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Spectral Decompositions of the Difference Operator Δ^m over the Sequence Space cs

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ABSTRACT

This study aims to bring together some studies on the spectra of difference operators in the literature over the cs sequence space and to provide a basis for related problems. So far, the problem has been solved up to a maximum of 2 orders on the sequence space cs. In this article, we discuss the difference operator Δ^m, represented by a (m+1) banded matrix, which generalizes the difference operators of the form Δ, Δ^2, B(r,s) and B(r,s,t) and we will give its boundedness, spectrum, fine spectrum and some spectral separations over the sequence space cs.

Keywords: Spectra, Fine spectra, Generalized difference operator, Band matrix.

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Introduction

The idea of defining the sequence space with the first-order difference operator was introduced in [1] and further generalized in [2]. Then, various sequence spaces have been studied by many authors by means of difference operators of different order (see for details [3-6]. Baliarsingh et al. in [7], also combined most of the difference operators previously described and found their fine spectra in a more general way.

Spectral theory has applications in many parts of mathematics and physics, including function theory, complex analysis, differential and integral equations, control theory, and quantum physics. In the literature, there are many important studies on the calculation of spectra of difference operators and generalized difference operators over various sequence spaces. For example, the fine spectrum of the difference operator on the sequence spaces c_0, c and ℓ_p $(0 \leq p < 1)$ was examined by Altay and Başar, [8-9]. The fine spectra of the operator $B(r,s)$ on c_0 and c were investigated by Altay and Başar in [10] and the fine spectra of the difference operator on the sequence spaces ℓ_p and bv_p $(1 < p < \infty)$ were calculated by Akhmedov and Başar in [11-12]. Srivastava and Kumar in [12-14] examined the fine spectra of the generalized difference operator Δ_v on sequence spaces c_0 and ℓ_1, where $a = (a_k)$ and $b = (b_k)$ are convergent sequences with certain properties, $v = (v_k)$ is a fixed or strictly decreasing sequence, and $r \in \mathbb{N}$. Recently, Akhmedov and El-Shabrawy [15], Dutta and Baliarsingh [16-17] examined the fine spectra of the generalized difference operators $\Delta_{a,b}$, Δ^2 and Δ_v^r on the sequence spaces c, c_0 and ℓ_1, respectively, where $a = (a_k)$ ve $b = (b_k)$ are convergent sequences with specific features, $v = (v_k)$ is a fixed or strictly decreasing sequence and $r \in \mathbb{N}$. In most of the studies mentioned above, the spectrum is discrete; point spectrum, continuous spectrum and residual spectrum. On the other hand, in [18] Durna and Yıldırım obtained spectral decomposition of factorable matrices on c_0 and in [19] Başar et al. studied the decomposition of the spectrum (approximate point spectrum, defect spectrum, compressed spectrum) of the generalized difference operator on some sequence spaces. In addition, in [20], Durna calculated the spectral decomposition of the Δ^{uv} generalized upper triangular double-band matrix over the sequence spaces c_0 and c. In [21] Das studied the spectrum and fine spectrum of the upper triangular matrix $U(r_1, r_2; s_1, s_2)$ on the c_0 sequence space. In [22], they studied the spectral decompositions of the generalized difference operator $B(r,s)$ on bv_0 and h sequence spaces. In [23], Tripathy and Das studied the spectrum and fine spectrum of the upper triangular matrix $U(r,0,0,s)$ on c_0 sequence space.

As mentioned above, there are many applications of spectra of bounded operators on Banach spaces. For this reason, the spectra of bounded linear operators have been studied by many people in recent years. So far, the problem on the cs sequence space has been solved up to a maximum of 2 orders. In this study, we will give the spectrum, point spectrum, continuous spectrum, residual spectrum and fine spectrum of the generalized difference operator Δ^m with $m+1$ band on the sequence space cs and calculate their spectral decomposition. Thus, the results obtained for the difference operator Δ studied in [8] and the results obtained for the generalized difference operator $B(1,-1)$ studied in [10] and [24] for $m = 1$ and the results obtained for the generalized difference operator $B(1,-2,1)$ studied in [25] for $m = 2$ will be included and generalized in this study.

Main Results

The generalized difference operator Δ^m $(m \in \mathbb{N})$ on the cs sequence space is defined by

$$(\Delta x)_k = x_k - x_{k-1}$$
$$(\Delta^2 x)_k = \Delta(\Delta x)_k = \Delta(x_k - x_{k-1})$$
$$= x_k - 2x_{k-1} + x_{k-2}$$
$$(\Delta^3 x)_k = x_k - 3x_{k-1} + 3x_{k-2} - x_{k-3}$$
$$\vdots$$
$$(\Delta^m x)_k = x_k - \binom{m}{1} x_{k-1} + \binom{m}{2} x_{k-2} + \cdots + (-1)^m x_{k-m}$$
$$= \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k-i}$$
$$= x_k - m x_{k-1} + \frac{m(m-1)}{2!} x_{k-2} + \cdots + (-1)^m x_{k-m},$$

for $k \in \mathbb{N}_0$, $\mathbb{N}_0 = \{0,1,2,\cdots\}$ and $x_k = 0$ for $k < 0$ ([7]).

It can be proven that the Δ^m operator can be represented by an (a_{nk}) matrix with $m+1$ bands. Here is

$$a_{nk} = \begin{cases} (-1)^{n-k}\binom{m}{k} & , \quad \max\{0, n-m\} \le k \le n \\ 0 & , \quad 0 \le k < \max\{0, n-m\} \text{or} k > n \end{cases}$$

for every $n, k \in \mathbb{N}_0$. Equivalently, it can be written as

$$\Delta^m = (a_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -m & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{m(m-1)}{2} & -m & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ (-1)^m & (-1)^{m-1}m & \cdots & -m & 1 & 0 & \cdots \\ 0 & (-1)^m & (-1)^{m-1}m & \cdots & -m & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1}$$

In this study, the fine spectrum and spectral decompositions of the Δ^m operator on the cs sequence space will be calculated, where cs is the sequence space of the form

$$cs: \left\{ x = (x_n) \in w : \lim_{n\to\infty} \sum_{i=0}^{n} x_i \text{exist} \right\}.$$

If $T: cs \to cs$ is a bounded linear operator with matrix representation A, then the matrix representation of the $T^*: cs^* \to cs^* \cong bv$ adjoint operator is the transpose of matrix A (see [26]).

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In fact, the Δ^m operator is represented by a $(m+1)$-th band matrix that generalizes the difference operators of the form Δ, Δ^2, $B(r,s)$ and $B(r,s,t)$ under different conditions. First, a few basic information will be given about the linearity and boundedness of the difference operator Δ^m. Then, spectrum and fine spectrum sets such as point spectrum, continuous spectrum and residual spectrum of the Δ^m operator will be determined in the sequence space cs. Let us now give two lemmas that will be very useful in determining the subdivision of the spectrum.

Lemma 1 *Necessary and sufficient condition for the linear operator T to have a dense range is that the adjoint operator T^* is one-to-one ([27], Theorem II 3.7).*

Lemma 2 *A necessary and sufficient condition for the linear operator T to have a bounded inverse is that the adjoint operator T^* is onto ([27], Theorem II 3.7).*

Since the definitions given above are related to the spectrum of the bounded linear operator, we first need to show the boundedness of the operator Δ^m on the sequence space cs. For this, the following lemmas will be used.

Lemma 3 *Necessary and sufficient condition for the matrix $A = (a_{nk})$ to represent a $T \in B(cs)$ operator*

i) For every k, the series $\sum_n a_{nk}$ is convergent,

ii) $\sup_N \sum_k \left|\sum_{n=1}^{N} (a_{nk} - a_{n,k-1})\right| < \infty$ ([26], 8.4.6B).

Lemma 4 $\sum_{n=0}^{\infty} \sum_{k=0}^{m} b_{nk} = \sum_{k=0}^{\infty} \sum_{n=0}^{m} b_{nk}$ *is valid.*

Proof.

$$\sum_{n=0}^{\infty} \sum_{k=0}^{m} b_{nk} = \sum_{k=0}^{m} b_{0k} + \sum_{k=0}^{m} b_{1k} + \sum_{k=0}^{m} b_{2k} + \cdots$$
$$= b_{00} + b_{01} + b_{02} + \cdots + b_{0m}$$
$$+ b_{10} + b_{11} + b_{12} + \cdots + b_{1m}$$
$$+ b_{20} + b_{21} + b_{22} + \cdots + b_{2m} + \cdots$$
$$= \sum_{n=0}^{\infty} b_{n0} + \sum_{n=0}^{\infty} b_{n1} + \cdots + \sum_{n=0}^{\infty} b_{nm}$$
$$= \sum_{k=0}^{\infty} \sum_{n=0}^{m} b_{nk}.$$

Theorem 1 *$\Delta^m \in B(cs)$ is valid.*

Proof. If we consider the matrix representation of the difference operator Δ^m in (1), Δ^m is a band matrix with $m+1$ bands. Therefore, it is clear that condition (i) of Lemma 3 is satisfied.

ii) We get

$$\sum_{n=1}^{N} (a_{nk} - a_{n,k-1}) = \sum_{n=1}^{N} (-1)^{n-k} \left[\binom{m}{k} - \binom{m}{k-1}\right]$$
$$= (-1)^k \left[\binom{m}{k} - \binom{m}{k-1}\right] \sum_{n=1}^{N} (-1)^n$$
$$= (-1)^k \left[\binom{m}{k} - \binom{m}{k-1}\right] A$$

where $A = \begin{cases} 0 & , \quad N \text{ is even} \\ 1 & , \quad N \text{ is odd} \end{cases}$. Then we have

$$\sum_k \left|\sum_{n=1}^{N} (a_{nk} - a_{n,k-1})\right| \le \sum_k \left[\binom{m}{k} + \binom{m}{k-1}\right]$$
$$= \sum_{k=0}^{m} \left[\binom{m}{k} + \binom{m}{k-1}\right]$$
$$\le 2^{m+1}.$$

So we get $\Delta^m \in B(cs)$.

Spectrum and fine spectrum of the difference operator $\boldsymbol{\Delta^m}$ on the $\boldsymbol{cs}$ sequence space

Theorem 2 *The point spectrum of Δ^m on cs is $\sigma_p(\Delta^m, cs) = \emptyset$.*

Proof. Think of the system of linear equations $\Delta^m x = \alpha x$, with $x \neq \theta = \{0,0,0,\cdots\}$ in cs. From (1) we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -m & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{m(m-1)}{2} & -m & 1 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^m & (-1)^{m-1}m & \cdots & -m & 1 & 0 & \cdots \\ 0 & (-1)^m & (-1)^{m-1}m & \cdots & -m & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = \alpha \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}$$

and so we get

$$\begin{aligned} &x_0 = \alpha x_0 \\ &-mx_0 + x_1 = \alpha x_1 \\ &\frac{m(m-1)}{2!} x_0 - mx_1 + x_2 = \alpha x_2 \\ &\frac{-m(m-1)(m-2)}{3!} x_0 + \frac{m(m-1)}{2!} x_1 - mx_2 + x_3 = \alpha x_3. \\ &\qquad\vdots \end{aligned} \tag{2}$$

If $x_0 \neq 0$ from the first equation of the (2) system of equations, then $\alpha = 1$. So from equation 2, $-mx_0 + x_1 = x_1$ and $x_0 = 0$, which is a contradiction. Let's assume that $x_0 = 0$, then from equation 2, if $x_1 = \alpha x_1$ and $x_1 \neq 0$ then $\alpha = 1$. So from equation 3, $-mx_1 + x_2 = x_2$ and $x_1 = 0$, which is a contradiction. If we continue like this, we get $x_1 = x_2 = x_3 = \cdots = 0$. From here, there is no $x \neq 0$ with $\Delta^m x = \alpha x$. Hence $\sigma_p(\Delta^m, cs) = \emptyset$.

Corollary 1 $I_3\sigma(\Delta^m, cs) = II_3\sigma(\Delta^m, cs) = III_3\sigma(\Delta^m, cs) = \emptyset$.

Proof. Since $\sigma_p(\Delta^m, cs) = I_3\sigma(\Delta^m, cs) \cup II_3\sigma(\Delta^m, cs) \cup III_3\sigma(\Delta^m, cs)$ from [19] Table 1.2, the desired result is obtained from Theorem 2.

Theorem 3 *The point spectrum of the $(\Delta^m)^*$ adjoint operator on $cs^* \cong bv$ is the set*

$\sigma_p((\Delta^m)^*, bv) = \{\alpha \in \mathbb{C}: |1-\alpha| \leq 2^m - 1\} \cup \{0\}$.

Proof. Let's assume that $(\Delta^m)^* x = \alpha x$ and $0 \neq x \in bv$. In this case,

$$\begin{aligned} &\binom{m}{0} x_0 - \binom{m}{1} x_1 + \binom{m}{2} x_2 - \binom{m}{3} x_3 + \cdots + (-1)^m \binom{m}{m} x_m = \alpha x_0 \\ &\binom{m}{0} x_1 - \binom{m}{1} x_2 + \binom{m}{2} x_3 + \cdots + (-1)^m \binom{m}{m-1} x_m + (-1)^m \binom{m}{m} x_{m+1} = \alpha x_1 \\ &\binom{m}{0} x_2 - \binom{m}{1} x_3 + \cdots + (-1)^m \binom{m}{m} x_{m+2} = \alpha x_2 \\ &\qquad\vdots \\ &\binom{m}{0} x_k - \binom{m}{1} x_{k+1} + \binom{m}{2} x_{k+2} + \cdots + (-1)^m x_{k+m} = \alpha x_k \\ &\qquad\vdots \end{aligned} \tag{3}$$

system of equations is obtained from the transpose of the matrix given in (1). If we write $k = n$ and $k = n+1$ in the equation (3) and subtract the sides,

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} (x_{k+n} - x_{k+n+1}) = \alpha(x_n - x_{n+1})$$

is obtained. Then

$$|x_n - x_{n+1}| \leq \frac{1}{|\alpha|} \sum_{k=0}^{m} \binom{m}{k} |x_{k+n} - x_{k+n+1}| \tag{4}$$

from the triangle inequality. If we write the inequality (4) for $n = 0,1,2,\cdots$ and add them side by side, we get

$$\sum_{n=0}^{\infty} |x_n - x_{n+1}| \le \frac{1}{|\alpha|}\sum_{n=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}|x_{k+n} - x_{k+n+1}|. \tag{5}$$

If we apply Lemma 4 to the right side of the inequality (5), we get

$$\sum_{n=0}^{\infty} |x_n - x_{n+1}| \le \frac{1}{|\alpha|}\sum_{k=0}^{\infty}\binom{m}{k}\sum_{n=0}^{m}|x_{k+n} - x_{k+n+1}|.$$

Since $\sum_{n=0}^{m}|x_{k+n} - x_{k+n+1}| \le \sum_{n=0}^{\infty}|x_{k+n} - x_{k+n+1}| \le \|x\|_{bv}$ and for $m < k$, $\binom{m}{k} = 0$,

$$\|x\|_{bv} \le \frac{\|x\|_{bv}}{|\alpha|}\sum_{k=0}^{m}\binom{m}{k} = \frac{2^m}{|\alpha|}\|x\|_{bv}$$

is obtained. Thus, $|\alpha| \le 2^m$ is found. Then $\{\alpha \in \mathbb{C}: |1-\alpha| < 2^m - 1\} \subseteq \sigma_p((\Delta^m)^*, bv)$ is obtained. Conversely, if the $|1-\alpha| < 2^m - 1$ case is considered, the $(\Delta^m)^* - \alpha I$ adjoint operator is not $1-1$, so from Lemma 1, $\Delta^m - \alpha I$ operator is not dense range in cs. So $(\Delta^m)^* - \alpha I$ is not invertible and

$$\sigma_p((\Delta^m)^*, bv) \subseteq \{\alpha \in \mathbb{C}: |1-\alpha| < 2^m - 1\}.$$

Also, in case of $m = 1$, $\alpha = 0$ for $x_0 \ne 0$ is an eigenvector corresponding to the eigenvalue $x = (x_0, 0, 0, \cdots)$. Hence $0 \in \sigma_p((\Delta^m)^*, bv)$ is obtained. If $m > 1$ anyway, 0 is contained by the set $\{\alpha \in \mathbb{C}: |1-\alpha| < 2^m - 1\}$.

Theorem 4 *The residue spectrum of the Δ^m operator on cs is the set $\sigma_r(\Delta^m, cs) = \{\alpha \in \mathbb{C}: |1-\alpha| < 2^m - 1\} \cup \{0\}$.*

Proof. Since $\sigma_r(\Delta^m, cs) = \sigma_p((\Delta^m)^*, bv) \backslash \sigma_p(\Delta^m, cs)$, the desired result is obtained from Theorem 2 and 3.

We will now calculate the spectrum of the Δ^m operator over the sequence space cs. For this we will need the $(\Delta^m - \alpha I)^{-1}$ operator. In the proof of [7, Theorem 5], under the condition $|1-\alpha| > 2^m - 1$, the operator $\Delta^m - \alpha I$ has an inverse, and its inverse is $(\Delta^m - \alpha I)^{-1} = (b_{nk})$, where

$$(b_{nk}) = \begin{pmatrix} \frac{1}{1-\alpha} & 0 & 0 & 0 & 0 & \cdots \\ \frac{m}{(1-\alpha)^2} & \frac{1}{1-\alpha} & 0 & 0 & 0 & \cdots \\ \frac{m^2}{(1-\alpha)^3} - \frac{m(m-1)}{2!(1-\alpha)^2} & \frac{m}{(1-\alpha)^2} & \frac{1}{1-\alpha} & 0 & 0 & \cdots \\ \frac{m^3}{(1-\alpha)^4} - \frac{m^2(m-1)}{(1-\alpha)^3} + \frac{m(m-1)(m-2)}{3!(1-\alpha)^2} & \frac{m^2}{(1-\alpha)^3} - \frac{m(m-1)}{2!(1-\alpha)^2} & \frac{m}{(1-\alpha)^2} & \frac{1}{1-\alpha} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6}$$

So we can prove the following theorem.

Theorem 5 The spectrum of Δ^m on cs is the set $\sigma(\Delta^m, cs) = \{\alpha \in \mathbb{C}: |1-\alpha| \le 2^m - 1\}$.

Proof. Since $(\Delta^m - \alpha I)$ is a triangular matrix, its inverse exists. The matrix $(\Delta^m - \alpha I)^{-1} = (b_{nk})$ is given in (6) and

$$\begin{aligned}
b_{nn} &= \frac{1}{1-\alpha}, \\
b_{n,n-1} &= \frac{m}{(1-\alpha)^2}, \\
b_{n,n-2} &= \frac{m^2}{(1-\alpha)^3} - \frac{m(m-1)}{2!\,(1-\alpha)^2}, \\
b_{n,n-3} &= \frac{m^3}{(1-\alpha)^4} - \frac{m^2(m-1)}{(1-\alpha)^3} + \frac{m(m-1)(m-2)}{3!\,(1-\alpha)^2} \\
b_{n,n-4} &= \frac{m^4}{(1-\alpha)^5} - \frac{m^3(m-1)}{(1-\alpha)^4} + \frac{m^2(m-1)(m-2)}{3!\,(1-\alpha)^3} - \frac{m^3(m-1)}{2!\,(1-\alpha)^4} \\
&\quad + \frac{m^2(m-1)^2}{2!2!(1-\alpha)^3} + \frac{m^2(m-1)(m-2)}{3!(1-\alpha)^3} - \frac{m(m-1)(m-2)(m-3)}{4!(1-\alpha)^2} \\
&\quad \vdots \\
b_{n,n-m} &= \frac{1}{1-\alpha}\Big[m b_{n,n-m+1} - \frac{m(m-1)}{2} b_{n,n-m+2} + \frac{m(m-1)(m-2)}{3!} b_{n,n-m+3} \\
&\quad + \cdots + (-1)^{m-1} b_{n,n}\Big] \\
&\quad \vdots
\end{aligned}$$

is obtained with $n \in \mathbb{N}_0$. Let us now show that $(\Delta^m - \alpha I)^{-1} \in B(cs)$. Lemma 3 will be used for this. In the proof of [7, Theorem 5], it was shown that the $\sum_{n=0}^{\infty} |b_{nk}|$ series converges for each k. Since every absolute convergent series converges, the $\sum_n b_{nk}$ series converges and the (i) condition of Lemma 3 is obtained.

Let us now show that the condition (ii) of Lemma 3 is $\sup_N \sum_k \left|\sum_{n=1}^{N} (b_{nk} - b_{n,k-1})\right| < \infty$. Since $b_{nk} = b_{n-1,k-1}$ for all n, k,

$$\sup_N \sum_{k=1} \left|(b_{1k} - b_{1,k-1}) + (b_{2k} - b_{2,k-1}) + (b_{3k} - b_{3,k-1}) + \ldots + (b_{Nk} - b_{N,k-1})\right|$$

$$= \sup_N \sum_k |b_{N,k}| = \sup_N \sum_{k=1}^{N} |b_{N,k}|$$

$$= |b_{N1}| + |b_{N2}| + |b_{N3}| + \ldots + |b_{N,N-1}| + |b_{NN}|$$

$$= |b_{N1}| + |b_{N-1,1}| + |b_{N-2,1}| + \ldots + |b_{21}| + |b_{11}| = \sum_{n=1}^{N} |b_{nk}|$$

is obtained. If we take $S_N = \sum_{n=1}^{N} |b_{nk}|$ then the sequence (S_N) is converget for $\left|\frac{2^m - 1}{1-\alpha}\right| < 1$ from [7, Theorem 5. So when $|1-\alpha| > 2^m - 1$, $(\Delta^m - \alpha I)^{-1} \in B(cs)$ is valid. Hence

$$\sigma(\Delta^m, cs) \subseteq \{\alpha \in \mathbb{C} : |1-\alpha| \leq 2^m - 1\} \tag{7}$$

is obtained. So $(\Delta^m - \alpha I)^{-1} \in B(cs)$.

Conversely, given that $\alpha \neq 1$ and $|1-\alpha| \leq 2^m - 1$, it is clear that $\Delta^m - \alpha I$ is a triangle and therefore $(\Delta^m - \alpha I)^{-1}$ exists. As a result of Theorem 2, the image of the unit sequence $y = (1,0,0,\cdots)$ under the $(\Delta^m - \alpha I)^{-1}$ transformation, that is, the sequence $x = (\Delta^m - \alpha I)^{-1} y$ is in cs. This suggests that under the condition $|1-\alpha| \leq 2^m - 1$ and from Theorem 3 the operator $(\Delta^m - \alpha I)^{-1}$ should not have an eigenvalue. From this it follows that $(\Delta^m)^* - \alpha I$ is not $1-1$. Therefore, under the condition $|1-\alpha| \leq 2^m - 1$, the $\Delta^m - \alpha I$ operator is not dense in cs. Finally, let's prove the result for the case $\alpha = 1$. If $\alpha = 1$ then

$$\Delta^m - I = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ -m & 0 & 0 & 0 & \cdots \\ \dfrac{m(m-1)}{2!} & -m & 0 & 0 & \cdots \\ -\dfrac{m(m-1)(1-2)}{3!} & \dfrac{m(m-1)}{2!} & -m & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is not reversible. Thus

$$\{\alpha \in \mathbb{C} : |1-\alpha| \leq 2^m - 1\} \subseteq \sigma(\Delta^m, c) \tag{8}$$

is obtained. The proof is completed using (7) and (8).

Theorem 6 *The continuous spectrum of the operator Δ^m on cs is the set*

$$\sigma_c(\Delta^m, cs) = \{\alpha \in \mathbb{C} : |1-\alpha| = 2^m - 1\} \backslash \{0\}.$$

Proof. The proof is obtained directly from the definition of the spectrum of the bounded linear operator and Theorems 2, 4 and 5.

Theorem 7 *If $\alpha \in (\{\alpha \in \mathbb{C} : |1-\alpha| \leq 2^m - 1\} \cup \{0\} \backslash \{1\})$ then $\alpha \in III_2\sigma(\Delta^m, cs)$ is valid.*

Proof. Let's assume that $\alpha \in (\{\alpha \in \mathbb{C} : |1-\alpha| \leq 2^m - 1\} \cup \{0\} \backslash \{1\})$. In this case, from Theorem 3, the operator $(\Delta^m)^* - \alpha I$ is not $1-1$, and hence $\alpha \in III\sigma(\Delta^m, cs)$ is obtained from Lemma 1. Moreover, it is obtained that for $\alpha \neq 1$ the operator $\Delta^m - \alpha I$ has an inverse, but for $\alpha = 0$ the operator $(\Delta^m)^* - \alpha I$ is not surjective. Thus, the operator $\Delta^m - \alpha I$ from Lemma 2 has no bounded inverse. This indicates that $\Delta^m - \alpha I$ is not continuous.

Corollary 2 $III_1\sigma(\Delta^m, cs) = \{1\}$.

Proof. From [19] Table 1.2, since $\sigma_r(\Delta^m, cs) = III_1\sigma(\Delta^m, cs) \cup III_2\sigma(\Delta^m, cs)$ and $III_1\sigma(\Delta^m, cs) \cap III_2\sigma(\Delta^m, cs) = \emptyset$, the desired result is obtained from Theorems 4 and 7.

Theorem 8 *If $\alpha \in (\{\alpha \in \mathbb{C} : |1-\alpha| > 2^m - 1\} \backslash \{1\})$ then $\alpha \in I_1\sigma(\Delta^m, cs)$ is valid.*

Proof. Let's assume that $\alpha \in (\{\alpha \in \mathbb{C} : |1-\alpha| > 2^m - 1\} \backslash \{1\})$. It is clear that $\alpha \neq 1$. Therefore, the $\Delta^m - \alpha I$ operator has an inverse. This indicates that the operator $\Delta^m - \alpha I$ is surjective, so the inverse operator $(\Delta^m - \alpha I)^{-1}$ in cs has a dense range. Therefore, from Lemma 1, the $(\Delta^m - \alpha I)$ operator is dense in cs. Hence $\alpha \in I\sigma(\Delta^m, cs)$.

Moreover, since $|1-\alpha|>2^m-1$, the operator $\Delta^m-\alpha I$ has a bounded inverse from Theorem 5. So $\alpha\in 1\sigma(\Delta^m,cs)$. Thus, $\alpha\in I_1\sigma(\Delta^m,cs)$ is obtained for α's satisfying the $|1-\alpha|>2^m-1$ inequality.

Spectral decompositions of the difference operator $\boldsymbol{\Delta^m}$ on the sequence space $\boldsymbol{cs}$ that do not need to be disjoint

Theorem 9 *For the spectral decomposition of the Δ^m operator on cs, the following are vald:*
a) $\sigma_{ap}(\Delta^m,cs)=\{\alpha\in\mathbb{C}:|1-\alpha|\le 2^m-1\}\backslash\{1\}$,

b) $\sigma_{\delta}(\Delta^m,cs)=\{\alpha\in\mathbb{C}:|1-\alpha|\le 2^m-1\}$,

c) $\sigma_{co}(\Delta^m,cs)=\{\alpha\in\mathbb{C}:|1-\alpha|<2^m-1\}\cup\{0\}$.

Proof. a) Since $\sigma_{ap}(\Delta^m,cs)=\sigma(\Delta^m,cs)\backslash III_1\sigma(\Delta^m,cs)$ from [19] Table 1.2, the desired result is obtained from Theorem 5 and Corollary 2.
b) Since $\sigma_{\delta}(\Delta^m,cs)=\sigma(\Delta^m,cs)\backslash I_3\sigma(\Delta^m,cs)$ from [19] Table 1.2, the desired result is obtained from Theorem 5 and Corollary 1.
c) Since

$$\begin{aligned}\sigma_{co}(\Delta^m,cs)&=III_1\sigma(\Delta^m,cs)\cup III_2\sigma(\Delta^m,cs)\cup III_3\sigma(\Delta^m,cs)\\&=\sigma_r(\Delta^m,cs)\cup III_3\sigma(\Delta^m,cs)\end{aligned}$$

from [10] Table 1.2, the desired result is obtained from Theorem 4 and Corollary 1.
Corollary 3 $\sigma_{ap}((\Delta^m)^*,bv)=\{\alpha\in\mathbb{C}:|1-\alpha|\le 2^m-1\}$ *and*

$\sigma_{\delta}((\Delta^m)^*,bv)=\{\alpha\in\mathbb{C}:|1-\alpha|\le 2^m-1\}\backslash\{1\}$ is valid.

Proof. It is obtained from [28] Proposition 1.3 and Theorem 9.

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Conflicts of interest

There are no conflicts of interest in this work.

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