Cumhuriyet Sci. J., 44(1) (2023) 170-180
DOI: https://doi.org/10.17776/csj.1208016

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

α −Integral Representation of The Solution for A Conformable Fractional Diffusion Operator and Basic Properties of The Operator

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Research Article

History
Received: 21/11/2022
Accepted: 06/03/2023

ABSTRACT

In this paper, we consider a diffusion operator which includes conformable fractional derivatives of order α (0<α≤1) instead of the ordinary derivatives in a traditional diffusion operator. We give an α-integral representation for the solution of this operator and obtain the conditions provided by the kernel functions in this representation. Also, by investigating the basic properties of this operator, we obtain the asymptotics of the data $\{\lambda_n, \alpha_n\}$, which are called the spectral data of the operator.

Keywords: Diffusion operator, Integral representation, Conformable fractional derivative, Spectral data.

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Introduction

In 2014, the concept of conformable fractional derivative is firstly introduced by Khalil et al. (see [1]). Then, many researchers showed the basic properties of this new derivative in [2-5]. In 2017, Zhao and Luo gave a physical interpretation of this derivative (see [6]). The conformable fractional derivative has been seen in various fields such as diffusion transport, Newton mechanics and arbitrary time scale problems (see [7-9]). In [10], it has been understood that this derivative is necessary and useful for generating new types of fractional operators. Recently, important studies for various operators with conformable fractional derivatives have been published (see [11-15]).

In 1981, Gasymov and Guseinov gave an integral representation of the solution of a diffusion operator and also showed the properties which provided by kernel functions (see [16]). This integral representation for the diffusion operator is so important that many researchers have made various spectral studies by using this representation. For example, in 2007, Koyunbakan and Panakhov gave the solution of the Half inverse problem in [17], and in 2010, Yang calculated the regularized trace in [18]. In current literature, there is no such integral representation for a diffusion operator with conformable fractional derivative and this study can be considered as α generalization of the representation in [16].

α −Integral Representation of the Solution

We consider a diffusion operator with discrete boundary conditions that include conformable fractional derivatives of order α $(0<\alpha\le 1)$ instead of the ordinary derivatives in a traditional diffusion operator. The operator $L_\alpha = L_\alpha\big(h, H, p(x), q(x)\big)$ is called as conformable fractional diffusion operator (CFDO) and is the form

$$\ell_\alpha y := -T_x^\alpha T_x^\alpha y + [2\lambda p(x) + q(x)]y = \lambda^2 y,\ \ 0<x<\pi \quad (1)$$

$$U_\alpha(y) := T_x^\alpha y(0) - hy(0) = 0 \quad (2)$$

$$V_\alpha(y) := T_x^\alpha y(\pi) + Hy(\pi) = 0 \quad (3)$$

where λ is the spectral parameter, $h,\ H \in \mathbb{R}$, $q(x) \in W_{2,\alpha}^1[0,\pi]$, $p(x) \in W_{2,\alpha}^2[0,\pi]$ are real valued functions, $p(x) \neq constant$ and $T_x^\alpha y$ is a conformable fractional derivative of order α of y at x, $\alpha \in (0,1]$.

In this section, we obtain an integral representation for the solution of this operator and show the conditions provided by the kernel functions in this representation.

Firstly, let's remember some important concepts of conformable fractional calculus. We note that more detailed knowledge about conformable fractional calculus can be seen in [1], [2], and [19].

Definition 2.1 Let $f : [0,\infty) \to \mathbb{R}$ be a given function. Then, the conformable fractional derivative of f of order α with respect to x is defined by

$$T_x^\alpha f(x) = \lim_{h\to 0} \frac{f(x+hx^{1-\alpha}) - f(x)}{h}, T_x^\alpha f(0) = \lim_{x\to 0^+} T_x^\alpha f(x),$$

for all $x>0$, $\alpha \in (0,1]$. If f is differentiable that is $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, then $T_x^\alpha f(x) = x^{1-\alpha} f'(x)$.

Definition 2.2 The conformable fractional integral from 0 to x of order α is defined as follows

$$I_\alpha f(x) = \int_0^x f(t) d_\alpha t = \int_0^x t^{\alpha-1} f(t) dt, \text{for all } x > 0.$$

Definition 2.3 (α −integration by parts) Let $f, g: [a, b] \to \mathbb{R}$ be α −differentiable functions. Then,

$$\int_a^b f(x) T_x^\alpha g(x) d_\alpha x = f(x)g(x)|_a^b - \int_a^b g(x) T_x^\alpha f(x) d_\alpha x.$$

Lemma 2.4 (α −Leibniz rule) Let $f(x,t)$ be a function such that $t^{\alpha-1}f(x,t)$ and $t^{\alpha-1}f_x(x,t)$ are continuous in t and x in some regions of the (x,t)-plane, including $a(x) \le t \le b(x)$, $x_0 \le x \le x_1$. If $a(x)$ and $b(x)$ are α −differentiable functions for $x_0 \le x \le x_1$, then,

$$T_x^\alpha \left(\int_{a(x)}^{b(x)} f(x,t) d_\alpha t \right) = T_x^\alpha b(x) f\big(x, b(x)\big) b^{\alpha-1}(x) - T_x^\alpha a(x) f\big(x, a(x)\big) a^{\alpha-1}(x) + \int_{a(x)}^{b(x)} T_x^\alpha f(x,t) d_\alpha t.$$

Definition 2.5 Let $1 \le p < \infty$, $a > 0$. We called the space $L_{p,\alpha}(0,a)$ if for all functions $f: [0,a] \to \mathbb{R}$ satisfies

$$\left(\int_0^a |f(x)|^p d_\alpha x \right)^{1/p} < \infty.$$

Lemma 2.6 The space $L_{p,\alpha}(0,a)$ associated with the norm function

$$\|f\|_{p,\alpha} := \left(\int_0^a |f(x)|^p d_\alpha x \right)^{1/p}$$

is a Banach space. Moreover if $p = 2$ then $L_{2,\alpha}(0,a)$ associated with the inner product for $f, g \in L_{2,\alpha}(0,a)$

$$\langle f, g \rangle := \int_0^a f(x) \overline{g(x)} d_\alpha x$$

is a Hilbert space.

Definition 2.7 Let $1 \le p < \infty$. We called the Sobolev space $W_{p,\alpha}^1[0,a]$ if for all functions on $[0,a]$ such that $f(x)$ is absolutely continuous and $T_x^\alpha f(x) \in L_{p,\alpha}(0,a)$.

Now, let $\varphi(x,\lambda;\alpha)$ be the solution of the equation (1) satisfying the initial conditions

$$\varphi(0,\lambda;\alpha) = 1, \; T_x^\alpha \varphi(0,\lambda;\alpha) = h. \tag{4}$$

Theorem 2.8 There are the functions $A\left(x, \frac{t^\alpha}{\alpha}\right)$ and $B\left(x, \frac{t^\alpha}{\alpha}\right)$ whose second order partial derivatives are summable on $[0,\pi]$ for each $x \in [0,\pi]$ and fixed α such that the representation

$$\varphi(x,\lambda;\alpha) = \cos\left(\lambda \frac{x^\alpha}{\alpha} - \theta(x)\right) + \int_0^x A\left(x, \frac{t^\alpha}{\alpha}\right) \cos\lambda \frac{t^\alpha}{\alpha} d_\alpha t + \int_0^x B\left(x, \frac{t^\alpha}{\alpha}\right) \sin\lambda \frac{t^\alpha}{\alpha} d_\alpha t \tag{5}$$

is provided, where the functions $A\left(x, \frac{t^\alpha}{\alpha}\right)$ and $B\left(x, \frac{t^\alpha}{\alpha}\right)$ satisfy the following system of partial differential equations

$$\begin{cases} T_x^\alpha T_x^\alpha A\left(x, \frac{t^\alpha}{\alpha}\right) - q(x) A\left(x, \frac{t^\alpha}{\alpha}\right) - 2p(x) T_t^\alpha B\left(x, \frac{t^\alpha}{\alpha}\right) = T_t^\alpha T_t^\alpha A\left(x, \frac{t^\alpha}{\alpha}\right) \\ T_x^\alpha T_x^\alpha B\left(x, \frac{t^\alpha}{\alpha}\right) - q(x) B\left(x, \frac{t^\alpha}{\alpha}\right) + 2p(x) T_t^\alpha A\left(x, \frac{t^\alpha}{\alpha}\right) = T_t^\alpha T_t^\alpha B\left(x, \frac{t^\alpha}{\alpha}\right). \end{cases} \tag{6}$$

Moreover, the following relations

$$B(x,0) = 0, \; T_t^\alpha A\left(x, \frac{t^\alpha}{\alpha}\right)\Big|_{t=0} = 0, \tag{7}$$

$$\theta(x)=\int_0^x p(t)d_\alpha t, \tag{8}$$

$$A(0,0)=h, \tag{9}$$

$$A\left(x,\frac{x^\alpha}{\alpha}\right)\cos\theta(x)+B\left(x,\frac{x^\alpha}{\alpha}\right)\sin\theta(x)=h+\frac{1}{2}\int_0^x \left(q(t)+p^2(t)\right)d_\alpha t, \tag{10}$$

$$\theta(x)=p(0)\frac{x^\alpha}{\alpha}+2\int_0^x \left(A\left(s,\frac{s^\alpha}{\alpha}\right)\sin\theta(s)-B\left(s,\frac{s^\alpha}{\alpha}\right)\cos\theta(s)\right)d_\alpha s \tag{11}$$

are satisfied. Conversely, if the second order derivatives of functions $A\left(x,\frac{t^\alpha}{\alpha}\right)$ and $B\left(x,\frac{t^\alpha}{\alpha}\right)$ are summable on $[0,\pi]$ for each $x\in[0,\pi]$, fixed α and these functions satisfy the equalities (6) and relations (7)-(11), then $\varphi(x,\lambda;\alpha)$ is a solution of equation (1) satifying the initial conditions (4).

Proof. From (5) and α −Leibniz rule, we get

$$\begin{aligned} T_x^\alpha\varphi(x,\lambda;\alpha)= & -\left(\lambda-T_x^\alpha\theta(x)\right)\sin\left(\lambda\frac{x^\alpha}{\alpha}-\theta(x)\right)+A\left(x,\frac{x^\alpha}{\alpha}\right)\cos\lambda\frac{x^\alpha}{\alpha}+B\left(x,\frac{x^\alpha}{\alpha}\right)\sin\lambda\frac{x^\alpha}{\alpha} \\ & +\int_0^x\left(T_x^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\right)\cos\lambda\frac{t^\alpha}{\alpha}d_\alpha t+\int_0^x\left(T_x^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\right)\sin\lambda\frac{t^\alpha}{\alpha}d_\alpha t \end{aligned} \tag{12}$$

and

$$\begin{aligned} T_x^\alpha T_x^\alpha\varphi(x,\lambda;\alpha)= & \left(T_x^\alpha T_x^\alpha\theta(x)\right)\sin\left(\lambda\frac{x^\alpha}{\alpha}-\theta(x)\right)-\left(\lambda-T_x^\alpha\theta(x)\right)^2\cos\left(\lambda\frac{x^\alpha}{\alpha}-\theta(x)\right) \\ & +\left(T_x^\alpha A\left(x,\frac{x^\alpha}{\alpha}\right)\right)\cos\lambda\frac{x^\alpha}{\alpha}-\lambda A\left(x,\frac{x^\alpha}{\alpha}\right)\sin\lambda\frac{x^\alpha}{\alpha} \\ & +T_x^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x}\cos\lambda\frac{x^\alpha}{\alpha}+\int_0^x\left(T_x^\alpha T_x^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\right)\cos\lambda\frac{t^\alpha}{\alpha}d_\alpha t \\ & +\left(T_x^\alpha B\left(x,\frac{x^\alpha}{\alpha}\right)\right)\sin\lambda\frac{x^\alpha}{\alpha}+\lambda B\left(x,\frac{x^\alpha}{\alpha}\right)\cos\lambda\frac{x^\alpha}{\alpha} \\ & +T_x^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x}\sin\lambda\frac{x^\alpha}{\alpha}+\int_0^x\left(T_x^\alpha T_x^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\right)\sin\lambda\frac{t^\alpha}{\alpha}d_\alpha t. \end{aligned} \tag{13}$$

On the other hand, using α −integration by parts twice for the integrals at (5), we obtain

$$\begin{aligned} \varphi(x,\lambda;\alpha)= & \cos\left(\lambda\frac{x^\alpha}{\alpha}-\theta(x)\right)+\frac{1}{\lambda}A\left(x,\frac{x^\alpha}{\alpha}\right)\sin\lambda\frac{x^\alpha}{\alpha}-\frac{1}{\lambda}B\left(x,\frac{x^\alpha}{\alpha}\right)\cos\lambda\frac{x^\alpha}{\alpha}+\frac{1}{\lambda}B(x,0) \\ & +\frac{1}{\lambda^2}T_t^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x}\cos\lambda\frac{x^\alpha}{\alpha}-\frac{1}{\lambda^2}T_t^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=0} \\ & -\frac{1}{\lambda^2}\int_0^x\left(T_t^\alpha T_t^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\right)\cos\lambda\frac{t^\alpha}{\alpha}d_\alpha t+\frac{1}{\lambda^2}T_t^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x}\sin\lambda\frac{x^\alpha}{\alpha} \\ & -\frac{1}{\lambda^2}\int_0^x\left(T_t^\alpha T_t^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\right)\sin\lambda\frac{t^\alpha}{\alpha}d_\alpha t. \end{aligned} \tag{14}$$

From the equalities (1), (5), (13), (14) and the following equalities

$$2\lambda p(x)\int_0^x A\left(x,\frac{t^\alpha}{\alpha}\right)\cos\lambda\frac{t^\alpha}{\alpha}d_\alpha t = 2p(x)A\left(x,\frac{x^\alpha}{\alpha}\right)\sin\lambda\frac{x^\alpha}{\alpha} - \int_0^x 2p(x)\left(T_t^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\right)\sin\lambda\frac{t^\alpha}{\alpha}d_\alpha t,$$

$$2\lambda p(x)\int_0^x B\left(x,\frac{t^\alpha}{\alpha}\right)\sin\lambda\frac{t^\alpha}{\alpha}d_\alpha t = -2p(x)B\left(x,\frac{x^\alpha}{\alpha}\right)\cos\lambda\frac{x^\alpha}{\alpha} + 2p(x)B(x,0) + \int_0^x 2p(x)\left(T_t^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\right)\cos\lambda\frac{t^\alpha}{\alpha}d_\alpha t,$$

we get

$$\begin{aligned}
&\lambda\cos\lambda\frac{x^\alpha}{\alpha}\left(-2T_x^\alpha\theta(x) + 2p(x)\right)\cos\theta(x) + \lambda\sin\lambda\frac{x^\alpha}{\alpha}\left(-2T_x^\alpha\theta(x) + 2p(x)\right)\sin\theta(x)\\
&+\cos\lambda\frac{x^\alpha}{\alpha}\Big\{\left(T_x^\alpha T_x^\alpha\theta(x)\right)\sin\theta(x) + \left(T_x^\alpha\theta(x)\right)^2\cos\theta(x) - T_x^\alpha A\left(x,\frac{x^\alpha}{\alpha}\right)\\
&-T_x^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x} + q(x)\cos\theta(x) - T_t^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x} - 2p(x)B\left(x,\frac{x^\alpha}{\alpha}\right)\Big\}\\
&+\sin\lambda\frac{x^\alpha}{\alpha}\Big\{-\left(T_x^\alpha T_x^\alpha\theta(x)\right)\cos\theta(x) + \left(T_x^\alpha\theta(x)\right)^2\sin\theta(x) - T_x^\alpha B\left(x,\frac{x^\alpha}{\alpha}\right)\\
&-T_x^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x} + q(x)\sin\theta(x) - T_t^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x} + 2p(x)A\left(x,\frac{x^\alpha}{\alpha}\right)\Big\}\\
&-\lambda B(x,0) + T_t^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=0} + 2p(x)B(x,0)\\
&+\int_0^x\left[T_t^\alpha T_t^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right) - T_x^\alpha T_x^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right) + q(x)A\left(x,\frac{t^\alpha}{\alpha}\right) + 2p(x)T_t^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\right]\cos\lambda\frac{t^\alpha}{\alpha}d_\alpha t\\
&+\int_0^x\left[T_t^\alpha T_t^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right) - T_x^\alpha T_x^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right) + q(x)B\left(x,\frac{t^\alpha}{\alpha}\right) - 2p(x)T_t^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\right]\sin\lambda\frac{t^\alpha}{\alpha}d_\alpha t = 0.
\end{aligned}$$

Since the set of $\left\{\cos\lambda\frac{x^\alpha}{\alpha}, \sin\lambda\frac{x^\alpha}{\alpha}\right\}$ is entire system for each fixed α and $p(x) \neq 0$, we immediately obtain equations (6) and (7).

From the system $\begin{cases}\left(-2T_x^\alpha\theta(x) + 2p(x)\right)\cos\theta(x) = 0\\ \left(-2T_x^\alpha\theta(x) + 2p(x)\right)\sin\theta(x) = 0\end{cases}$, the equation $T_x^\alpha\theta(x) - p(x) = 0$ and hence (8) is taken.

From the equalities $T_x^\alpha\theta(x) = p(x)$, $d_\alpha A\left(x,\frac{t^\alpha}{\alpha}\right) = T_x^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)d_\alpha x + T_t^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)d_\alpha t$ and the following system

$$\begin{cases}
\left(T_x^\alpha T_x^\alpha\theta(x)\right)\sin\theta(x) + \left[\left(T_x^\alpha\theta(x)\right)^2 + q(x)\right]\cos\theta(x) - 2p(x)B\left(x,\frac{x^\alpha}{\alpha}\right)\\
-T_x^\alpha A\left(x,\frac{x^\alpha}{\alpha}\right) - T_x^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x} - T_t^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x} = 0\\
-\left(T_x^\alpha T_x^\alpha\theta(x)\right)\cos\theta(x) + \left[\left(T_x^\alpha\theta(x)\right)^2 + q(x)\right]\sin\theta(x) + 2p(x)A\left(x,\frac{x^\alpha}{\alpha}\right)\\
-T_x^\alpha B\left(x,\frac{x^\alpha}{\alpha}\right) - T_x^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x} - T_t^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{t=x} = 0
\end{cases}$$

we get

$$\begin{cases}
T_x^\alpha A\left(x,\frac{x^\alpha}{\alpha}\right) + p(x)B\left(x,\frac{x^\alpha}{\alpha}\right) - \frac{1}{2}\left(T_x^\alpha p(x)\right)\sin\theta(x) - \frac{1}{2}\left(q(x) + p^2(x)\right)\cos\theta(x) = 0\\
T_x^\alpha B\left(x,\frac{x^\alpha}{\alpha}\right) - p(x)A\left(x,\frac{x^\alpha}{\alpha}\right) + \frac{1}{2}\left(T_x^\alpha p(x)\right)\cos\theta(x) - \frac{1}{2}\left(q(x) + p^2(x)\right)\sin\theta(x) = 0
\end{cases}.$$

In the above system, if the first equation is multiplied by $\cos\theta(x)$ and the second equation by $\sin\theta(x)$ and added side by side, we obtain

$$T_x^\alpha\left[A\left(x,\frac{x^\alpha}{\alpha}\right)\cos\theta(x) + B\left(x,\frac{x^\alpha}{\alpha}\right)\sin\theta(x)\right] = \frac{1}{2}\left(q(x) + p^2(x)\right)$$

or

$$A\left(x,\frac{x^{\alpha}}{\alpha}\right)\cos\theta(x)+B\left(x,\frac{x^{\alpha}}{\alpha}\right)\sin\theta(x)-A(0,0)=\frac{1}{2}\int_{0}^{x}\left(q(t)+p^{2}(t)\right)d_{\alpha}t.$$

Moreover, from equalities (4), (12), (7) and (8), we have that

$$T_x^{\alpha}\varphi(0,\lambda;\alpha)=A(0,0)=h.$$

Thus, equations (9) and (10) are obtained.

On the other hand, if the first equation is multiplied by $\sin\theta(x)$ and the second equation by $\left(-\cos\theta(x)\right)$ and added side by side, we obtain

$$T_x^{\alpha}\left[A\left(x,\frac{x^{\alpha}}{\alpha}\right)\sin\theta(x)-B\left(x,\frac{x^{\alpha}}{\alpha}\right)\cos\theta(x)\right]=\frac{1}{2}T_x^{\alpha}p(x)$$

or

$$A\left(x,\frac{x^{\alpha}}{\alpha}\right)\sin\theta(x)-B\left(x,\frac{x^{\alpha}}{\alpha}\right)\cos\theta(x)+B(0,0)=\frac{1}{2}\left(p(x)-p(0)\right).$$

From equality (7), we have that

$$B(0,0)=0.$$

Thus, equation (11) is obtained.

Basic Properties of the Operator L_{α}

In this section, we investigate some spectral properties of the operator L_{α} by supposing that the function $q(x)$ satisfies the additional condition

$$\int_0^{\pi}[|T_x^{\alpha}y(x)|^2+q(x)|y(x)|^2]d_{\alpha}x>0 \tag{15}$$

for all $y(x)\in W_{2,\alpha}^2[0,\pi]$ such that $y(x)\neq 0$ and

$$\left(T_x^{\alpha}y(0)\right)\overline{y}(0)-\left(T_x^{\alpha}y(\pi)\right)\overline{y}(\pi)=0. \tag{16}$$

Let the functions $\varphi:=\varphi(x,\lambda;\alpha)$ and $\psi:=\psi(x,\lambda;\alpha)$ be the solutions of the equation (1) satisfying the initial conditions (4) and

$$\psi(\pi,\lambda;\alpha)=1,T_x^{\alpha}\psi(\pi,\lambda;\alpha)=-H \tag{17}$$

respectively. It is clear that

$$U_{\alpha}(\varphi)=0,V_{\alpha}(\psi)=0.$$

We denote

$$\Delta_{\alpha}(\lambda)=W_{\alpha}[\psi,\varphi]=\begin{vmatrix}\psi & \varphi\\ T_x^{\alpha}\psi & T_x^{\alpha}\varphi\end{vmatrix}=\psi T_x^{\alpha}\varphi-\varphi T_x^{\alpha}\psi. \tag{18}$$

The function $\Delta_{\alpha}(\lambda)$ is called the characteristic function for the operator L_{α}, where $W_{\alpha}[\psi,\varphi]$ is the fractional Wronskian of the functions ψ and φ. Obviously, the function $\Delta_{\alpha}(\lambda)$ is entire function in λ.

Lemma 3.1 For each fixed α, $\Delta_{\alpha}(\lambda)$ does not depend on x and can be written as

$$\Delta_{\alpha}(\lambda)=V_{\alpha}(\varphi)=-U_{\alpha}(\psi). \tag{19}$$

Proof. It is clear from (18) that

$$T_x^\alpha \Delta_\alpha(\lambda) = T_x^\alpha(\psi T_x^\alpha \varphi - \varphi T_x^\alpha \psi) = \psi T_x^\alpha T_x^\alpha \varphi - \varphi T_x^\alpha T_x^\alpha \psi. \quad (20)$$

On the other hand, since the functions φ and ψ are the solutions of equation (1), the following equations are obtained

$$T_x^\alpha T_x^\alpha \varphi = (2\lambda p(x) + q(x) - \lambda^2)\varphi,$$

$$T_x^\alpha T_x^\alpha \psi = (2\lambda p(x) + q(x) - \lambda^2)\psi.$$

If these equations are substituted in (20)

$$T_x^\alpha \Delta_\alpha(\lambda) = 0$$

is obtained. Thus, the function $\Delta_\alpha(\lambda)$ is a constant with respect to x in $[0, \pi]$.

Moreover, if $x = 0$ and $x = \pi$ are substituted in equation (18) and conditions (4) and (17) are taken into account, then (19) is immediately taken.

Lemma 3.2 The zeros $\{\lambda_n\}$ of the function $\Delta_\alpha(\lambda)$ coincide with the eigenvalues of the operator L_α and for eigenfunctions $\psi_n := \psi(x, \lambda_n; \alpha)$ and $\varphi_n := \varphi(x, \lambda_n; \alpha)$ there exists a sequence $\{\beta_n\}$, such that the following relations are satisfied for each fixed α

$$\psi_n = \beta_n \varphi_n, \ \beta_n \neq 0. \quad (21)$$

Proof. Let λ_0 be an eigenvalue of the operator L_α, we show that $\Delta_\alpha(\lambda_0) = 0$. We suppose that $\Delta_\alpha(\lambda_0) \neq 0$. Then the functions $\varphi(x, \lambda_0; \alpha)$ and $\psi(x, \lambda_0; \alpha)$ are linearly independent. Thus

$$y(x, \lambda_0; \alpha) = c_1 \psi(x, \lambda_0; \alpha) + c_2 \varphi(x, \lambda_0; \alpha)$$

is a general solution of the operator L_α corresponding to $\lambda = \lambda_0$ for constants c_1 and c_2. Hence, the above equation can written for $c_1 \neq 0$ as

$$\psi(x, \lambda_0; \alpha) = \frac{1}{c_1} y(x, \lambda_0; \alpha) - \frac{c_2}{c_1} \varphi(x, \lambda_0; \alpha).$$

Thus, we get

$$\Delta_\alpha(\lambda_0) = W_\alpha[\psi(x, \lambda_0; \alpha), \varphi(x, \lambda_0; \alpha)] = \frac{1}{c_1} W_\alpha[y(x, \lambda_0; \alpha), \varphi(x, \lambda_0; \alpha)].$$

If the initial conditions (4) and especially $x = 0$ are taken into account in this equation, $\Delta_\alpha(\lambda_0) = 0$ contradiction is obtained for each fixed α.

On the other hand, let λ_0 be a zero of the function $\Delta_\alpha(\lambda)$. Then $\Delta_\alpha(\lambda_0) = 0$. So, we get $\psi(x, \lambda_0; \alpha) = \beta_0 \varphi(x, \lambda_0; \alpha)$ for $\beta_0 \neq 0$. Furthermore, the functions $\varphi(x, \lambda_0; \alpha)$ and $\psi(x, \lambda_0; \alpha)$ satisfy the initial conditions (4) and (17), respectively. So, the functions $\varphi(x, \lambda_0; \alpha)$ and $\psi(x, \lambda_0; \alpha)$ are eigenfunctions related to λ_0.

Since the eigenfunctions corresponding to each eigenvalue differ from each other by a multiplicative constant, there is a sequence $\{\beta_n\}$ such that the equality (21) is satisfied for each $n \in \mathbb{N}$ and fixed α.

Lemma 3.3 The eigenvalues of the operator L_α are real and nonzero for each fixed α.

Proof. This lemma can be proved similarly as in [20].

Denote a linear operator $L_{\alpha,0}$ with the following differential expression

$$\ell_{\alpha,0} y(x) := -T_x^\alpha T_x^\alpha y(x) + q(x) y(x)$$

and boundary conditions (2) and (3), where $y(x) \in W_{2,\alpha}^2[0, \pi]$.

Using α −integration by part and the condition (15), we get

$$\langle \ell_{\alpha,0} y, y \rangle = \int_0^\pi (\ell_{\alpha,0} y) \overline{y} d_\alpha x = \int_0^\pi [|T_x^\alpha y(x)|^2 + q(x)|y(x)|^2] d_\alpha x > 0.$$

Let λ_0 be an eigenvalue of the operator L_α and $y_0 = y(x, \lambda_0; \alpha)$ an eigenfunction corresponding to this eigenvalue and normalized by $\langle y_0, y_0 \rangle = 1$.

It is clear that

$$-T_x^\alpha T_x^\alpha y_0 + \big(2\lambda_0 p(x) + q(x)\big) y_0 = \lambda_0^2 y_0$$

or

$$\lambda_0^2 y_0 - 2\lambda_0 p(x) y_0 - \ell_{\alpha,0} y_0 = 0. \tag{22}$$

If we take the inner product of both sides of the equation (22) by y_0, we obtain

$$\lambda_0^2 - 2\lambda_0 \langle p(x) y_0, y_0 \rangle - \langle \ell_{\alpha,0} y_0, y_0 \rangle = 0.$$

Hence

$$\lambda_0 = \langle p(x) y_0, y_0 \rangle - \sqrt{\langle p(x) y_0, y_0 \rangle^2 + \langle \ell_{\alpha,0} y_0, y_0 \rangle}.$$

Since $p(x)$ is real and $\langle \ell_{\alpha,0} y, y \rangle > 0$, the proof is completed from the last relation.

Lemma 3.4 The eigenfunctions $y_n := y(x, \lambda_n; \alpha)$ and $y_k := y(x, \lambda_k; \alpha)$ corresponding to the eigenvalues λ_n and λ_k *($\lambda_n \neq \lambda_k$)* of the operator L_α are orthogonal in the sense of

$$(\lambda_n + \lambda_k) \int_0^\pi y_n y_k d_\alpha x - 2 \int_0^\pi p(x) y_n y_k d_\alpha x = 0. \tag{23}$$

Proof. Take into our account that the operator $L_{\alpha,0}$ is symmetric and

$$\ell_{\alpha,0} y(x) := -T_x^\alpha T_x^\alpha y(x) + q(x) y(x) = \lambda^2 y(x) - 2\lambda p(x) y(x).$$

We have that

$$\langle \ell_{\alpha,0} y_n, y_k \rangle = \langle y_n, \ell_{\alpha,0} y_k \rangle,$$

$$\int_0^\pi (\lambda_n^2 y_n - 2\lambda_n p(x) y_n) y_k d_\alpha x = \int_0^\pi y_n (\lambda_k^2 y_k - 2\lambda_k p(x) y_k) d_\alpha x,$$

$$(\lambda_n^2 - \lambda_k^2) \int_0^\pi y_n y_k d_\alpha x - 2(\lambda_n - \lambda_k) \int_0^\pi p(x) y_n y_k d_\alpha x = 0.$$

By virtue of $\lambda_n \neq \lambda_k$ the equality (23) is obtained.

Definition 3.5 Let φ_n be the eigenfunction of the operator L_α corresponding to the eigenvalues λ_n. The numbers

$$\alpha_n = \int_0^\pi \varphi_n^2 d_\alpha x - \frac{1}{\lambda_n} \int_0^\pi p(x) \varphi_n^2 d_\alpha x \tag{24}$$

are called the normalizing numbers of the operator L_α and the data $\{\lambda_n, \alpha_n\}$ are also called the spectral data of the operator L_α.

Lemma 3.6 The equality $\dot{\Delta}_\alpha(\lambda_n) = -2\lambda_n \beta_n \alpha_n$ is valid and the eigenvalues of the operator L_α are simple, i.e., $\dot{\Delta}_\alpha(\lambda_n) \neq 0$, where $\dot{\Delta}_\alpha(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \Delta_\alpha(\lambda)$.

Proof. Since ψ and φ_n are the solutions of the equation (1), the following equalities

$$-T_x^\alpha T_x^\alpha \psi + \big(2\lambda p(x) + q(x)\big) \psi = \lambda^2 \psi,$$

$$-T_x^\alpha T_x^\alpha \varphi_n + \big(2\lambda_n p(x) + q(x)\big) \varphi_n = \lambda_n^2 \varphi_n$$

are held.

If the first equation is multiplied by φ_n, the second equation is multiplied by ψ and substracting them side by side, then the equality

$$T_x^\alpha W_\alpha[\psi,\varphi_n] = (\lambda^2-\lambda_n^2)\varphi_n\psi - 2(\lambda-\lambda_n)p(x)\varphi_n\psi$$

is obtained. By conformable fractional integrating above equation over $[0,\pi]$ and taking into account (17) and (19), we get

$$(\lambda^2-\lambda_n^2)\int_0^\pi \varphi_n\psi d_\alpha x - 2(\lambda-\lambda_n)\int_0^\pi p(x)\varphi_n\psi d_\alpha x = W_\alpha[\psi,\varphi_n]|_0^\pi = -\Delta_\alpha(\lambda).$$

Hence

$$(\lambda+\lambda_n)\int_0^\pi \varphi_n\psi d_\alpha x - 2\int_0^\pi p(x)\varphi_n\psi d_\alpha x = -\frac{\Delta_\alpha(\lambda)}{\lambda-\lambda_n}.$$

Passing to the limit as $\lambda\to\lambda_n$ in the last equation, it yields

$$2\lambda_n\int_0^\pi \varphi_n\psi_n d_\alpha x - 2\int_0^\pi p(x)\varphi_n\psi_n d_\alpha x = -\lim_{\lambda\to\lambda_n}\frac{\Delta_\alpha(\lambda)}{\lambda-\lambda_n} = -\dot{\Delta}_\alpha(\lambda_n).$$

From (21) and (24), we have that

$$\dot{\Delta}_\alpha(\lambda_n) = -2\lambda_n\beta_n\alpha_n. \tag{25}$$

It is obvious from (25) that $\dot{\Delta}(\lambda_n)\neq 0$ for $\lambda_n\neq 0$, that is, the eigenvalues are simple. Therefore, the proof is completed.

Theorem 3.7 The operator L_α has a countable set of eigenvalues $\{\lambda_n\}$ and the following estimate holds:

$$\lambda_n = \frac{n\alpha}{\pi^{\alpha-1}} + c_{\alpha,0} + \frac{c_{\alpha,1}}{n} + o\left(\frac{1}{n}\right), |n|\to\infty, \tag{26}$$

where

$$c_{\alpha,0} = \frac{\alpha}{\pi^\alpha}\int_0^\pi p(x)d_\alpha x,\quad c_{\alpha,1} = \frac{1}{\pi}\left[h + H + \frac{1}{2}\int_0^\pi \left(q(x)+p^2(x)\right)d_\alpha x\right].$$

Proof. According to (19) the relation $\Delta_\alpha(\lambda)$, with the help of (5) and (12), as

$$\begin{aligned}\Delta_\alpha(\lambda) &= -(\lambda-p(\pi))\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right) + A\left(\pi,\frac{\pi^\alpha}{\alpha}\right)\cos\lambda\frac{\pi^\alpha}{\alpha}\\ &+B\left(\pi,\frac{\pi^\alpha}{\alpha}\right)\sin\lambda\frac{\pi^\alpha}{\alpha} + H\cos\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right)\\ &+\int_0^\pi T_x^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{x=\pi}\cos\lambda\frac{t^\alpha}{\alpha}d_\alpha t + \int_0^\pi T_x^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{x=\pi}\sin\lambda\frac{t^\alpha}{\alpha}d_\alpha t\\ &+H\int_0^\pi A\left(\pi,\frac{t^\alpha}{\alpha}\right)\cos\lambda\frac{t^\alpha}{\alpha}d_\alpha t + H\int_0^\pi B\left(\pi,\frac{t^\alpha}{\alpha}\right)\sin\lambda\frac{t^\alpha}{\alpha}d_\alpha t\end{aligned} \tag{27}$$

is written.

It follows from (10) and (11)

$$\begin{cases} A\left(\pi,\frac{\pi^\alpha}{\alpha}\right)\cos\theta(\pi)+B\left(\pi,\frac{\pi^\alpha}{\alpha}\right)\sin\theta(\pi)=h+\frac{1}{2}\int_0^\pi\left(q(t)+p^2(t)\right)d_\alpha t \\ A\left(\pi,\frac{\pi^\alpha}{\alpha}\right)\sin\theta(\pi)-B\left(\pi,\frac{\pi^\alpha}{\alpha}\right)\cos\theta(\pi)=\frac{p(\pi)-p(0)}{2} \end{cases}$$

which implies that

$$\begin{cases} A\left(\pi,\frac{\pi^\alpha}{\alpha}\right)=\frac{p(\pi)-p(0)}{2}\sin\theta(\pi)+\left[h+\frac{1}{2}\int_0^\pi\left(q(t)+p^2(t)\right)d_\alpha t\right]\cos\theta(\pi) \\ B\left(\pi,\frac{\pi^\alpha}{\alpha}\right)=\frac{p(0)-p(\pi)}{2}\cos\theta(\pi)+\left[h+\frac{1}{2}\int_0^\pi\left(q(t)+p^2(t)\right)d_\alpha t\right]\sin\theta(\pi) \end{cases}$$

and hence

$$\begin{aligned} &A\left(\pi,\frac{\pi^\alpha}{\alpha}\right)\cos\lambda\frac{\pi^\alpha}{\alpha}+B\left(\pi,\frac{\pi^\alpha}{\alpha}\right)\sin\lambda\frac{\pi^\alpha}{\alpha}=\frac{p(0)-p(\pi)}{2}\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right) \\ &+\left[h+\frac{1}{2}\int_0^\pi\left(q(t)+p^2(t)\right)d_\alpha t\right]\cos\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right). \end{aligned} \tag{28}$$

Take into account (28) in (27), we obtain

$$\begin{aligned} \Delta_\alpha(\lambda)&=-\left(\lambda-p(\pi)\right)\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right)+\frac{p(0)-p(\pi)}{2}\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right) \\ &+\left[h+H+\frac{1}{2}\int_0^\pi\left(q(t)+p^2(t)\right)d_\alpha t\right]\cos\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right) \\ &+\int_0^\pi T_x^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{x=\pi}\cos\lambda\frac{t^\alpha}{\alpha}d_\alpha t+\int_0^\pi T_x^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{x=\pi}\sin\lambda\frac{t^\alpha}{\alpha}d_\alpha t \\ &+H\left[\int_0^\pi A\left(\pi,\frac{t^\alpha}{\alpha}\right)\cos\lambda\frac{t^\alpha}{\alpha}d_\alpha t+\int_0^\pi B\left(\pi,\frac{t^\alpha}{\alpha}\right)\sin\lambda\frac{t^\alpha}{\alpha}d_\alpha t\right]. \end{aligned} \tag{29}$$

Thus, for $\lambda\neq p(\pi)$ the equation $\Delta_\alpha(\lambda)=0$ as

$$\begin{aligned} &\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right)+\frac{p(\pi)-p(0)}{2}\frac{1}{\lambda-p(\pi)}\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right) \\ &-\frac{1}{\lambda-p(\pi)}\left[h+H+\frac{1}{2}\int_0^\pi\left(q(t)+p^2(t)\right)d_\alpha t\right]\cos\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right)-\frac{\Delta_{\alpha,1}(\lambda)}{\lambda-p(\pi)}=0 \end{aligned}$$

is taken, where

$$\begin{aligned} \Delta_{\alpha,1}(\lambda)&=\int_0^\pi\left[T_x^\alpha A\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{x=\pi}+HA\left(\pi,\frac{t^\alpha}{\alpha}\right)\right]\cos\lambda\frac{t^\alpha}{\alpha}d_\alpha t \\ &+\int_0^\pi\left[T_x^\alpha B\left(x,\frac{t^\alpha}{\alpha}\right)\Big|_{x=\pi}+HB\left(\pi,\frac{t^\alpha}{\alpha}\right)\right]\sin\lambda\frac{t^\alpha}{\alpha}d_\alpha t=o\left(e^{|\operatorname{Im}\lambda|\frac{\pi^\alpha}{\alpha}}\right). \end{aligned}$$

Taking Taylor's expansion formula for the $\frac{1}{\lambda-p(\pi)}=\frac{1}{\lambda}+\frac{p(\pi)}{\lambda^2}+o\left(\frac{1}{\lambda^2}\right)$, $|\lambda|\to\infty$ into account, we get

$$\begin{aligned} &\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right)+\frac{p(\pi)-p(0)}{2\lambda}\sin\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right)-\frac{1}{\lambda}\left[h+H+\frac{1}{2}\int_0^\pi\left(q(t)+p^2(t)\right)d_\alpha t\right]\cos\left(\lambda\frac{\pi^\alpha}{\alpha}-\theta(\pi)\right) \\ &+o\left(\frac{1}{\lambda}e^{|\operatorname{Im}\lambda|\frac{\pi^\alpha}{\alpha}}\right)=0 \end{aligned} \tag{30}$$

We take a circle $\Gamma_n = \left\{\lambda \mid \left|\lambda - c_{\alpha,0}\right| \le \frac{\alpha}{\pi^{\alpha-1}}\left(n+\frac{1}{2}\right), n = 0,1,2,\dots\right\}$ in the λ −plane and define $\Delta_\alpha(\lambda) = f(\lambda;\alpha) + g(\lambda;\alpha)$, where

$$f(\lambda;\alpha) = \sin\left(\lambda\frac{\pi^\alpha}{\alpha} - \theta(\pi)\right)$$

and

$$g(\lambda;\alpha) = \frac{p(\pi)-p(0)}{2\lambda}\sin\left(\lambda\frac{\pi^\alpha}{\alpha} - \theta(\pi)\right) - \frac{1}{\lambda}\left[h + H + \frac{1}{2}\int_0^\pi \left(q(t)+p^2(t)\right)d_\alpha t\right]\cos\left(\lambda\frac{\pi^\alpha}{\alpha} - \theta(\pi)\right) + o\left(\frac{1}{\lambda}e^{|\operatorname{Im}\lambda|\frac{\pi^\alpha}{\alpha}}\right)$$

.

For sufficiently large n and each fixed α, the number of zeros of the function $\Delta_\alpha(\lambda) = f(\lambda;\alpha) + g(\lambda;\alpha)$ in the region Γ_n is the same as the number of zeros of the function $f(\lambda;\alpha)$ from Rouche's theorem (see, e.g., [21]).

Thus, if $\sin\left(\lambda\frac{\pi^\alpha}{\alpha} - \theta(\pi)\right) = 0$, then $\lambda_n = \frac{n\alpha}{\pi^{\alpha-1}} + c_{\alpha,0}$, $n \in \mathbb{Z}\backslash\{0\}$ is taken, where $c_{\alpha,0} = \frac{\alpha}{\pi^\alpha}\theta(\pi) = \frac{\alpha}{\pi^\alpha}\int_0^\pi p(x)d_\alpha x$.

We conclude that $\lambda_n = \frac{n\alpha}{\pi^{\alpha-1}} + c_{\alpha,0} + \varepsilon_n$, $\varepsilon_n = o\left(\frac{1}{n}\right)$ as $|n| \to \infty$. By substituting this into (30), we obtain

$$\sin\left(n\pi + \varepsilon_n\frac{\pi^\alpha}{\alpha}\right) + \frac{p(\pi)-p(0)}{2}\frac{1}{\frac{n\alpha}{\pi^{\alpha-1}} + c_{\alpha,0} + \varepsilon_n}\sin\left(n\pi + \varepsilon_n\frac{\pi^\alpha}{\alpha}\right)$$
$$-\frac{1}{\frac{n\alpha}{\pi^{\alpha-1}} + c_{\alpha,0} + \varepsilon_n}\left[h + H + \frac{1}{2}\int_0^\pi \left(q(t)+p^2(t)\right)d_\alpha t\right]\cos\left(n\pi + \varepsilon_n\frac{\pi^\alpha}{\alpha}\right) + o\left(\frac{1}{n}\right) = 0.$$

This implies that

$$\sin\left(\varepsilon_n\frac{\pi^\alpha}{\alpha}\right) + \frac{p(\pi)-p(0)}{2}\frac{1}{\frac{n\alpha}{\pi^{\alpha-1}} + c_{\alpha,0} + \varepsilon_n}\sin\left(\varepsilon_n\frac{\pi^\alpha}{\alpha}\right) -$$
$$\frac{1}{\frac{n\alpha}{\pi^{\alpha-1}} + c_{\alpha,0} + \varepsilon_n}\left[h + H + \frac{1}{2}\int_0^\pi \left(q(t)+p^2(t)\right)d_\alpha t\right]\cos\left(\varepsilon_n\frac{\pi^\alpha}{\alpha}\right) + o\left(\frac{1}{n}\right) = 0.$$

From the following asymptotics

$$\cos\left(\varepsilon_n\frac{\pi^\alpha}{\alpha}\right) = 1 + o\left(\frac{1}{n^2}\right), \qquad \sin\left(\varepsilon_n\frac{\pi^\alpha}{\alpha}\right) = \varepsilon_n\frac{\pi^\alpha}{\alpha} + o\left(\frac{1}{n^3}\right), \qquad \frac{1}{\frac{n\alpha}{\pi^{\alpha-1}} + c_{\alpha,0} + \varepsilon_n} = \frac{\pi^{\alpha-1}}{n\alpha} + o\left(\frac{1}{n^2}\right)$$

and the above last equation, we get

$$\varepsilon_n = \frac{c_{\alpha,1}}{n} + o\left(\frac{1}{n}\right),$$

where

$$c_{\alpha,1} = \frac{1}{\pi}\left[h + H + \frac{1}{2}\int_0^\pi \left(q(t)+p^2(t)\right)d_\alpha t\right].$$

Thus, (26) is valid, i.e., the proof is completed.

Lemma 3.8 The normalizing numbers α_n of the operator L_α holds the following asymptotic formula

$$\alpha_n = \frac{\pi^\alpha}{2\alpha} + \frac{d_{\alpha,0}}{n} + o\left(\frac{1}{n}\right), \qquad |n| \to \infty, \tag{31}$$

where

$$d_{\alpha,0} = -\frac{\pi^\alpha}{2\alpha}p(0).$$

Proof. The formula (31) obtains from (24) by using the asymptotic formula (26) for λ_n.

Conflicts of interest

The authors declare that they have no conflict of interest.

Acknowledgements

This study will be included in MSc Thesis of Esengül Koç.

References

[1] Khalil R., Al Horania M., Yousefa A., et al., A New Definition of Fractional Derivative, *J. Comput. Appl. Math.*, 264 (2014) 65-70.

[2] Abdeljawad T., On Conformable Fractional Calculus, *J. Comput. Appl. Math.*, 279 (2015) 57-66.

[3] Atangana A., Baleanu D., Alsaedi A., New Properties of Conformable Derivative, *Open Math.*, 13 (2015) 889-898.

[4] Abu Hammad M., Khalil R., Abel's Formula and Wronskian for Conformable Fractional Differential Equations, *Int. J. Differ. Equ. Appl.*, 13 (3) (2014) 177-183.

[5] Birgani O.T., Chandok S., Dedovic N., Radenoviç S., A Note on Some Recent Results of the Conformable Derivative, *Adv. Theory Nonlinear Anal. Appl.*, 3 (1) (2019) 11-17.

[6] Zhao D., Luo M., General Conformable Fractional Derivative and its Physical Interpretation, *Calcolo*, 54 (2017) 903–917

[7] Zhou H.W., Yang S., Zhang S.Q., Conformable Derivative Approach to Anomalous Diffusion, *Physica A*, 491 (2018) 1001-1013.

[8] Chung W.S., Fractional Newton Mechanics with Conformable Fractional Derivative, *J. Comput. Appl. Math.*, 290 (2015) 150-158.

[9] Benkhettou N., Hassani S., Torres D.F.M., A Conformable Fractional Calculus on Arbitrary Time Scales, *J. King Saud Univ. Sci.*, 28 (2016) 93–98.

[10] Jarad F., Uğurlu E., Abdeljawad T., Baleanu D., On a New Class of Fractional Operators, *Adv. in Differ. Equ.*, 2017 (2017) Article ID 247 .

[11] Mortazaasl H., Jodayree Akbarfam A,. Trace Formula and Inverse Nodal Problem for a Conformable Fractional Sturm-Liouville Problem, *Inverse Problems in Science and Engineering*, 28 (4) (2020) 524-555.

[12] Keskin B., Inverse Problems for one Dimentional Conformable Fractional Dirac Type Integro Differential System, *Inverse Problems*, 36 (6) (2020) 065001.

[13] Adalar I., Ozkan A.S., Inverse Problems for a Conformable Fractional Sturm-Liouville Operators, *Journal of Inverse and Ill-posed Problems*, 28 (6) (2020).

[14] Çakmak Y., Inverse Nodal Problem for a Conformable Fractional Diffusion Operator, *Inverse Problems in Science and Engineering*, 29 (9) (2021) 1308-1322.

[15] Allahverdiev B.P., Tuna H., Yalçinkaya Y., Conformable Fractional Sturm-Liouville Equation, *Mathematical Methods in the Applied Sciences*, 42 (10) (2019) 3508-3526.

[16] Gasymov M.G., Gusejnov G.S., Determination of a Diffusion Operator from Spectral Data, *Akad. Nauk Azerb. SSR. Dokl.*, 37 (2) (1981) 19-23.

[17] Koyunbakan H., Panakhov E.S., Half-Inverse Problem for Diffusion Operators on the Finite Interval, *J. Math. Anal. Appl.*, 326 (2) (2007) 1024-1030.

[18] Yang C.F., New Trace Formulae for a Quadratic Pencil of the Schrodinger Operator, *Journal of Mathematical Physics*, 51 (3) (2010) 33506-33506-10.

[19] Wang Y., Zhou J., Li Y., Fractional Sobolev's Spaces on Time Scales via Conformable Fractional Calculus and Their Application to a Fractional Differential Equation on Time Scales, *Adv. Math. Phys.*, 2016 (2016) 21.

[20] Guseinov G.Sh., Inverse Spectral Problems for a Quadratic Pencil of Sturm-Liouville Operators on a Finite Interval, *Spectral Theory of Operators and Its Applications, Elm, Baku, Azerbaijan,* 7, (1986) 51–101. (In Russian)

[21] Freiling G., Yurko V.A., Inverse Sturm–Liouville Problems and Their Applications, New York: Nova Science Publishers, (2001).