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Inverse Nodal Problem for a Conformable Fractional Diffusion Operator With Parameter-Dependent Nonlocal Boundary Condition

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ABSTRACT
In this paper, we consider the inverse nodal problem for the conformable fractional diffusion operator with parameter-dependent Bitsadze–Samarskii type nonlocal boundary condition. We obtain the asymptotics for the eigenvalues, the eigenfunctions, and the zeros of the eigenfunctions (called nodal points or nodes) of the considered operator, and provide a constructive procedure for solving the inverse nodal problem, i.e., we reconstruct the potential functions $p(x)$ and $q(x)$ by using a dense subset of the nodal points.

***Keywords:** Diffusion operator, Inverse nodal problem, Conformable fractional derivative, Nonlocal boundary condition.*

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Introduction

Inverse nodal problem consists in reconstructing operators from given a dense set of zeros of eigenfunctions called nodal points or nodes. Inverse nodal problems for different differential operators have been studied for years. In 1988, McLaughlin gave a solution to inverse nodal problem for the Sturm–Liouville operator and sought to recover the potential function $\boldsymbol{q}(\boldsymbol{x})$ by using the zeros of the eigenfunctions (see [1]). In 1989, Hald and McLaughlin showed that it is sufficient to know the nodal points to uniquely determine the potential function of the Sturm–Liouville problem (see [2]). In 1997, Yang gave an algorithm to recover the potential function and boundary condition from any dense subset of the nodal points (see [3]). Inverse nodal problems have been investigated by several researchers for the Sturm−Liouville operators and the diffusion operators with the usual derivative (see [4]-[17] and references therein).

As known, there are two types of nonlocal boundary conditions, Bitsadze–Samarskii type conditions and integral type conditions. These conditions appear when data cannot be measured directly at the boundary and have many applications (see [18]-[19] and references therein). In 1969, firstly Bitsadze and Samarskii applied nonlocal boundary conditions to elliptic equations (see [20]). Some studies on inverse nodal problems for various types of operators with nonlocal boundary conditions can be seen in [21]-[27].

In 2014, Khalil et al. introduced new definition of fractional derivative called conformable fractional derivative of order α∈(0,1] (see [28]). In 2015, the basic properties and main results of this derivative were given by Abdeljawad and Atangana et al. ([29]-[30]). In recent years, the direct and inverse problems for the various operators which include conformable fractional derivatives have been studied (see [31]-[36] and references therein).

In current literature, there are any results on the inverse nodal problems for a diffusion operator with parameter-dependent Bitsadze–Samarskii-type nonlocal boundary condition, which include conformable fractional derivative.

Preliminaries

We give known some concepts of the conformable fractional calculus that more detail knowledge can be seen in [28]-[30] and [37].

Definition 2.1 Let $f:[0,\infty)\to\mathbb{R}$ be a given function. Then, the conformable fractional derivative of f of order α with respect to x is defined by

$$T_\alpha f(x)=\lim_{h\to 0}\frac{f(x+hx^{1-\alpha})-f(x)}{h},$$

$T_\alpha f(0)=\lim\limits_{x\to 0^+} T_\alpha f(x)$, for all $x>0$, $\alpha\in(0,1]$.

If f is differentiable that is $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$, then, $T_\alpha f(x)=x^{1-\alpha}f'(x)$.

Theorem 2.2 Let f, g be α −differentiable at x, $x>0$.

i) $T_\alpha(c_1 f+c_2 g)=c_1 T_\alpha f+c_2 T_\alpha g,\ \forall\, c_1,c_2\in\mathbb{R}$,
ii) $T_\alpha(x^r)=rx^{r-\alpha},\ \forall\, r\in\mathbb{R}$,
iii) $T_\alpha(c)=0,\ (c-const.)$
iv) $T_\alpha(fg)=T_\alpha(f)g+fT_\alpha(g)$,
v) $T_\alpha\left(\frac{f}{g}\right)=\frac{T_\alpha(f)g-fT_\alpha(g)}{g^2},\ (g\neq 0)$.

Definition 2.3 The conformable fractional integral is defined by

$$I_\alpha f(x) = \int_0^x f(t) d_\alpha t = \int_0^x t^{\alpha-1} f(t) dt, \text{for all } x > 0.$$

Theorem 2.4 ***(**α **−chain rule)*** Let $f, g: [0, \infty) \to \mathbb{R}$ be α −differentiable functions. Then, $f\big(g(x)\big)$ is α −differentiable function and for all $x, x \neq 0, g(x) \neq 0$

$$\Big(T_\alpha\big(f(g)\big)\Big)(x) = (T_\alpha f)\big(g(x)\big)(T_\alpha g)(x) g^{\alpha-1}(x).$$

Definition 2.5 (α −integration by parts) Let $f, g: [a, b] \to \mathbb{R}$ be α −differentiable functions. Then,

$$\int_a^b f(x) T_\alpha g(x) d_\alpha x = f(x) g(x)|_a^b - \int_a^b g(x) T_\alpha f(x) d_\alpha x.$$

Lemma 2.6 ***(**α **−Leibniz rule)*** Let $f(x,t)$ be a function such that $t^{\alpha-1} f(x,t)$ and $t^{\alpha-1} f_x(x,t)$ are continuous in t and x in some regions of the (x,t)-plane, including $a(x) \leq t \leq b(x)$, $x_0 \leq x \leq x_1$. If $a(x)$ and $b(x)$ are α −differentiable functions for $x_0 \leq x \leq x_1$, then,

$$T_\alpha \left(\int_{a(x)}^{b(x)} f(x,t) d_\alpha t \right) = T_\alpha b(x) f\big(x, b(x)\big) b^{\alpha-1}(x) - T_\alpha a(x) f\big(x, a(x)\big) a^{\alpha-1}(x) + \int_{a(x)}^{b(x)} T_\alpha f(x,t) d_\alpha t.$$

Definition 2.7 Let $1 \leq p < \infty$, $a > 0$. The space $L_{p,\alpha}(0,a)$ consists of all functions $f: [0,a] \to \mathbb{R}$ satisfying the condition

$$\left(\int_0^a |f(x)|^p d_\alpha x\right)^{1/p} < \infty.$$

Lemma 2.8 The space $L_{p,\alpha}(0,a)$ associated with the norm function

$$\|f\|_{p,\alpha} := \left(\int_0^a |f(x)|^p d_\alpha x\right)^{1/p}$$

is a Banach space. Moreover if $p = 2$ then $L_{2,\alpha}(0,a)$ associated with the inner product for $f, g \in L_{2,\alpha}(0,a)$

$$\langle f, g \rangle := \int_0^a f(x) \overline{g(x)} d_\alpha x$$

is a Hilbert space.

Definition 2.9 Let $1 \leq p < \infty$. The Sobolev space $W_{p,\alpha}^1[0,a]$ consists of all functions on $[0,a]$ such that $f(x)$ is absolutely continuous and $T_\alpha f(x) \in L_{p,\alpha}(0,a)$.

Asymptotics of the Eigenvalues and Eigenfunctions

In this section, we consider a diffusion operator with parameter-dependent Bitsadze–Samarskii-type nonlocal boundary condition which includes conformable fractional derivatives of order α instead of the ordinary derivatives in a traditional diffusion operator. The operator $L_\alpha = L_\alpha(p(x), q(x), \beta)$ is called a conformable fractional diffusion operator (CFDO) and is the form

$$\ell_\alpha y := -T_\alpha T_\alpha y + [2\lambda p(x) + q(x)] y = \lambda^2 y, \quad 0 < x < 1 \tag{1}$$

$$U_\alpha(y) := y(0) = 0 \tag{2}$$

$$V_\alpha(y) := \lambda y(1) - y(\beta) = 0 \tag{3}$$

where λ is the spectral parameter, $\alpha \in (0,1]$, $q(x) \in W_{2,\alpha}^1[0,1]$, $p(x) \in W_{2,\alpha}^2[0,1]$ are real-valued functions, $p(x) \neq const.$, $\beta \in (0,1)$, and for $\gamma = \beta, 1$

$$\int_0^\gamma p(x) d_\alpha x = 0. \tag{4}$$

From [35], the general solution of equation (1)

$$y(x,\lambda;\alpha)=c_1\cos\left(\frac{\lambda}{\alpha}x^{\alpha}-Q(x)\right)+c_2\sin\left(\frac{\lambda}{\alpha}x^{\alpha}-Q(x)\right)$$
$$+\int_0^x \frac{\sin\left(\frac{\lambda}{\alpha}(x^{\alpha}-t^{\alpha})-Q(x)+Q(t)\right)}{\lambda-p(t)}\left((q(t)+p^2(t))y(t,\lambda;\alpha)+\frac{T_\alpha p(t)}{\lambda-p(t)}T_\alpha y(t,\lambda;\alpha)\right)d_\alpha t, \quad (5)$$

where,

$$Q(x):=\int_0^x p(t)d_\alpha t. \quad (6)$$

Let the functions $C=C(x,\lambda;\alpha)$ and $S=S(x,\lambda;\alpha)$ be solutions of equation (1) under the initial conditions

$$C(0,\lambda;\alpha)=1,\ T_\alpha C(0,\lambda;\alpha)=0 \text{ and } S(0,\lambda;\alpha)=0,\ T_\alpha S(0,\lambda;\alpha)=1 \quad (7)$$

respectively.

Thus, from (5), following solutions

$$C(x,\lambda;\alpha)=\cos\left(\frac{\lambda}{\alpha}x^{\alpha}-Q(x)\right)$$
$$+\int_0^x \frac{\sin\left(\frac{\lambda}{\alpha}(x^{\alpha}-t^{\alpha})-Q(x)+Q(t)\right)}{\lambda-p(t)}\left((q(t)+p^2(t))C(t,\lambda;\alpha)+\frac{T_\alpha p(t)}{\lambda-p(t)}T_\alpha C(t,\lambda;\alpha)\right)d_\alpha t \quad (8)$$

and

$$S(x,\lambda;\alpha)=\frac{\sin\left(\frac{\lambda}{\alpha}x^{\alpha}-Q(x)\right)}{\lambda-p(0)}$$
$$+\int_0^x \frac{\sin\left(\frac{\lambda}{\alpha}(x^{\alpha}-t^{\alpha})-Q(x)+Q(t)\right)}{\lambda-p(t)}\left((q(t)+p^2(t))S(t,\lambda;\alpha)+\frac{T_\alpha p(t)}{\lambda-p(t)}T_\alpha S(t,\lambda;\alpha)\right)d_\alpha t, \quad (9)$$

are obtained.

From [14], [35] and [38], for the asymptotic representations of the functions $C(x,\lambda;\alpha)$ and $S(x,\lambda;\alpha)$, the following lemma can be given.

Lemma 3.1 For $|\lambda|\to\infty$ and each fixed α, the following asymptotic formulae is valid:

$$C(x,\lambda;\alpha)=\cos\left(\frac{\lambda}{\alpha}x^{\alpha}-Q(x)\right)+O\left(\frac{1}{\lambda}\exp\left(|\mathrm{Im}\lambda|\frac{x^{\alpha}}{\alpha}\right)\right) \quad (10)$$

and

$$S(x,\lambda;\alpha)=\frac{1}{\lambda}\sin\left(\frac{\lambda}{\alpha}x^{\alpha}-Q(x)\right)$$
$$+\frac{1}{2\lambda^2}\left\{(p(x)+p(0))\sin\left(\frac{\lambda}{\alpha}x^{\alpha}-Q(x)\right)-c_1(x)\cos\left(\frac{\lambda}{\alpha}x^{\alpha}-Q(x)\right)\right.$$
$$+\int_0^x (q(t)+p^2(t))\cos\left(\frac{\lambda}{\alpha}(x^{\alpha}-2t^{\alpha})-Q(x)+2Q(t)\right)d_\alpha t$$
$$\left.+\int_0^x (T_\alpha p(t))\sin\left(\frac{\lambda}{\alpha}(x^{\alpha}-2t^{\alpha})-Q(x)+2Q(t)\right)d_\alpha t\right\} \quad (11)$$
$$+\frac{1}{4\lambda^3}\left\{c_3(x)\sin\left(\frac{\lambda}{\alpha}x^{\alpha}-Q(x)\right)-c_4(x)\cos\left(\frac{\lambda}{\alpha}x^{\alpha}-Q(x)\right)\right\}$$
$$+O\left(\frac{1}{\lambda^4}\exp\left(|\mathrm{Im}\lambda|\frac{x^{\alpha}}{\alpha}\right)\right),$$

where,

$c_1(x) = \int_0^x (q(t) + p^2(t))d_\alpha t,\ c_2(x) = \int_0^x (q(t) + p^2(t))p(t)d_\alpha t,$

$$c_3(x) = 4p^2(0) + \frac{2(p(x) + p(0))^{1+\alpha} - 2^{2+\alpha}p^{1+\alpha}(0) + (p(x) - p(0))^{1+\alpha}}{1+\alpha} - \frac{1}{2}\left(\int_0^x (q(t) + p^2(t))d_\alpha t\right)^2,$$

$$c_4(x) = \int_0^x (q(t) + p^2(t))(p(x) + p(0) + 2p(t))d_\alpha t = (p(x) + p(0))c_1(x) + 2c_2(x).$$

The eigenvalues of the problem L_α coincide with the zeros of its characteristic function given by

$$\Delta_\alpha(\lambda) = \begin{vmatrix} U_\alpha(C) & U_\alpha(S) \\ V_\alpha(C) & V_\alpha(S) \end{vmatrix} = \lambda S(1, \lambda; \alpha) - S(\beta, \lambda; \alpha). \tag{12}$$

Thus, using the formulae (4), (11), and (12), we obtain the following asymptotic formula for $\Delta_\alpha(\lambda)$

$$\begin{aligned}\Delta_\alpha(\lambda) = {} & \sin\frac{\lambda}{\alpha} + \frac{1}{2\lambda}\Big\{(p(1) + p(0))\sin\frac{\lambda}{\alpha} - c_1(1)\cos\frac{\lambda}{\alpha} - 2\sin\frac{\lambda}{\alpha}\beta^\alpha \\ & + \int_0^1 (q(t) + p^2(t))\cos\left(\frac{\lambda}{\alpha}(1 - 2t^\alpha) + 2Q(t)\right) d_\alpha t \\ & + \int_0^1 (T_\alpha p(t))\sin\left(\frac{\lambda}{\alpha}(1 - 2t^\alpha) + 2Q(t)\right) d_\alpha t\Big\} \\ & + \frac{1}{4\lambda^2}\left\{c_3(1)\sin\frac{\lambda}{\alpha} - c_4(1)\cos\frac{\lambda}{\alpha} - 2(p(\beta) + p(0))\sin\frac{\lambda}{\alpha}\beta^\alpha + 2c_1(\beta)\cos\frac{\lambda}{\alpha}\beta^\alpha\right\} \\ & + O\left(\frac{1}{\lambda^3}\exp\frac{|\operatorname{Im}\lambda|}{\alpha}\right), |\lambda| \to \infty.\end{aligned} \tag{13}$$

By the method in [10], using (13) and Rouchè theorem and taking $\Delta_\alpha(\lambda_n) = 0$ we can prove that the eigenvalues λ_n have the form

$$\begin{aligned}\lambda_n = {} & n\alpha\pi + \frac{c_1(1) - A_n^n + 2(-1)^n \sin n\beta^\alpha\pi}{2n\pi} \\ & + \frac{(p(1)+p(0))c_1(1) + 2c_2(1) + 2(-1)^n(p(\beta)+p(0))\sin n\beta^\alpha\pi - 2(-1)^n c_1(\beta)\cos n\beta^\alpha\pi}{4n^2\alpha\pi^2} \\ & + o\left(\frac{1}{n^2}\right), |n| \to \infty,\end{aligned} \tag{14}$$

where, for $n \in \mathbb{Z}\backslash\{0\}$, $x_n^0 = 0$, $x_n^n = 1$, $j \in \mathbb{Z}$,

$$A_n^j = \int_0^{x_n^j} (q(t) + p^2(t))\cos(2n\pi t^\alpha - 2Q(t))d_\alpha t - \int_0^{x_n^j} (T_\alpha p(t))\sin(2n\pi t^\alpha - 2Q(t))d_\alpha t.$$

Inverse Nodal Problem

In this section, under condition (7) we obtain the asymptotics for the zeros of the function $\varphi(x, \lambda_n; \alpha)$ called the nodal points of the operator L_α and develop a constructive procedure for solving the inverse nodal problem.

It is clear from (14) that for sufficiently large $|n|$, there is exactly one eigenvalue λ_n in the domain $\Gamma_n = \{\lambda: |\lambda - n\alpha\pi| \le \delta\}$, $\delta > 0$ and since the functions $p(x)$ and $q(x)$ are real-valued, λ_n are real. Thus, the functions $\varphi(x, \lambda_n; \alpha)$ are real-valued and

$$\varphi(x, \lambda_n; \alpha) = U_\alpha(C(x, \lambda_n; \alpha))S(x, \lambda_n; \alpha) - U_\alpha(S(x, \lambda_n; \alpha))C(x, \lambda_n; \alpha) = S(x, \lambda_n; \alpha) \tag{15}$$

are the eigenfunctions corresponding to the eigenvalues λ_n for sufficiently large $|n|$.

Thus, substituting (14) in (11), we get

$$\begin{aligned}\lambda_n\varphi(x, \lambda_n; \alpha) = {} & \sin(n\pi x^\alpha - Q(x)) \\ & + \frac{1}{2n\alpha\pi}\{[(c_1(1) - A_n^n + 2(-1)^n \sin n\beta^\alpha\pi)x^\alpha - c_1(x)]\cos(n\pi x^\alpha - Q(x)) \\ & + (p(x) + p(0))\sin(n\pi x^\alpha - Q(x))\end{aligned}$$

$$+\int_0^x (q(t)+p^2(t))\cos(n\pi(x^\alpha-2t^\alpha)-Q(x)+2Q(t))d_\alpha t$$
$$+\int_0^x (T_\alpha p(t))\sin(n\pi(x^\alpha-2t^\alpha)-Q(x)+2Q(t))d_\alpha t\}$$
$$+\frac{1}{4n^2\alpha^2\pi^2}\{[(p(1)+p(0))c_1(1)x^\alpha+2c_2(1)x^\alpha \tag{16}$$
$$+2(-1)^n(p(\beta)+p(0))x^\alpha \sin n\beta^\alpha\pi-2(-1)^n c_1(\beta)x^\alpha \cos n\beta^\alpha\pi$$
$$+(p(x)+p(0))(c_1(1)+2(-1)^n\sin n\beta^\alpha\pi)x^\alpha-c_4(x)]\cos(n\pi x^\alpha-Q(x))$$
$$+[c_1(x)(c_1(1)+2(-1)^n\sin n\beta^\alpha\pi)x^\alpha$$
$$-(c_1(1)+2(-1)^n\sin n\beta^\alpha\pi)^2x^{2\alpha}+c_3(x)]\sin(n\pi x^\alpha-Q(x))$$
$$+o\left(\frac{1}{n^2}\right),|n|\to\infty,$$

uniformly in $x\in[0,1]$.

We can see from (16) that for sufficiently large $|n|$ and each fixed α, the eigenfunctions $\varphi(x,\lambda_n;\alpha)$ has exactly $|n|-1$ nodal points x_n^j, $j\in\mathbb{Z}$ in $(0,1)$ as

$$0<x_n^1<x_n^2<\ldots<x_n^{n-1}<1 \text{ for } n>0$$

and

$$0<x_n^{-1}<x_n^{-2}<\ldots<x_n^{n+1}<1 \text{ for } n<0.$$

Lemma 4.1 The numbers x_n^j satisfy the following asymptotic formula for sufficiently large $|n|$ and each fixed α:

$$\begin{aligned}\left(x_n^j\right)^\alpha&=\frac{j}{n}+\frac{Q(x_n^j)}{n\pi}\\&+\frac{1}{2n^2\alpha\pi^2}\left[c_1(x_n^j)-c_1(1)(x_n^j)^\alpha-\left(A_n^j-A_n^n(x_n^j)^\alpha\right)-2(-1)^n(x_n^j)^\alpha\sin n\beta^\alpha\pi\right]\\&+\frac{1}{2n^3\alpha^2\pi^3}\Big[c_2(x_n^j)-\left(c_2(1)+\frac{(p(1)+p(0))c_1(1)}{2}\right)(x_n^j)^\alpha\\&+(-1)^n(p(\beta)+p(0))(x_n^j)^\alpha\sin n\beta^\alpha\pi-(-1)^n c_1(\beta)(x_n^j)^\alpha\cos n\beta^\alpha\pi\Big]\\&+o\left(\frac{1}{n^3}\right),\end{aligned}\tag{17}$$

uniformly with respect to *j*.

Proof. From (15) and taking $\varphi(x_n^j,\lambda_n;\alpha)=0$, we get

$$\sin\left(n\pi\left(x_n^j\right)^\alpha-Q(x_n^j)\right)+\frac{1}{2n\alpha\pi}\Big\{[(c_1(1)-A_n^n+2(-1)^n\sin n\beta^\alpha\pi)(x_n^j)^\alpha-c_1(x_n^j)]\cos\left(n\pi\left(x_n^j\right)^\alpha-Q(x_n^j)\right)$$
$$+(p(x_n^j)+p(0))\sin\left(n\pi\left(x_n^j\right)^\alpha-Q(x_n^j)\right)+\int_0^{x_n^j}(q(t)+p^2(t))\cos\left(n\pi\left(\left(x_n^j\right)^\alpha-2t^\alpha\right)-Q(x_n^j)+2Q(t)\right)d_\alpha t$$
$$+\int_0^{x_n^j}(T_\alpha p(t))\sin\left(n\pi\left(\left(x_n^j\right)^\alpha-2t^\alpha\right)-Q(x_n^j)+2Q(t)\right)d_\alpha t\Big\}+\frac{1}{4n^2\alpha^2\pi^2}\Big\{\Big[(p(1)+p(0))c_1(1)\left(x_n^j\right)^\alpha+2c_2(1)\left(x_n^j\right)^\alpha$$
$$+2(-1)^n(p(\beta)+p(0))\left(x_n^j\right)^\alpha\sin n\beta^\alpha\pi-2(-1)^n c_1(\beta)\left(x_n^j\right)^\alpha\cos n\beta^\alpha\pi$$
$$+\left(p(x_n^j)+p(0)\right)(c_1(1)+2(-1)^n\sin n\beta^\alpha\pi)\left(x_n^j\right)^\alpha-c_4(x_n^j)\Big]\cos\left(n\pi\left(x_n^j\right)^\alpha-Q(x_n^j)\right)$$
$$+\Big[c_1(x_n^j)(c_1(1)+2(-1)^n\sin n\beta^\alpha\pi)\left(x_n^j\right)^\alpha-(c_1(1)+2(-1)^n\sin n\beta^\alpha\pi)^2\left(x_n^j\right)^{2\alpha}$$
$$+c_3(x_n^j)]\sin\left(n\pi\left(x_n^j\right)^\alpha-Q(x_n^j)\right)\Big\}+o\left(\frac{1}{n^2}\right)=0,\ |n|\to\infty.$$

If last equality is divided by $\cos\left(n\pi\left(x_n^j\right)^\alpha-Q(x_n^j)\right)$ and necessary arrangements are made, for $|n|\to\infty$, we obtain that

$$\tan\left(n\pi\left(x_n^j\right)^\alpha - Q\left(x_n^j\right)\right) = \left\{1 + \frac{1}{2n\alpha\pi}\left[p\left(x_n^j\right) + p(0) + B_n^j\right] + \frac{1}{4n^2\alpha^2\pi^2}\left[c_1\left(x_n^j\right)\left(c_1(1) + 2(-1)^n \sin n\beta^\alpha\pi\right)\left(x_n^j\right)^\alpha - \left(c_1(1) + 2(-1)^n \sin n\beta^\alpha\pi\right)^2\left(x_n^j\right)^{2\alpha} - c_3\left(x_n^j\right)\right]\right\}^{-1} \times \left\{-\frac{1}{2n\alpha\pi}\left[\left(c_1(1) - A_n^n + 2(-1)^n \sin n\beta^\alpha\pi\right)\left(x_n^j\right)^\alpha - c_1\left(x_n^j\right) + A_n^j\right] - \frac{1}{4n^2\alpha^2\pi^2}\left[\left(p(1) + p(0)\right)c_1(1)\left(x_n^j\right)^\alpha + 2c_2(1)\left(x_n^j\right)^\alpha + 2(-1)^n\left(p(\beta) + p(0)\right)\left(x_n^j\right)^\alpha \sin n\beta^\alpha\pi - 2(-1)^n c_1(\beta)\left(x_n^j\right)^\alpha \cos n\beta^\alpha\pi + \left(p\left(x_n^j\right) + p(0)\right)\left(c_1(1) + 2(-1)^n \sin n\beta^\alpha\pi\right)\left(x_n^j\right)^\alpha - c_4\left(x_n^j\right)\right] + o\left(\frac{1}{n^2}\right)\right\},$$

where,

$$B_n^j = \int_0^{x_n^j} \left(q(t) + p^2(t)\right)\sin\left(2n\pi t^\alpha - 2Q(t)\right)d_\alpha t + \int_0^{x_n^j} \left(T_\alpha p(t)\right)\cos\left(2n\pi t^\alpha - 2Q(t)\right)d_\alpha t.$$

Hence, if Taylor's expansion formula is taken into account for the function $\frac{1}{1+u}$ as $u \to 0$ then,

$$\tan\left(n\pi\left(x_n^j\right)^\alpha - Q\left(x_n^j\right)\right) = \frac{1}{2n\alpha\pi}\left[c_1\left(x_n^j\right) - c_1(1)\left(x_n^j\right)^\alpha - \left(A_n^j - A_n^n\left(x_n^j\right)^\alpha\right) - 2(-1)^n\left(x_n^j\right)^\alpha \sin n\beta^\alpha\pi\right] + \frac{1}{2n^2\alpha^2\pi^2}\left[c_2\left(x_n^j\right) - \left(c_2(1) + \frac{\left(p(1)+p(0)\right)c_1(1)}{2}\right)\left(x_n^j\right)^\alpha + (-1)^n\left(p(\beta) + p(0)\right)\left(x_n^j\right)^\alpha \sin n\beta^\alpha\pi - (-1)^n c_1(\beta)\left(x_n^j\right)^\alpha \cos n\beta^\alpha\pi\right] + o\left(\frac{1}{n^2}\right),\ |n| \to \infty$$

and if Taylor's expansion formula for Arctangent is taken into account then,

$$n\pi\left(x_n^j\right)^\alpha - Q\left(x_n^j\right) = j\pi + \frac{1}{2n\alpha\pi}\left[c_1\left(x_n^j\right) - c_1(1)\left(x_n^j\right)^\alpha - \left(A_n^j - A_n^n\left(x_n^j\right)^\alpha\right) - 2(-1)^n\left(x_n^j\right)^\alpha \sin n\beta^\alpha\pi\right] + \frac{1}{2n^2\alpha^2\pi^2}\left[c_2\left(x_n^j\right) - \left(c_2(1) + \frac{\left(p(1)+p(0)\right)c_1(1)}{2}\right)\left(x_n^j\right)^\alpha + (-1)^n\left(p(\beta) + p(0)\right)\left(x_n^j\right)^\alpha \sin n\beta^\alpha\pi - (-1)^n c_1(\beta)\left(x_n^j\right)^\alpha \cos n\beta^\alpha\pi\right] + o\left(\frac{1}{n^2}\right),\ |n| \to \infty$$

is obtained. From the last equality, we arrive at (17).

Corollary 4.2 From (17) it is obvious that for each fixed α, the set X of all nodal points is dense in the interval $[0,1]$.

Let X be the set of nodal points and $\beta^\alpha = \frac{k}{\ell}$, $k, \ell \in \mathbb{Z}$. For each fixed $x \in [0,1]$ and $\alpha \in (0,1]$, we can choose a sequence $\{j_n\} \subset X$ so that $\lim_{|n|\to\infty} x_n^{j_n} = x$. Clearly the subsequence $\{x_m^{j_m}\}$ converges also to x for $m = 2n\ell$. Then, there exist finite limits and corresponding equalities hold:

$$\pi \lim_{|m|\to\infty}\left(m\left(x_m^{j_m}\right)^\alpha - j_m\right) := Q(x), \tag{18}$$

$$2\alpha\pi \lim_{|m|\to\infty} m\left[\pi\left(m\left(x_m^{j_m}\right)^\alpha - j_m\right) - Q\left(x_m^{j_m}\right)\right] := f(x), \tag{19}$$

$$\alpha\pi \lim_{|m|\to\infty} m\left\{2\alpha m\pi\left[\pi\left(m\left(x_m^{j_m}\right)^\alpha - j_m\right) - Q\left(x_m^{j_m}\right)\right] - f\left(x_m^{j_m}\right) + A_m^{j_m} - A_m^m\left(x_m^{j_m}\right)^\alpha + 2(-1)^m\left(x_m^{j_m}\right)^\alpha \sin m\beta^\alpha\pi\right\} := g(x) \tag{20}$$

and

$$f(x) = c_1(x) - c_1(1)x^\alpha, \tag{21}$$

$$g(x) = c_2(x) - \left(c_2(1) + \frac{\left(p(1)+p(0)\right)c_1(1)}{2}\right)x^\alpha - c_1(\beta)x^\alpha. \tag{22}$$

Therefore, we can prove the following theorem for the solution of the inverse nodal problem.

Theorem 4.3 Given any dense subset of nodal points $X_0 \subset X$ uniquely determines the functions $p(x)$ and $q(x)$ a.e. on $[0,1]$. Moreover, these functions can be found by the following procedure.

Step-1. Denote $m = 2n\ell$ and for each fixed x and α, choose a sequence $\{x_m^{j_m}\} \subset X_0$ such that $\lim_{|m|\to\infty} x_m^{j_m} = x$,

Step-2. Find the function $Q(x)$ from (18) and taking into account (6) calculate

$$p(x) = T_\alpha Q(x), \tag{23}$$

Step-3. Find the function $f(x)$ from (19) and determine

$$q(x) - \alpha \int_0^1 q(t) d_\alpha t := r(x) = T_\alpha f(x) - p^2(x) + \alpha \int_0^1 p^2(t) d_\alpha t, \tag{24}$$

Step-4. $2\alpha Q(x) - (p(1) + p(0))x^\alpha - 2\beta^\alpha x^\alpha \neq 0$ and for each fixed x, α, find $g(x)$ from (20) and calculate

$$\int_0^1 q(t) d_\alpha t = \frac{2}{2\alpha Q(x) - (p(1) + p(0))x^\alpha - 2\beta^\alpha x^\alpha}\Big[g(x) - \int_0^x (r(t) + p^2(t))p(t) d_\alpha t$$
$$+ x^\alpha \int_0^1 (r(t) + p^2(t))p(t) d_\alpha t + \frac{(p(1)+p(0))x^\alpha}{2} \int_0^1 (r(t) + p^2(t)) d_\alpha t \tag{25}$$
$$+ x^\alpha \int_0^\beta (r(t) + p^2(t)) d_\alpha t\Big],$$

Step-5. Calculate the function $q(x)$ via the formula

$$q(x) = r(x) + \alpha \int_0^1 q(t) d_\alpha t. \tag{26}$$

Proof. From (6), it is obvious that formula (23) is provided.

α-differentiating (21), we get

$$T_\alpha f(x) = q(x) + p^2(x) - \alpha \int_0^1 (q(t) + p^2(t)) d_\alpha t.$$

If denote $r(x) := q(x) - \alpha \int_0^1 q(t) d_\alpha t$, then we obtain (24). If we substitute $q(x) = r(x) + \alpha \int_0^1 q(t) d_\alpha t$ in (22) and take (4) into account, we get formula (25).

Finally, from (24) and (25), we arrive at (26).

Conflicts of interest

There are no conflicts of interest in this work.

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