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The Slab Critical Thickness Problem with Reflecting Boundary Condition for the Anlı-Güngör Scattering Function

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ABSTRACT

The criticality equation, which defines the relation between the secondary neutron number and the thickness of the slab, and the numerical solutions of this equation are investigated with reflecting boundary condition for the recently studied the Anlı-Güngör (AG) scattering function. The analytical calculations are performed by HN method. The numerical results are calculated with Wolfram Mathematica software for the varying secondary neutron number, the varying scattering parameter, and the varying reflection coefficient. The critical slab thickness values decrease for increasing reflection coefficient as expected.

Keywords: Legendre expansion of scattering function, Anlı-Güngör scattering function, HN method, The critical slab problem, Reflecting boundary condition.

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Introduction

In this study the critical slab problem is investigated with the reflecting boundary condition for the Anlı-Güngör (AG) scattering function [1] with quadratic term. The criticality equation defines the relation between the secondary neutron number and the thickness of medium.

The critical slab problem was investigated by Mitsis [2] with the Case method [3,4], Carlvik [5], Sahni and Sjöstrand [6-8], Sahni [9], Dahl and Sjöstrand [10]. It was investigated for the reflecting boundary conditions by Garis [11], Garis and Sjöstrand [12], Atalay [13], Türeci *et al.* [14]. The problem was investigated with different methods such as the Case method, C_N Method [15], F_N method [16] and H_N method [17]. The effects of the anisotropic scatterings were also investigated with İnönü's scattering function [18]. Siewert and Williams [19] searched the effect of the anisotropy for the critical slab problem. The different types of the Legendre expansion of scattering function [20] were studied by different researchers [21-26].

The AG scattering function is

$$f(\mu,\mu')=\sum_{n=0}^{N} t^n P_n(\mu) P_n(\mu') \quad (1)$$

where t is the scattering parameter, $P_n(\mu)$ is the Legendre polynomial with n^th order. Here the defining interval of the scattering parameter for all scattering situations is given as $|t| \le 1$. The Legendre expansion of scattering function is

$$f(\mu,\mu')=\sum_{n=0}^{N}(2n+1) f_n P_n(\mu) P_n(\mu'), \quad (2)$$

where f_n is the scattering coefficients, and $P_n(\mu)$ is the Legendre polynomial with n^th order.

It is obvious that both scattering functions are the same for only linear anisotropic scattering. Although both scattering functions are similar, these functions have different properties. One of differences between the Legendre expansion of scattering function and AG anisotropic scattering function is about the scattering coefficients. t parameter in AG scattering is defined in $t \in [-1,1]$. But the determination of the defining interval of the scattering coefficients in Legendre expansion of scattering function is a difficult job. A detailed analysis was performed by Gülderen et al. [27] in the Milne problem for linear-triplet Legendre expansion of scattering function. A different analysis was given by Köklü and Özer [28] for the tetra Legendre expansion of scattering function. Although the scattering parameter is defined in the interval $t \in [-1,1]$, the parameter is defined in $t \in [-0.54, 0.54]$ for the quadratic AG scattering because of the rule that the sum of probabilities is equal to one. This result can be determined by the solving of an inequality relation of Eq.(1), which defines the rule that the sum of probabilities is equal to one.

The other difference appears for the further scattering terms. For instance, if we take $f_1=0$, $f_2 \neq 0$ and $f_{n\ge3}$, then the scattering is called as pure-quadratic anisotropic scattering in Legendre expansion of scattering function. But a similar selection is not possible in the AG scattering function. If any researcher wants to study the quadratic AG scattering as in this study, then the linear anisotropic scattering term will automatically be in the scattering function.

The AG scattering function which has been recently studied with P_N method [29] for the criticality problem. It also has been studied for half-space albedo problem [30], the Milne problem [31], slab albedo problem [32] and the criticality problem [33] for quadratic AG scattering with H_N method. Maleki [34] used the scattering function to solve the half-space albedo and slab albedo problems with Monte Carlo method.

In this study the reflection situation is taken into account. Thus, the effect of the reflection could be investigated. The results are interesting for being closed to unity of the scattering term values for fixed c. The thickness of the medium represents the parabolic behaviour for this situation.

The Case Method Solution for the Anlı-Güngör Scattering Function

The analytical calculations are performed by H_N method. The method is based on the usage the Case method relations. Therefore, the relations of the Case method must be derived for the AG scattering function. This analysis was performed by Türeci and Bülbül [35].

First, we can start with the source-free, one-speed, time-independent, homogeneous medium and plane geometry neutron transport equation:

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\psi(x,\mu)=\frac{c}{2}\int_{-1}^{1} f(\mu,\mu')\psi(x,\mu')d\mu' , \tag{3}$$

where x is the spatial variable in mfp unit, μ is the direction cosine, c is the secondary neutron number, and $f(\mu,\mu')$ is the AG scattering function. The Case eigenfunction for the AG scattering is

$$\phi(\nu,\mu)=\frac{c\nu}{2}\frac{K_n(\nu,\mu)}{\nu-\mu} , \tag{4}$$

where

$$K_n(\nu,\mu)=\sum_{n=0}^{N} t^n P_n(\mu)J_n(\nu) , \tag{5}$$

and

$$J_n(\nu)=\int_{-1}^{1} P_n(\mu)\phi(\nu,\mu)d\mu. \tag{6}$$

$J_n(\nu)$ obeys the recursion relation, and it is given by

$$J_{k+1}(\nu)=\frac{\nu}{k+1}\left[(2k+1)-ct^k\right]J_k(\nu)-\frac{k}{k+1}J_{k-1}(\nu). \tag{7}$$

$J_0(\nu)$ corresponds to the normalization of the Case eigenfunction. The normalization condition is

$$\int_{-1}^{1} \phi(\nu,\mu)d\mu=1. \tag{8}$$

If $\nu\notin[-1,1]$, then the solution of the Eq. (8) gives one pair discrete eigenvalues, $\pm\nu_0$. If $\nu\in[-1,1]$, then we have a singular point at $\nu=\mu$. We are interested in the quadratic AG scattering here. Therefore, the upper limit of the scattering is $N=2$ in the expansion of Eq. (2), and the fourth and beyond terms, $N>2$, are assumed to be small enough to be neglected. Thus, the scattering function studied in this study is

$$f(\mu,\mu')=1+t\ P_1(\mu)P_1(\mu')+t^2P_2(\mu)P_2(\mu') . \tag{9}$$

The explicit form of Eq. (5) is

$$K_2(\xi,\mu)=1+t\xi(1-c)\mu+\frac{t^2}{4}\left(3\mu^2-1\right)\left(\xi^2(1-c)(3-ct)-1\right), \tag{10}$$

and the Case eigenfunction is

$$\phi(\nu,\mu)=\frac{c\nu}{2}\frac{1+t\nu(1-c)\mu+\frac{t^2}{4}\left(\nu^2(1-c)(3-ct)-1\right)\left(3\mu^2-1\right)}{\nu-\mu} . \tag{11}$$

The normalization condition for the quadratic AG scattering given in Eq. (8) is

$$\ln\left(\frac{1+\nu_0}{1-\nu_0}\right)=\frac{2}{c\nu_0}\frac{1+c\nu_0 tJ_1(\nu_0)+\frac{3c\nu_0^2}{2}t^2J_2(\nu_0)}{1+\nu_0 tJ_1(\nu_0)+t^2J_2(\nu_0)\frac{3\nu_0^2-1}{2}} . \tag{12}$$

$$J_0(\xi)=1, \tag{13a}$$

$$J_1(\xi)=\xi(1-c), \tag{13b}$$

$$J_2(\xi)=\frac{1}{2}\left(\xi^2(1-c)(3-ct)-1\right) . \tag{13c}$$

The discrete eigenvalues, $\pm\nu_0$, are the numerical solutions of Eq. (12). Equation (12) is a transcendental equation, and it can be only solved as numerical methods such as Newton-Raphson method [36, 37] or Muller's method [36, 38]. Both methods can give real and complex roots of any function. The criticality problem is studied for $c>1$, and Eq. (12) has complex roots, $\pm i\ \nu_0$ where $i^2=-1$. Finally, the discrete and the continuum eigenfunctions are

$$\phi(\pm\nu_0,\mu)=\frac{c\nu_0}{2}\frac{1\pm t\nu_0(1-c)\mu+\frac{t^2}{4}\left(\nu_0^2(1-c)(3-ct)-1\right)\left(3\mu^2-1\right)}{\nu_0\mp\mu}, \tag{14}$$

$$\phi(\nu,\mu)=\frac{c\nu}{2}P\frac{1+t\nu(1-c)\mu+\frac{t^2}{4}\left(\nu^2(1-c)(3-ct)-1\right)\left(3\mu^2-1\right)}{\nu-\mu}+\lambda(\nu)\delta(\nu-\mu) . \tag{15}$$

where P corresponds to the Cauchy principal value, and $\lambda(\nu)$ is

$$\lambda(\nu)=1-\frac{c\nu}{2}P\int_{-1}^{1}\frac{K_n(\nu,\mu)}{\nu-\mu}d\mu . \tag{16}$$

The explicit form of Eq. (16) is

$$\lambda(\nu)=1+c\nu tJ_1(\nu)+\frac{3c\nu^2}{2}t^2J_2(\nu)-c\nu\left[1+t\nu J_1(\nu)+t^2J_2(\nu)\frac{3\nu^2-1}{2}\right]\tanh^{-1}(\nu) . \tag{17}$$

These eigenfunctions obey the orthogonality relations:

$$\int_{-1}^{1}\mu\phi(\pm\nu_0,\mu)\phi(\pm\nu_0,\mu)d\mu=M(\pm\nu_0),\quad M(\pm\nu_0)=-M(\mp\nu_0), \tag{18}$$

$$\int_{-1}^{1} \mu\phi(\pm\nu_0,\mu)\phi(\mp\nu_0,\mu)d\mu=0, \tag{19}$$

$$\int_{-1}^{1} \mu\phi(\nu,\mu)\phi(\nu,\mu)d\mu=M(\nu) \tag{20}$$

Finally, the solution of Eq. (3) for the AG scattering function is

$$\psi(x,\mu)=A(\nu_0)\phi(\nu_0,\mu)e^{-x/\nu_0}+A(-\nu_0)\phi(-\nu_0,\mu)e^{x/\nu_0}+\int_{-1}^{1} A(\nu)\phi(\nu,\mu)e^{-x/\nu}d\nu \tag{21}$$

where $A(\pm\nu_0)$ and $A(\nu)$ are the arbitrary expansion coefficients. The explicit form of Eq. (18) for the scattering function is

$$\begin{aligned} M(\nu_0) = \left(\frac{c\nu_0}{2}\right)^2 \frac{2}{\nu_0^2-1}\Big[&(\alpha_0+\nu_0\beta_0)\left(\alpha_0\nu_0-2\beta_0+3\nu_0^2\beta_0\right) \\ &+\frac{2}{3}\gamma_0\left(-6\nu_0\alpha_0+9\nu_0^3\alpha_0-\beta_0\left(1-8\nu_0^2\right)+12\beta_0\nu_0^4\right)+\frac{\nu_0\gamma_0^2}{3}\left(-2-10\nu_0^2+15\nu_0^4\right)\Big] \\ &-\left(\frac{c\nu_0}{2}\right)^2\left(\alpha_0+\beta_0\nu_0+\gamma_0\nu_0^2\right)\left(\alpha_0+3\beta_0\nu_0+5\gamma_0\nu_0^2\right)\ln\left(\frac{\nu_0+1}{\nu_0-1}\right) \end{aligned} \tag{22}$$

where

$$\alpha_0\equiv\alpha(\nu_0)=1-\frac{1}{2}t^2J_2(\nu_0), \tag{23a}$$

$$\beta_0\equiv\beta(\nu_0)=\nu_0^2t(1-c), \tag{23b}$$

$$\gamma_0\equiv\gamma(\nu_0)=\frac{3}{2}t^2J_2(\nu_0). \tag{23c}$$

and the explicit form of Eq. (20) is

$$M(\nu)=\nu\lambda^2(\nu)+\frac{c^2\pi^2\nu^3}{4}\left[1+tJ_1(\nu)P_1(\nu)+t^2J_2(\nu)P_2(\nu)\right]^2. \tag{24}$$

where $P_1(\nu)$ and $P_2(\nu)$ are the Legendre polynomials in terms of the continuum eigenvalue, ν.

The criticality equation with HN method

We are interesting a slab reactor which is placed in $x\in[-a,a]$. Thus, the thickness of the slab is $\tau=2a$. It is assumed that the inside of the slab is the medium, and the outside is the vacuum. The interaction of the neutrons with the medium is thought as the quadratic AG scattering.

The neutron flux has a symmetry condition over the right and left walls over the boundaries of the medium. This condition implies that the outgoing fluxes are the same. The outgoing neutron fluxes are defined as a power series expansion:

$$\Psi(-a,-\mu)=\Psi(a,\mu)=\sum_{l=0}^{G} a_l\mu^l, \quad \mu\in[0,1] \tag{25}$$

The reflection boundary condition means that there are reflected neutrons from the boundaries to the medium. The reflection is defined with the reflection coefficient. Therefore, the reflected neutron flux from the boundaries is proportional with the outgoing flux with the reflection coefficient, R, $R\in[0,1]$. Thus, the reflection boundary conditions are

$$\Psi(a,-\mu)=R\Psi(a,\mu), \quad \mu>0, \tag{26}$$

$$\Psi(-a,\mu)=R\Psi(-a,-\mu), \quad \mu>0. \tag{27}$$

If the symmetry condition given in Eq. (25) is used in the solution, Eq. (21), then the arbitrary expansion coefficients in the solution of the transport equation become

$$A(\nu_0)=A(-\nu_0) \quad \text{and} \quad A(\nu)=A(-\nu). \tag{28}$$

Thus, the solution of Eq. (3) turns into

$$\begin{aligned}\Psi(x,\mu) &= A(\nu_0)\left[\phi(\nu_0,\mu)e^{-x/\nu_0}+\phi(-\nu_0,\mu)e^{x/\nu_0}\right] \\ &+\int_0^1 A(\nu)\left[\phi(\nu,\mu)e^{-x/\nu}+\phi(-\nu,\mu)e^{x/\nu}\right]d\nu, \quad \mu\in[-1,1].\end{aligned} \tag{29}$$

If Eq. (29) is written for $x=a$, then we get the following equation:

$$\begin{aligned}\Psi(a,\mu) &= A(\nu_0)\left[\phi(\nu_0,\mu)e^{-a/\nu_0}+\phi(-\nu_0,\mu)e^{a/\nu_0}\right] \\ &+\int_0^1 A(\nu)\left[\phi(\nu,\mu)e^{-a/\nu}+\phi(-\nu,\mu)e^{a/\nu}\right]d\nu, \quad \mu\in[-1,1].\end{aligned} \tag{30}$$

Now, the aim is to determine the arbitrary expansion coefficients in Eq. (30) by using the neutron flux definitions in Eqs. (25-27), and the orthogonality properties given in Eqs. (18-20). Since the neutron flux must go to zero when x goes to infinity, we want to eliminate the positive exponential terms in the application of the orthogonality relations. Therefore, the equation is multiplied by $\mu\phi(-\nu_0,\mu)$, and integrated over $\mu\in[-1,1]$. Thus, we find

$$A(\nu_0)M(-\nu_0)e^{a/\nu_0}=\int_{-1}^{1}\mu\phi(-\nu_0,\mu)\Psi(a,\mu)d\mu, \tag{31}$$

$$A(\nu_0)M(-\nu_0)e^{a/\nu_0}=\int_0^1\mu\phi(-\nu_0,\mu)\underbrace{\Psi(a,\mu)}_{\sum_{l=0}^{G}a_l\mu^l}d\mu-\int_0^1\mu\phi(-\nu_0,-\mu)\underbrace{\Psi(a,-\mu)}_{R\sum_{l=0}^{G}a_l\mu^l}d\mu, \tag{32}$$

$$A(\nu_0)=-\frac{e^{-a/\nu_0}}{M(\nu_0)}\frac{c\nu_0}{2}\sum_l a_l\left[A_l(\nu_0)-RB_l(\nu_0)\right]. \tag{33}$$

Similarly, if Eq. (30) is multiplied by $\mu\phi(-\nu,\mu)$ and integrated over $\mu\in[-1,1]$, then we find the arbitrary coefficient for the continuum part:

$$A(\nu)=-\frac{e^{-a/\nu}}{M(\nu)}\frac{c\nu}{2}\sum_l a_l\left[A_l(\nu)-RB_l(\nu)\right]. \tag{34}$$

where $A_l(\xi)$ and $B_l(\xi)$, $\xi=\nu,\ \nu_0$ are the moments of Case's eigenfunctions in interval $\mu\in[0,1]$:

$$A_l(\xi)=\frac{2}{c\xi}\int_0^1\mu^{l+1}\phi(\nu_0,-\mu)d\mu, \tag{35}$$

$$B_l(\xi)=\frac{2}{c\xi}\int_0^1\mu^{l+1}\phi(\nu_0,\mu)d\mu, \tag{36}$$

$$A_0(\xi)=\alpha_\xi-\frac{\beta_\xi}{2}+\frac{\gamma_\xi}{3}-\xi\left[\alpha_\xi+\xi\beta_\xi+\xi^2\gamma_\xi\right]\ln\left(1+\frac{1}{\xi}\right)+\xi\beta_\xi-\frac{\xi\gamma_\xi}{2}+\xi^2\gamma_\xi, \tag{37}$$

$$B_0(\xi) = \frac{2}{c}\left(1+c\xi\beta_\xi+c\xi^2\gamma_\xi\right) - \xi\left[\alpha_\xi+\xi\beta_\xi+\xi^2\gamma_\xi\right]\ln\left(1+\frac{1}{\xi}\right)$$
$$-\xi\beta_\xi - \xi^2\gamma_\xi - \frac{\xi\gamma_\xi}{2} - \alpha_\xi - \frac{\beta_\xi}{2} - \frac{\gamma_\xi}{3}, \tag{38}$$

where α_ξ, β_ξ and γ_ξ for $\xi = \pm\nu_0, \nu$ are defined in Eqs. (23). $A_l(\xi)$ and $B_l(\xi)$ satisfy own recursion relations:

$$A_l(\xi) = \frac{\alpha_\xi}{l+1} - \frac{\beta_\xi}{l+2} + \frac{\gamma_\xi}{l+3} - \xi A_{l-1}(\xi) \tag{39}$$

$$B_l(\xi) = \xi A_{l-1}(\xi) - \frac{\alpha_\xi}{l+1} - \frac{\beta_\xi}{l+2} - \frac{\gamma_\xi}{l+3} \tag{40}$$

Now, if Eq. (25) is written for $\mu \in [0,1]$, then we have

$$\Psi(a,\mu) = A(\nu_0)\left[\phi(\nu_0,\mu)e^{-a/\nu_0} + \phi(-\nu_0,\mu)e^{a/\nu_0}\right]$$
$$+\int_0^1 A(\nu)\left[\phi(\nu,\mu)e^{-a/\nu} + \phi(-\nu,\mu)e^{a/\nu}\right]d\nu, \quad \mu \in [0,1]. \tag{41}$$

If the expansion coefficients given in Eqs. (33, 34) are written in Eq. (41), and equation is multiplied to μ^{m+1} and integrated over $\mu \in [0,1]$, then we get the following equation system. m is an integer number, and it takes its value from 0 to G.

$$\sum_{l=0}^{G} a_l \left[\frac{1}{l+m+2} + \left(\frac{c\nu_0}{2}\right)^2 \frac{\left[A_l(\nu_0) - RB_l(\nu_0)\right]B_m(\nu_0)}{M(\nu_0)} e^{-2a/\nu_0}\right.$$
$$+\left(\frac{c\nu_0}{2}\right)^2 \frac{\left[A_l(\nu_0) - RB_l(\nu_0)\right]A_m(\nu_0)}{M(\nu_0)}$$
$$+\left(\frac{c}{2}\right)^2 \int_0^1 \nu^2 \frac{\left[A_l(\nu) - RB_l(\nu)\right]B_m(\nu)}{M(\nu)} e^{-2a/\nu} d\nu$$
$$\left.+\left(\frac{c}{2}\right)^2 \int_0^1 \nu^2 \frac{\left[A_l(\nu) - RB_l(\nu)\right]A_m(\nu)}{M(\nu)} d\nu\right] = 0. \tag{42}$$

This last equation can be written as

$$\sum_{l=0}^{G} a_l T_{lm} = 0 \tag{43}$$

Now, we can define a square matrix with the elements of T_{lm} so that since a_l can't become zero, the determinant of T matrix must be equal to zero. This defines the criticality equation:

$$\det T = 0 \tag{44}$$

Results

The tabulated results are the numerical solutions of Eq. (44) for varying c, t and R. The results are calculated with Wolfram Mathematica [39] software. Equation (42) contains two integral term and one of them, first integral term, includes the unknown slab thickness variable. Gaussian-quadrature method [37] is used to calculate the numerical value of this integral. Thus, this integral term is written as sum relation by using the Gaussian-quadrature. The *WorkingPrecision* option in Mathematica was selected as 32. Therefore, all calculations are performed with this precision in the background.

Table 1 represents the critical thickness results, $\tau = 2a$, for varying c for varying scattering parameter and varying reflection coefficients. Figures 1-3 represent the critical thickness values for fixed secondary neutron number, varying scattering parameter and reflection coefficient for $c = 1.1,\ 1.2$ and 1.5, respectively. Figures 4-5 represent the 3D plot for fixed secondary numbers, $c = 1.1$ and 1.5, respectively.

When the scattering parameter increases for fixed c and R, the critical thickness increases. While the critical thickness values decrease for the increasing reflection coefficient and the fixed c and t as we expected.

As a result of the calculation for the critical thickness, it is seen that for small c values the critical thickness shows a concave downward decreasing behaviour for increasing R values, but for large c values it shows a concave upward decreasing behaviour. Accordingly, it can be concluded that reflection is more dominant for small c values than for large c values.

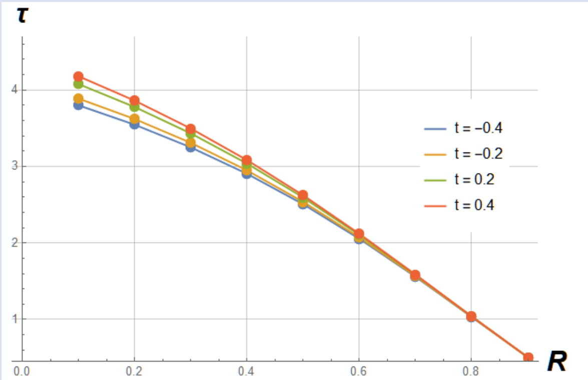

Figure 1. The critical thickness values for *c=1.1* and varying reflection coefficient.

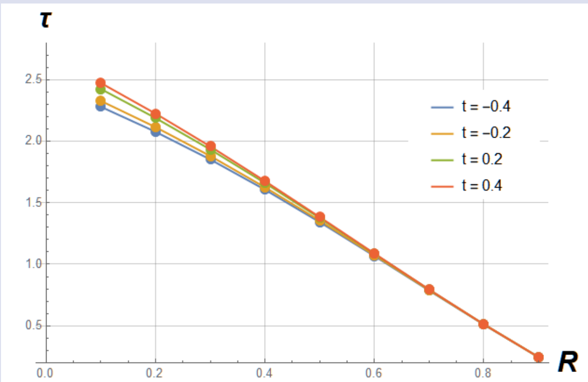

Figure 2. The critical thickness values for *c=1.2* and varying reflection coefficient.

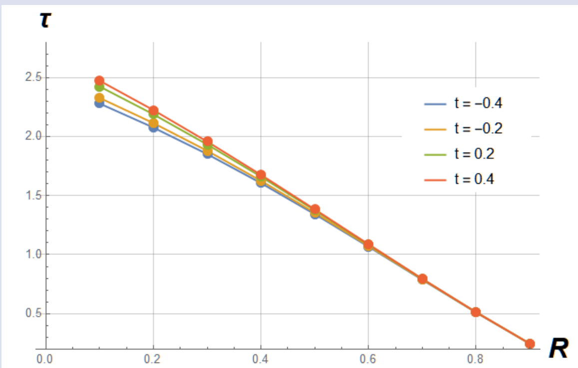

Figure 3. The critical thickness values for *c=1.5* and varying reflection coefficient.

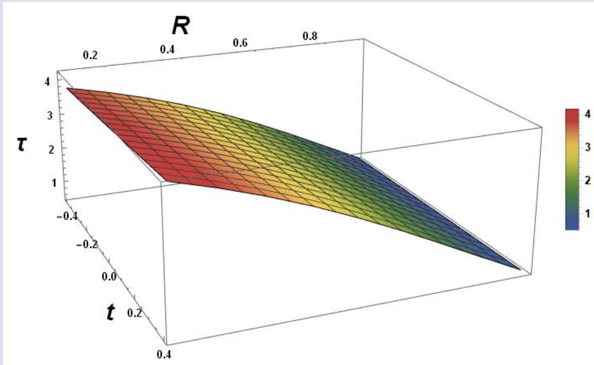

Figure 4. The critical thickness values for *c=1.1* and varying *t* and *R*.

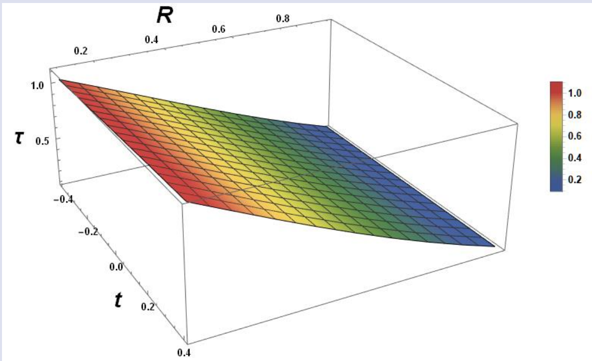

Figure 5. The critical thickness values for *c=1.5* and varying *t* and *R*.

Table 1. The critical thickness values for varying *c* and for varying *t* and *R* with only 6th approximation

c	t/R	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1.1	-0.40	3.80104467	3.54729782	3.25098206	2.90581248	2.50736783	2.05572136	1.55856109	1.03308608	0.50440557
	-0.20	3.88935824	3.62157117	3.31035936	2.95003729	2.53717277	2.07313449	1.56682989	1.03597076	0.50503525
	0.20	4.07880035	3.77846691	3.43336829	3.03938983	2.59543075	2.10561110	1.58109247	1.04014183	0.50552139
	0.40	4.18126050	3.86187248	3.49734950	3.08458338	2.62381989	2.12059592	1.58704169	1.04141439	0.50537625
1.2	-0.40	2.27973382	2.07674808	1.85196894	1.60639728	1.34311558	1.06755242	0.78717660	0.51053495	0.24589472
	-0.20	2.32758131	2.11439170	1.87989904	1.62563335	1.35517752	1.07428209	0.79043220	0.51186420	0.24632210
	0.20	2.42418263	2.18834355	1.93289863	1.66050297	1.37567202	1.08460617	0.79457662	0.51299478	0.24644826
	0.40	2.47299333	2.22452930	1.95777578	1.67595626	1.38397783	1.08813144	0.79543707	0.51278839	0.24614617
1.3	-0.40	1.63931760	1.47091460	1.29089088	1.10170522	0.90690300	0.71084003	0.51817916	0.33329141	0.15974203
	-0.20	1.67155373	1.49532402	1.30834922	1.11337952	0.91412681	0.71493846	0.52029633	0.33427534	0.16011436
	0.20	1.73291589	1.53997387	1.33868243	1.13229104	0.92468808	0.72002323	0.52226397	0.33479809	0.16017173
	0.40	1.76172571	1.55989287	1.35129162	1.13934326	0.92791662	0.72095630	0.52209387	0.33433148	0.15985613
1.4	-0.40	1.27709152	1.13375149	0.98435875	0.83144467	0.67799934	0.52716053	0.38186106	0.24452771	0.11690145
	-0.20	1.30110321	1.15151896	0.99683695	0.83971643	0.68315441	0.53017559	0.38351364	0.24536187	0.11724244
	0.20	1.34426557	1.18181555	1.01669268	0.85167953	0.68963288	0.53321489	0.38466629	0.24566381	0.11727524
	0.40	1.36296781	1.19397747	1.02380653	0.85520440	0.69086466	0.53319621	0.38415019	0.24512731	0.11696650
1.5	-0.40	1.04230861	0.91794640	0.79080504	0.66313873	0.53731702	0.41558037	0.29982540	0.19146657	0.09139188
	-0.20	1.06129148	0.93179758	0.80045315	0.66954117	0.54136417	0.41802057	0.30122588	0.19221184	0.09170958
	0.20	1.09357304	0.95385306	0.81453772	0.67782785	0.54576006	0.42004838	0.30198515	0.19240896	0.09173081
	0.40	1.10638080	0.96169113	0.81872881	0.67956489	0.54603061	0.41960018	0.30133060	0.19185727	0.09143383
1.6	-0.40	0.87739918	0.76788463	0.65760987	0.54849023	0.44237478	0.34087438	0.24524057	0.15631335	0.07453353
	-0.20	0.89301557	0.77918514	0.66546139	0.55373000	0.44574013	0.34295851	0.24647900	0.15699598	0.07483198
	0.20	0.91818923	0.79602579	0.67600645	0.55982647	0.44892655	0.34441107	0.24701809	0.15713506	0.07484684
	0.40	0.92725150	0.80121654	0.67847566	0.56055276	0.44867982	0.34374885	0.24630737	0.15658845	0.07456272
1.7	-0.40	0.75518926	0.65758994	0.56051625	0.46556655	0.37418030	0.28752435	0.20643193	0.13139592	0.06260446
	-0.20	0.76840801	0.66711136	0.56713665	0.47001879	0.37708444	0.28936390	0.20755425	0.13202988	0.06288625
	0.20	0.78864907	0.68042495	0.57534549	0.47470115	0.37950453	0.29045746	0.20795748	0.13213342	0.06289721
	0.40	0.79518965	0.68388960	0.57672966	0.47481501	0.37896089	0.28968475	0.20722847	0.13160027	0.06262585
1.8	-0.40	0.66105918	0.57321874	0.48673951	0.40294590	0.32295954	0.24762910	0.17750688	0.11286608	0.05374462
	-0.20	0.67249103	0.58143513	0.49246763	0.40683050	0.32552974	0.24928801	0.17853979	0.11345994	0.05401160
	0.20	0.68915399	0.59224378	0.49904953	0.41054499	0.32743291	0.25014220	0.17885320	0.11354012	0.05402001
	0.40	0.69392430	0.59453124	0.49971714	0.41027050	0.32671290	0.24931382	0.17812488	0.11302392	0.05376084
1.9	-0.40	0.58640447	0.50669193	0.42888889	0.35408971	0.28316927	0.21674402	0.15517213	0.09858288	0.04692188
	-0.20	0.59645745	0.51391331	0.43394201	0.35754548	0.28548511	0.21826234	0.15613270	0.09914250	0.04717555
	0.20	0.61043480	0.52287495	0.43934349	0.36056751	0.28702271	0.21894877	0.15638358	0.09920649	0.04718220
	0.40	0.61392240	0.52433224	0.43952170	0.36003943	0.28619690	0.21809568	0.15566593	0.09870849	0.04693448

We can also discuss the scattering functions. If we take the first two terms of these scattering functions, then we have

$$f_{Legendre}(\mu,\mu')=f_0+3f_1P_1(\mu)P_1(\mu') , \tag{45}$$

$$f_{AG}(\mu,\mu')=f_0+tP_1(\mu)P_1(\mu') . \tag{46}$$

By using an analogue between Eqs. (45 and 46) we can take that $3f_1 \equiv t$. Thus, we can define a relation between the scattering coefficients. If we take the first three terms of the scattering functions, then we have

$$5f_2 = t^2 = (3f_1)^2 \Rightarrow f_2 = \frac{(3f_1)^2}{5} . \tag{47}$$

Therefore, both scattering functions will give the same result for the above condition. But the results will be different except this condition. If we go on over the scattering functions, a similar result will be valid for the triplet scattering situation. Then, we can define a condition for f_3 :

$$7f_3=t^3=(3f_1)^3 \Rightarrow f_3=\frac{(3f_1)^3}{7} . \tag{48}$$

But it is important that we cannot separate the scattering terms for AG scattering function. If any researcher wants to study triplet anisotropic scattering, for instance, then the linear and the quadratic scatterings will automatically be in the scattering function, except for the Legendre expansion of scattering function.

Conclusions

In this study the critical slab problem was investigated for the reflection boundary condition. The criticality equation defines the relation between the secondary neutron number, c, and the critical thickness, $\tau = 2a$, in mfp unit. Therefore, c and the scattering parameter, t, are the independent variable, and τ is the dependent variable. But, since we take into account the reflection boundary condition, another independent variable is the reflection boundary condition, R.

The critical thickness values are investigated for varying c, t and R. The results are given in tables for only 6th approximation. The critical thickness values decrease for increasing c for certain t and R values as we expected. Similarly, the critical thickness values decrease for increasing R values.

Another result is that both Legendre expansion of scattering and AG scattering will give the same result for certain scattering situations. If the scattering is taken quadratic AG scattering; then the situations must be the same for $t = 3f_1$, and $t^2 = (3f_1)^2$. But other values of t will give different results.

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Conflicts of interest

The author states no conflict of interests.

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