2(z^2 + \bar{z}^2).$$

Expected value if $\alpha = 1$, $\beta = 2$ are selected specifically,

$$E[w(z)] = 2(z + \bar{z})$$

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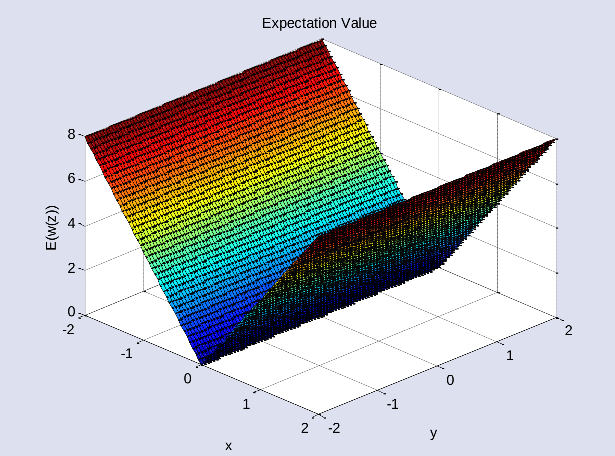

Figure 4. The expected value of equation (16) for values $\alpha = 1$, $\beta = 2$

The expectations can be given in a single graph for a comparison with the deterministic results of equation (1) as above (Figure 4). Maximum and minimum values of expected values of the random variables are obtained as follows: $w(z)$ takes its maximum value 8 and its minimum value 0. Variance,

$$Var[w(z)] = 4(z^2 + \bar{z}^2)$$

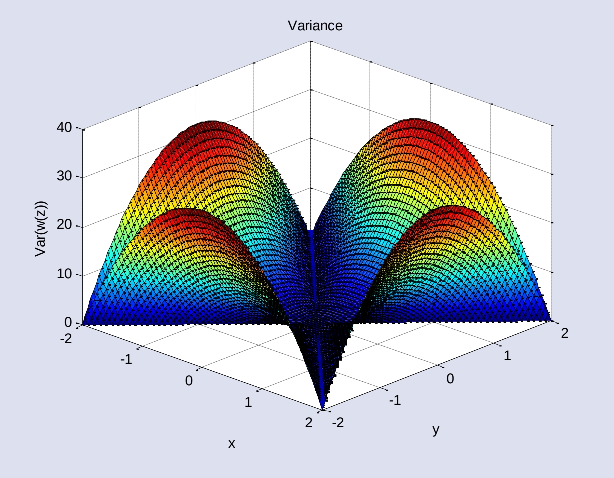

Figure 5. Variance of equation (16) for values $\alpha = 1$, $\beta = 2$

The variance of $w(z)$ is given above (Figure 5). Extremum values of the variances of the random variables are obtained as follows: $min[Var(w(z))] = 0$ and $max[Var(w(z))] = 32$

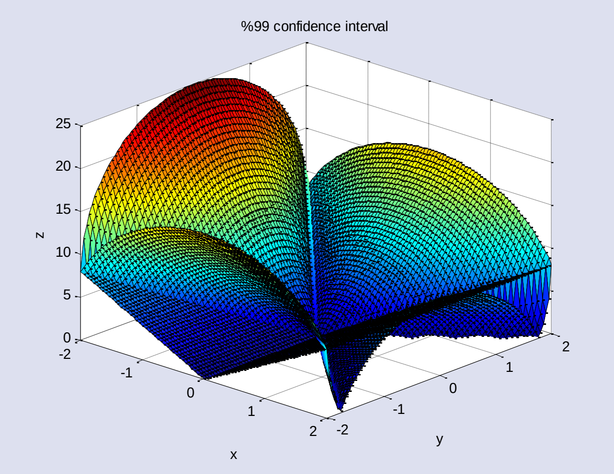

Figure 6. Confidence interval(%99) of equation (14) for values $\alpha = 1$, $\beta = 2$

The confidence intervals of $w(z)$ are given in Figure 3. The extremum values of the confidence intervals are as follows: $min\big(E(w(z)) - 3std(w(z))\big) = 0$ and $max(E(w(z)) + 3std(w(z))) = 24.9706$. Here, $K = 3$ gives an approximate 99% confidence interval.

Conclusion

With the aid of random variables chosen from the initial conditions, a random complex differential equation was used in this work. With the aid of transformations, a system of random partial differential equations was created from a normal and gamma distribution. The two-dimensional Sumudu and inverse Sumudu transformations have been used to analytically solve the resulting system of equations. Several examples demonstrate approximations to the solution stochastic process's mean and standard deviation functions. Calculated and graphically displayed are the found solution's probability characteristics.

Conflict of interest

The author declares no conflicts of interest

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