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The Intersection of Two Ruled Surfaces Corresponding to Spherical Indicatrix Curves on the Unit Dual Sphere

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ABSTRACT

In this study, we first investigate the intersection of two different ruled surfaces in $\mathbb{R}^3$ for two different tangential spherical indicatrix curves on DS^2 using the E. Study mapping. The conditions for the intersection of these ruled surfaces in $\mathbb{R}^3$ are expressed by theorems with bivariate functions. Secondly, considering two different principal normal spherical indicatrix curves on DS^2, we examine the intersection of two different ruled surfaces in $\mathbb{R}^3$ by using E. Study mapping. Similarly, the conditions for the intersection of these ruled surfaces in $\mathbb{R}^3$ are indicated by theorems with bivariate functions. Thirdly, using E. Study mapping, we explore the intersection of two different ruled surfaces in $\mathbb{R}^3$ by considering two different binormal spherical indicatrix curves on DS^2. Likewise, the conditions for the intersection of these ruled surfaces in $\mathbb{R}^3$ are denoted by theorems with bivariate functions. Fourthly, considering two different pole spherical indicatrix curves on DS^2, we study the intersection of two different ruled surfaces in $\mathbb{R}^3$ by using E. Study mapping. In the same way, the conditions for the intersection of these ruled surfaces in $\mathbb{R}^3$ are specified by theorems with bivariate functions. Finally, we provide some examples that support the main results.

Keywords: *Dual space, Dual sphere, Spherical indicatrix curve, Ruled surface, Surface intersection.*

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Introduction

In geometry, surface theory has significant concepts in many disciplines such as surface modeling, kinematics, computer science, etc. One of the most important of these surfaces is the ruled surface. A ruled surface is represented by a straight line that moves along a curve [1]. Many authors have explored the application and geometric interpretation of ruled surfaces, see [2-4]. The surface-surface intersection problem (SSI) is also one of the significant research topics in geometry. B. U. Düldül and M. Çalışkan examined the unit tangent vector of the tangential intersection curve of two surfaces in all three types of SSI problems, see [5]. The intersections of parametric-parametric, implicit-implicit and parametric-implicit combinations of two surfaces are studied in detail, see [6]. For ruled surfaces, which are special cases of surfaces, the intersection of two ruled surfaces has been investigated. Furthermore, each connected component of the surface intersection curve corresponds to a connected component in the zero-set except for some singular points, redundant solutions and degenerate cases, see [7].

Dual numbers were first introduced by W. K. Clifford in 1873. Later, E. Study constructed the correspondence between the geometry of lines and the points of the unit dual sphere. E. Study says that there exists one-to-one correspondence between the oriented lines in Euclidean space and the points on the unit dual sphere. For more details on dual numbers, see [8]. Using the E. Study mapping, the curve on DS^2 corresponds to the ruled surfaces in $\mathbb{R}^3$, see [9-10]. Similarly, the spherical curves on DS^2 correspond to the ruled surfaces in $\mathbb{R}^3$, see [11].

In this article, we examine the intersections of two ruled surfaces corresponding to two different tangential spherical indicatrix curves, two different principal normal spherical indicatrix curves, two different binormal spherical indicatrix curves and two different pole spherical indicatrix curves on DS^2, respectively. Firstly, we consider the intersections of ruled surfaces corresponding to two different tangential spherical indicatrix curves on DS^2. Here, the intersection of the parameter curves of two ruled surfaces is shown by some theorems with the help of bivariate functions. Secondly, we investigate the intersections of ruled surfaces corresponding to two different principal normal spherical indicatrix curves on DS^2. In the same way, the intersection of the parameter curves of two ruled surfaces is given by some theorems with the help of bivariate functions. Thirdly, we discover the intersections of ruled surfaces corresponding to two different binormal spherical indicatrix curves on DS^2. Likewise, the intersection of the parameter curves of two ruled surfaces is expressed by some theorems with the help of bivariate functions. Fourthly, we explore the intersections of ruled surfaces corresponding to two different pole spherical indicatrix curves on DS^2. Similarly, the intersection of the parameter curves of two ruled surfaces is denoted by some theorems with the help of bivariate functions.

This study is organized as follows: In Section 2, we present the properties of dual vectors, dual spherical indicatrix curves, ruled surfaces and the intersection of two ruled surfaces. In Section 3, we examine the intersections of two ruled surfaces corresponding to two different tangential spherical indicatrix curves, two different principal normal spherical indicatrix curves, two different binormal spherical indicatrix curves, two different pole spherical indicatrix curves on the unit dual sphere DS^2, respectively. Additionally, we give some examples to illustrate the main theorems. In Section 4, we discussed the results obtained.

Preliminaries

In this section, we recall some basic definitions and theorems about dual vectors, dual spherical indicatrix curves, ruled surfaces and the intersection of two ruled surfaces.

The set of dual numbers is defined as

$$D = \{X = x + \varepsilon x^*: (x, x^*) \in \mathbb{R} \times \mathbb{R}, \varepsilon^2 = 0\}.$$

The combination of $\vec{x}$ and $\vec{x}^*$is called dual vectors in $\mathbb{R}^3$. These vectors are real part and dual part of $\vec{X}$, respectively. If $\vec{x}$ and $\vec{x}^*$are vectors in $\mathbb{R}^3$, then $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$is defined as dual vector. Assume that $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$and $\vec{Y} = \vec{y} + \varepsilon\vec{y}^*$ are dual vectors. The addition, inner product and vector product are presented as follows:
The addition is

$$\vec{X} + \vec{Y} = (\vec{x} + \vec{y}) + \varepsilon(\vec{x}^* + \vec{y}^*)$$

and their inner product is

$$\langle \vec{X}, \vec{Y} \rangle = \langle \vec{x}, \vec{y} \rangle + \varepsilon(\langle \vec{x}^*, \vec{y} \rangle + \langle \vec{x}, \vec{y}^* \rangle).$$

Also, the vector product is given as

$$\vec{X} \times \vec{Y} = \vec{x} \times \vec{y} + \varepsilon(\vec{x} \times \vec{y}^* + \vec{x}^* \times \vec{y}).$$

The norm of $\vec{X} = \vec{x} + \varepsilon\vec{x}^*$is defined as

$$|\vec{X}| = \sqrt{\langle \vec{x}, \vec{x} \rangle} + \varepsilon \frac{\langle \vec{x}, \vec{x}^* \rangle}{\sqrt{\langle \vec{x}, \vec{x} \rangle}}.$$

The norm of $\vec{X}$ exists only for $\vec{x} \neq 0$. If the norm of $\vec{X}$ is equal to 1, the dual vector is called unit dual vector. The unit dual sphere which consists of the all unit dual vectors is defined as

$$DS^2 = \{\vec{X} = \vec{x} + \varepsilon\vec{x}^* \in D^3: |\vec{X}| = 1\}, \tag{1}$$

for detailed information on dual vectors, see [8].

Theorem 1 (E. Study mapping) There exists one-to-one correspondence between the oriented lines in $\mathbb{R}^3$ and the points of DS^2, see [8].

Theorem 2 Let $\bar{\alpha}(u) = \alpha(u) + \varepsilon\alpha^*(u)$ be a curve on DS^2. In $\mathbb{R}^3$, the ruled surface obtained by the curve $\bar{\alpha}(u)$ can be represented as

$$\phi(u, s) = \alpha(u) \times \alpha^*(u) + s\alpha(u), \tag{2}$$

where $C(u) = \alpha(u) \times \alpha^*(u)$ is the base curve of ϕ, see [9-10].

Definition 1 Let $\bar{\alpha} = \alpha + \varepsilon\alpha^*$ be a curve on DS^2. The dual Frenet vectors, dual curvature, dual torsion, dual Darboux vectors and dual pole vectors of this curve, respectively, can be presented as follows [11]:

$$\bar{T} = \bar{\alpha}', \ \ \bar{N} = \frac{\bar{\alpha}''}{\|\bar{\alpha}''\|}, \ \ \bar{B} = \bar{T} \times \bar{N}, \ \ \bar{\kappa} = \langle \bar{T}', \bar{N} \rangle,$$
$$\bar{\tau} = \langle \bar{N}', \bar{B} \rangle, \ \ \bar{W} = \bar{\tau}.\bar{T} + \bar{\kappa}.\bar{B}, \ \ \bar{C} = \frac{\bar{W}}{\|\bar{W}\|}.$$

Theorem 3 Let $\bar{T}(u) = T(u) + \varepsilon T^*(u)$ be a tangential spherical indicatrix curve on DS^2. In $\mathbb{R}^3$, the ruled surface obtained by the curve $\bar{T}(u)$ can be shown as

$$\phi_{\bar{T}}(u, s) = T(u) \times T^*(u) + sT(u), \tag{3}$$

where $C_{\bar{T}}(u) = T(u) \times T^*(u)$ is the base curve of $\phi_{\bar{T}}$, see [9-11].

Theorem 4 Let $\bar{N}(u) = N(u) + \varepsilon N^*(u)$ be a principal normal spherical indicatrix curve on DS^2. In $\mathbb{R}^3$, the ruled surface obtained by the curve $\bar{N}(u)$ can be given as

$$\phi_{\bar{N}}(u, s) = N(u) \times N^*(u) + sN(u), \tag{4}$$

where $C_{\bar{N}}(u) = N(u) \times N^*(u)$ is the base curve of $\phi_{\bar{N}}$, see [9-11].

Theorem 5 Let $\bar{B}(u) = B(u) + \varepsilon B^*(u)$ be a binormal spherical indicatrix curve on DS^2. In $\mathbb{R}^3$, the ruled surface obtained by the curve $\bar{B}(u)$ can be expressed as

$$\phi_{\bar{B}}(u, s) = B(u) \times B^*(u) + sB(u), \tag{5}$$

where $C_{\bar{B}}(u) = B(u) \times B^*(u)$ is the base curve of $\phi_{\bar{B}}$, see [9-11].

Theorem 6 Let $\bar{C}(u) = C(u) + \varepsilon C^*(u)$ be a pole spherical indicatrix curve on DS^2. In $\mathbb{R}^3$, the ruled surface obtained by the curve $\bar{C}(u)$ can be denoted as

$$\phi_{\bar{C}}(u, s) = C(u) \times C^*(u) + sC(u), \tag{6}$$

where $C_{\bar{C}}(u) = C(u) \times C^*(u)$ is the base curve of $\phi_{\bar{C}}$, see [9-11].

Now, we give some proporties for the intersection curve of two ruled surfaces in $\mathbb{R}^3$.

Let $\phi^1(u, s)$ and $\phi^2(v, t)$ be two ruled surfaces defined by

$$\phi^1(u, s) = \eta_1(u) + se_1(u), \tag{7}$$

$$\phi^2(v, t) = \eta_2(v) + se_2(v). \tag{8}$$

Here, $\eta_1(u)$ and $\eta_2(v)$ are the base curves of surfaces $\phi^1(u,s)$ and $\phi^2(v,t)$, respectively.

The u-parameter curve of $\phi^1(u,s)$ with a constant s_0-parameter and the v-parameter curve of $\phi^2(v,t)$ with a constant t_0-parameter are denoted by $L_1(u)=\phi^1(u,s_0)$ and $L_2(v)=\phi^2(v,t_0)$, respectively, see [7].
If the ruled surfaces $\phi^1(u,s)$ and $\phi^2(v,t)$ intersect,

$$\phi^1(u,s)=\phi^2(v,t),$$

and we get

$$\eta_1(u)-\eta_2(v)=-se_1(u)+te_2(v).$$

Since these three vectors are linearly dependent, the following equation can be represented by [7]:

$$\lambda(u,v)=det\{e_1(u),e_2(v),[\eta_1(u)-\eta_2(v)]\}=0.$$

Theorem 7 Let $\phi^1(u,s)$ and $\phi^2(v,t)$ be two ruled surfaces in $\mathbb{R}^3$. If $\lambda(u,v)=0$, then the parameter curves $L_1(u)$ and $L_2(v)$ intersect, see [7].

In order to $\mu(u,v)=0$, $L_1(u)$ and $L_2(v)$ intersect. But, there are some points in the solution set of the equation $\mu(u,v)=0$ that is not at the intersection of these two ruled surfaces. The intersection contains these points when the directors of the ruled surfaces are $\phi^1(u,s)$ and $\phi^2(v,t)$ parallel. In this way, in the condition of parallelism of the direction vectors of two ruled surfaces, there may be some points that are not at the intersection of two ruled surfaces, see [7].

The parallelism of the direction vectors of the ruled surfaces $\phi^1(u,s)$ and $\phi^2(v,t)$ is expressed with bivariate functions as follows [7] :

$$\Delta(u,v)=\|e_1(u)\times e_2(v)\|^2.$$

The parallelism of $\eta_1(u)-\eta_2(v)$ and the direction vector of $\phi^1(u,s)$ are given by bivariate functions as follows [7] :

$$\delta_1(u,v)=\|e_1(u)\times[\eta_1(u)-\eta_2(v)]\|^2.$$

Similarly, the parallelism of $\eta_1(u)-\eta_2(v)$ and the direction vector of the $\phi^2(v,t)$ are presented with bivariate functions as follows [7] :

$$\delta_2(u,v)=\|e_2(v)\times[\eta_1(u)-\eta_2(v)]\|^2.$$

Theorem 8 The parameter curves $L_1(u)$ and $L_2(v)$ intersect if and only if

$$\Delta(u,v)=\delta_1(u,v)=\delta_2(u,v)=0,$$

see [7].

Intersection of Two Ruled Surfaces Corresponding to Spherical Indicatrix Curves on the Unit Dual Sphere

In this section, we examine the intersections of two ruled surfaces corresponding to two different tangential spherical indicatrix curves, two different principal normal spherical indicatrix curves, two different binormal spherical indicatrix curves, and two different pole spherical indicatrix curves on DS^2, respectively. Furthermore, we give some examples to support the main results.

Let $\bar{\alpha}_1=\alpha_1+\varepsilon\alpha_1^*$ and $\bar{\alpha}_2=\alpha_2+\varepsilon\alpha_2^*$ be two curves on DS^2. The tangential spherical indicatrix curves of these curves are $\bar{T}_1=T_1+\varepsilon T_1^*$ and $\bar{T}_2=T_2+\varepsilon T_2^*$, respectively. According to the E. Study mapping, $\phi_{\bar{T}_1}(u,s)$ and $\phi_{\bar{T}_2}(v,t)$ are the ruled surfaces corresponding to these tangential spherical indicatrix curves shown above in $\mathbb{R}^3$. These ruled surfaces are expressed as follows:

$$\phi_{\bar{T}_1}(u,s)=T_1(u)\times T_1^*(u)+sT_1(u), \tag{9}$$

$$\phi_{\bar{T}_2}(v,t)=T_2(v)\times T_2^*(v)+tT_2(v). \tag{10}$$

Here, the base curves of the ruled surfaces $\phi_{\bar{T}_1}(u,s)$ and $\phi_{\bar{T}_2}(v,t)$ are $C_{\bar{T}_1}(u)=T_1(u)\times T_1^*(u)$ and $C_{\bar{T}_2}(v)=T_2(v)\times T_2^*(v)$, respectively.

The u-parameter curve of $\phi_{\bar{T}_1}(u,s)$ with a constant s_0-parameter is represented by $L_{\bar{T}_1}(u)=\phi_{\bar{T}_1}(u,s_0)$. Similarly, the v-parameter curve of $\phi_{\bar{T}_2}(v,t)$ with a constant t_0-parameter is represented by $L_{\bar{T}_2}(v)=\phi_{\bar{T}_2}(v,t_0)$.

If $\phi_{\bar{T}_1}(u,s)$ and $\phi_{\bar{T}_2}(v,t)$ intersect, we can write

$$\phi_{\bar{T}_1}(u,s)=\phi_{\bar{T}_2}(v,t),$$

and

$$C_{\bar{T}_1}(u)-C_{\bar{T}_2}(v)=-sT_1(u)+tT_2(v).$$

Since these three vectors are linearly dependent, the following equation can be given

$$\lambda_{\bar{T}}(u,v)=det\{T_1(u),T_2(v),C_{\bar{T}_1}(u)-C_{\bar{T}_2}(v)\}=0.$$

Theorem 9 Let $\phi_{\bar{T}_1}(u,s)$ and $\phi_{\bar{T}_2}(v,t)$ be the two ruled surfaces corresponding to two different tangential spherical indicatrix curves on the unit dual sphere in $\mathbb{R}^3$. If $\lambda_{\bar{T}}(u,v)=0$, then the parameter curves $L_{\bar{T}_1}(u)$ and $L_{\bar{T}_2}(v)$ intersect.

According to the parallelism of these three vectors with each other, the bivariate functions are denoted as follows, respectively:

$$\Delta_{\bar{T}}(u,v)=\|T_1(u)\times T_2(v)\|^2,$$

$$\delta_{\bar{T}}^{1}(u,v)=\left\|T_1(u)\times\left[C_{\bar{T}_1}(u)-C_{\bar{T}_2}(v)\right]\right\|^2,$$

$$\delta_{\bar{T}}^{2}(u,v)=\left\|T_2(v)\times\left[C_{\bar{T}_1}(u)-C_{\bar{T}_2}(v)\right]\right\|^2.$$

Theorem 10 Let $L_{\bar{T}_1}(u)$ and $L_{\bar{T}_2}(v)$ be the parameter curves of the ruled surfaces $\phi_{\bar{T}_1}(u,s)$ and $\phi_{\bar{T}_2}(v,t)$, which correspond to two different tangential spherical indicatrix curves on DS^2. In this case, the parameter curves $L_{\bar{T}_1}(u)$ and $L_{\bar{T}_2}(v)$ intersect if and only if

$$\Delta_{\bar{T}}(u,v)=\delta_{\bar{T}}^{1}(u,v)=\delta_{\bar{T}}^{2}(u,v)=0.$$

Let $\bar{\alpha}_1=\alpha_1+\varepsilon\alpha_1^*$ and $\bar{\alpha}_2=\alpha_2+\varepsilon\alpha_2^*$ be two curves on DS^2. The principal spherical indicatrix curves of these curves are $\bar{N}_1=N_1+\varepsilon N_1^*$ and $\bar{N}_2=N_2+\varepsilon N_2^*$, respectively. By the E. Study mapping, $\phi_{\bar{N}_1}(u,s)$ and $\phi_{\bar{N}_2}(v,t)$ are the ruled surfaces corresponding to these principal spherical indicatrix curves given above in $\mathbb{R}^3$. These ruled surfaces are denoted as follows:

$$\phi_{\bar{N}_1}(u,s)=N_1(u)\times N_1^*(u)+sN_1(u), \tag{11}$$

$$\phi_{\bar{N}_2}(v,t)=N_2(v)\times N_2^*(v)+tN_2(v). \tag{12}$$

Here, the base curves of the ruled surfaces $\phi_{\bar{N}_1}(u,s)$ and $\phi_{\bar{N}_2}(v,t)$ are $C_{\bar{N}_1}(u)=N_1(u)\times N_1^*(u)$ and $C_{\bar{N}_2}(v)=N_2(v)\times N_2^*(v)$.

The u-parameter curve of $\phi_{\bar{N}_1}(u,s)$ with a constant s_0-parameter is presented by $L_{\bar{N}_1}(u)=\phi_{\bar{N}_1}(u,s_0)$. Similarly, the v-parameter curve of $\phi_{\bar{N}_2}(v,t)$ with a constant t_0-parameter is given by $L_{\bar{N}_2}(v)=\phi_{\bar{N}_2}(v,t_0)$.

If $\phi_{\bar{N}_1}(u,s)$ and $\phi_{\bar{N}_2}(v,t)$ intersect, we can write
$\phi_{\bar{N}_1}(u,s)=\phi_{\bar{N}_2}(v,t)$,
and

$$C_{\bar{N}_1}(u)-C_{\bar{N}_2}(v)=-sN_1(u)+tN_2(v)$$

Since these three vectors are linearly dependent, the following equation can be expressed

$$\lambda_{\bar{N}}(u,v)=det\left\{N_1(u),N_2(v),C_{\bar{N}_1}(u)-C_{\bar{N}_2}(v)\right\}=0.$$

Theorem 11 Let $\phi_{\bar{N}_1}(u,s)$ and $\phi_{\bar{N}_2}(v,t)$ be the two ruled surfaces corresponding to two different principal spherical indicatrix curves on the unit dual sphere in $\mathbb{R}^3$. If $\lambda_{\bar{N}}(u,v)=0$, then the parameter curves $L_{\bar{N}_1}(u)$ and $L_{\bar{N}_2}(v)$ intersect.

According to the parallelism of these three vectors with each other, the bivariate functions are indicated as follows, respectively:

$$\Delta_{\bar{N}}(u,v)=\|N_1(u)\times N_2(v)\|^2,$$

$$\delta_{\bar{N}}^{1}(u,v)=\left\|N_1(u)\times\left[C_{\bar{N}_1}(u)-C_{\bar{N}_2}(v)\right]\right\|^2,$$

$$\delta_{\bar{N}}^{2}(u,v)=\left\|N_2(v)\times\left[C_{\bar{N}_1}(u)-C_{\bar{N}_2}(v)\right]\right\|^2.$$

Theorem 12 Let $L_{\bar{N}_1}(u)$ and $L_{\bar{N}_2}(v)$ be the parameter curves of the ruled surfaces $\phi_{\bar{N}_1}(u,s)$ and $\phi_{\bar{N}_2}(v,t)$, which correspond to two different principal spherical indicatrix curves on DS^2. In this case, the parameter curves $L_{\bar{N}_1}(u)$ and $L_{\bar{N}_2}(v)$ intersect if and only if

$$\Delta_{\bar{N}}(u,v)=\delta_{\bar{N}}^{1}(u,v)=\delta_{\bar{N}}^{2}(u,v)=0.$$

Let $\bar{\alpha}_1=\alpha_1+\varepsilon\alpha_1^*$ and $\bar{\alpha}_2=\alpha_2+\varepsilon\alpha_2^*$ be two curves on DS^2. The binormal spherical indicatrix curves of these curves are $\bar{B}_1=B_1+\varepsilon B_1^*$ and $\bar{B}_2=B_2+\varepsilon B_2^*$, respectively. According to the E. Study mapping, $\phi_{\bar{B}_1}(u,s)$ and $\phi_{\bar{B}_2}(v,t)$ are the ruled surfaces corresponding to these binormal spherical indicatrix curves shown above in $\mathbb{R}^3$. These ruled surfaces are indicated as follows:

$$\phi_{\bar{B}_1}(u,s)=B_1(u)\times B_1^*(u)+sB_1(u), \tag{13}$$

$$\phi_{\bar{B}_2}(v,t)=B_2(v)\times B_2^*(v)+tB_2(v). \tag{14}$$

Here, the base curves of the ruled surfaces $\phi_{\bar{B}_1}(u,s)$ and $\phi_{\bar{B}_2}(v,t)$ are $C_{\bar{B}_1}(u)=B_1(u)\times B_1^*(u)$ and $C_{\bar{B}_2}(v)=B_2(v)\times B_2^*(v)$.

The u-parameter curve of $\phi_{\bar{B}_1}(u,s)$ with a constant s_0-parameter is given $L_{\bar{B}_1}(u)=\phi_{\bar{B}_1}(u,s_0)$. Similarly, The v-parameter curve of $\phi_{\bar{B}_2}(v,t)$ with a constant t_0-parameter is represented $L_{\bar{B}_2}(v)=\phi_{\bar{B}_2}(v,t_0)$.

If $\phi_{\bar{B}_1}(u,s)$ and $\phi_{\bar{B}_2}(v,t)$ intersect, we can write

$$\phi_{\bar{B}_1}(u,s)=\phi_{\bar{B}_2}(v,t),$$

and

$$C_{\bar{B}_1}(u)-C_{\bar{B}_2}(v)=-sB_1(u)+tB_2(v)$$

Since these three vectors are linearly dependent, the following equation can be denoted

$$\lambda_{\bar{B}}(u,v)=det\left\{B_1(u),B_2(v),C_{\bar{B}_1}(u)-C_{\bar{B}_2}(v)\right\}=0.$$

Theorem 13 Let $\phi_{\bar{B}_1}(u,s)$ and $\phi_{\bar{B}_2}(v,t)$ be the two ruled surfaces corresponding to two different binormal spherical indicatrix curves on the unit dual sphere in $\mathbb{R}^3$. If $\lambda_{\bar{B}}(u,v)=0$, then the parameter curves $L_{\bar{B}_1}(u)$ and $L_{\bar{B}_2}(v)$ intersect.

According to the parallelism of these three vectors with each other, the bivariate functions are expressed as follows, respectively:

$$\Delta_{\bar{B}}(u,v)=\|B_1(u)\times B_2(v)\|^2,$$

$$\delta_{\bar{B}}^{1}(u,v)=\left\|N_1(u)\times\left[C_{\bar{B}_1}(u)-C_{\bar{B}_2}(v)\right]\right\|^2,$$

$$\delta_{\bar{B}}^{2}(u,v)=\left\|B_2(v)\times\left[C_{\bar{B}_1}(u)-C_{\bar{B}_2}(v)\right]\right\|^2.$$

Theorem 14 Let $L_{\bar{B}_1}(u)$ and $L_{\bar{B}_2}(v)$ be the parameter curves of the ruled surfaces $\phi_{\bar{B}_1}(u,s)$ and $\phi_{\bar{B}_2}(v,t)$, which correspond to two different binormal spherical indicatrix curves on DS^2. In this case, the parameter curves $L_{\bar{B}_1}(u)$ and $L_{\bar{B}_2}(v)$ intersect if and only if

$$\Delta_{\bar{B}}(u,v) = \delta_{\bar{B}}^1(u,v) = \delta_{\bar{B}}^2(u,v) = 0.$$

Let $\bar{\alpha}_1 = \alpha_1 + \varepsilon\alpha_1^*$ and $\bar{\alpha}_2 = \alpha_2 + \varepsilon\alpha_2^*$ be two curves on the unit dual sphere DS^2. The pole spherical indicatrix curves of these curves are $\bar{C}_1 = C_1 + \varepsilon C_1^*$ and $\bar{C}_2 = C_2 + \varepsilon C_2^*$, respectively. By the E. Study mapping, $\phi_{\bar{C}_1}(u,s)$ and $\phi_{\bar{C}_2}(v,t)$ are the ruled surfaces corresponding to these pole spherical indicatrix curves taken above in $\mathbb{R}^3$. These ruled surfaces are given as follows:

$$\phi_{\bar{C}_1}(u,s) = C_1(u) \times C_1^*(u) + sC_1(u), \tag{15}$$

$$\phi_{\bar{C}_2}(v,t) = C_2(v) \times C_2^*(v) + tC_2(v). \tag{16}$$

Here, the base curves of the ruled surfaces $\phi_{\bar{C}_1}(u,s)$ and $\phi_{\bar{C}_2}(v,t)$ are $C_{\bar{C}_1}(u) = C_1(u) \times C_1^*(u)$ and $C_{\bar{C}_2}(v) = C_2(v) \times C_2^*(v)$.

The u-parameter curve of $\phi_{\bar{C}_1}(u,s)$ with a constant s_0-parameter is shown $L_{\bar{C}_1}(u) = \phi_{\bar{C}_1}(u,s_0)$. Similarly, the v-parameter curve of $\phi_{\bar{C}_2}(v,t)$ with a constant t_0-parameter is indicated $L_{\bar{C}_2}(v) = \phi_{\bar{C}_2}(v,t_0)$.
If $\phi_{\bar{C}_1}(u,s)$ and $\phi_{\bar{C}_2}(v,t)$ intersect, we can write

$$\phi_{\bar{C}_1}(u,s) = \phi_{\bar{C}_2}(v,t),$$

and

$$C_{\bar{C}_1}(u) - C_{\bar{C}_2}(v) = -sC_1(u) + tC_2(v)$$

Since these three vectors are linearly dependent, the following equation can be represented

$$\lambda_{\bar{C}}(u,v) = det\{C_1(u), C_2(v), C_{\bar{C}_1}(u) - C_{\bar{C}_1}(v)\} = 0.$$

Theorem 15 Let $\phi_{\bar{C}_1}(u,s)$ and $\phi_{\bar{C}_2}(v,t)$ be the two ruled surfaces corresponding to two different pole spherical indicatrix curves on the unit dual sphere in $\mathbb{R}^3$. If $\lambda_{\bar{C}}(u,v) = 0$, then the parameter curves $L_{\bar{C}_1}(u)$ and $L_{\bar{C}_2}(v)$ intersect.

According to the parallelism of these three vectors with each other, the bivariate functions are given as follows, respectively:

$$\Delta_{\bar{C}}(u,v) = \|C_1(u) \times C_2(v)\|^2,$$

$$\delta_{\bar{C}}^1(u,v) = \left\|C_1(u) \times \left[C_{\bar{C}_1}(u) - C_{\bar{C}_2}(v)\right]\right\|^2,$$

$$\delta_{\bar{C}}^2(u,v) = \left\|C_2(v) \times \left[C_{\bar{C}_1}(u) - C_{\bar{C}_2}(v)\right]\right\|^2.$$

Theorem 16 Let $L_{\bar{C}_1}(u)$ and $L_{\bar{C}_2}(v)$ be the parameter curves of the ruled surfaces $\phi_{\bar{C}_1}(u,s)$ and $\phi_{\bar{C}_2}(v,t)$, which correspond to two different pole spherical indicatrix curves on DS^2. In this case, the parameter curves $L_{\bar{C}_1}(u)$ and $L_{\bar{C}_2}(v)$ intersect if and only if

$$\Delta_{\bar{C}}(u,v) = \delta_{\bar{C}}^1(u,v) = \delta_{\bar{C}}^2(u,v) = 0.$$

Example 17. Let us consider $\bar{\alpha}_1(u) = (cos(u), sin(u), 0) + \varepsilon(-sin(u), cos(u), u^4)$ and $\bar{\alpha}_2(v) = (sin(v), cos(v), 0) + \varepsilon(-cos(v), sin(v), v^4)$. Since $\langle\alpha_1(u), \alpha_1^*(u)\rangle = 0$, $\langle\alpha_2(v), \alpha_2^*(v)\rangle = 0$ and $|\alpha_1(u)| = |\alpha_2(v)| = 1$, the $\bar{\alpha}_1(u)$ and $\bar{\alpha}_2(v)$ are on DS^2.

The dual tangential spherical indicatrix curves of the $\bar{\alpha}_1(u)$ and $\bar{\alpha}_2(v)$ are

$$\bar{T}_1(u) = (-sin(u), cos(u), 0) + \varepsilon(-cos(u), -sin(u), 4u^3),$$

$$\bar{T}_2(v) = (cos(v), -sin(v), 0) + \varepsilon(sin(v), cos(v), 4v^3).$$

The ruled surface corresponding to the $\bar{T}_1(u)$ is

$$\bar{\phi}_{\bar{T}_1}(u,s) = (4u^3\cos(u), 4u^3\sin(u), 1) + s(-\sin(u), \cos(u), 0)$$

where the base curve is $C_{\bar{T}_1}(u) = (4u^3\cos(u), 4u^3\sin(u), 1)$.

The ruled surface corresponding to the $\bar{T}_2(v)$ is

$$\bar{\phi}_{\bar{T}_2}(v,t) = (-4v^3\sin(v), -4v^3\cos(v), 1) + t(\cos(v), -\sin(v), 0)$$

where the base curve is $C_{\bar{T}_2}(v) = (-4v^3\sin(v), -4v^3\cos(v), 1)$.

Let's examine the intersection of two ruled surfaces corresponding to two different tangential spherical indicatrix curves on DS^2. Let $\bar{\phi}_{\bar{T}_1}(u,s) = \bar{\phi}_{\bar{T}_2}(v,t)$, the following equations can be written by

$$(4u^3\cos(u) + 4v^3\sin(v), 4u^3\sin(u) + 4v^3\cos(v), 0) = -s(-\sin(u), \cos(u), 0) + t(\cos(v), -\sin(v), 0).$$

Since $C_{\bar{T}_1}(u) - C_{\bar{T}_2}(v)$, $T_1(u)$ and $T_2(v)$ are linearly dependent, $\lambda_{\bar{T}}(u,v) = 0$. Hence, the parameter curves $L_{\bar{T}_1}(u)$ and $L_{\bar{T}_2}(v)$ intersect.
If $\Delta_{\bar{T}}(u,v)$, $\delta_{\bar{T}}^1(u,v)$, $\delta_{\bar{T}}^2(u,v)$ are calculated, we obtain

$$\Delta_{\bar{T}}(u,v) = (\cos(u+v))^2,$$

$$\delta_{\bar{T}}^1(u,v) = 16(u^3 + v^3\sin(u+v))^2,$$

$$\delta_{\bar{T}}^2(u,v) = 16(v^3 + u^3\sin(u+v))^2.$$

For the $\lambda_{\bar{T}}(u,v) = 0$, the solution of the $(u,v) = \left(\frac{3\pi}{4}, \frac{3\pi}{4}\right)$, $\Delta_{\bar{T}}(u,v) = \delta_{\bar{T}}^1(u,v) = \delta_{\bar{T}}^2(u,v) = 0$. Hence, the parameter curves $L_{\bar{T}_1}(u)$ and $L_{\bar{T}_2}(v)$ intersect. Then, $\bar{\phi}_{\bar{T}_1}(u,s)$ and $\bar{\phi}_{\bar{T}_2}(v,t)$ ruled surfaces intersect.

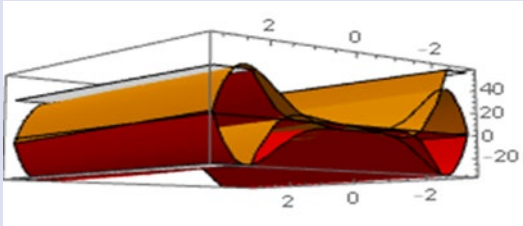

Figure 1. The intersection of $\bar{\phi}_{\bar{T}_1}$ and $\bar{\phi}_{\bar{T}_2}$ ruled surfaces.

The dual principal spherical indicatrix curves of the $\bar{\alpha}_1(u)$ and $\bar{\alpha}_2(v)$ are

$\bar{N}_1(u) = (-\cos(u), -sin(u), 0) + \varepsilon(\sin(u), -\cos(u), 12u^2)$,

$\bar{N}_2(v) = (-\sin(v), -\cos(v), 0) + \varepsilon(\cos(v), -\sin(v), 12v^2)$.

The ruled surface corresponding to the $\bar{N}_1(u)$ is

$$\bar{\phi}_{\bar{N}_1}(u,s) = (-12u^2\sin(u), 12u^2\cos(u), 1) + s(-\cos(u), -\sin(u), 0)$$

where the base curve is

$$C_{\bar{N}_1}(u) = (-12u^2\sin(u), 12u^2\cos(u), 1).$$

The ruled surface corresponding to the $\bar{N}_2(v)$ is

$$\bar{\phi}_{\bar{N}_2}(v,t) = (-12v^2\cos(v), 12v^2\sin(v), 1) + t(-\sin(v), -\cos(v), 0)$$

where the base curve is

$$C_{\bar{N}_2}(v) = (-12v^2\cos(v), 12v^2\sin(v), 1).$$

Let's examine the intersection of two ruled surfaces correspondig to two different principal normal spherical indicatrix curves DS^2. Let $\bar{\phi}_{\bar{N}_1}(u,s) = \bar{\phi}_{\bar{N}_2}(v,t)$, the following equation can be written by

$$(-12u^2\sin(u) + 12v^2\cos(v), 12u^2\cos(u) - 12v^2\sin(v), 0) = -s(-\cos(u), -\sin(u), 0) + t(-\sin(v), -\cos(v), 0).$$

Since $C_{\bar{N}_1}(u) - C_{\bar{N}_2}(v)$, $\bar{N}_1(u)$ and $\bar{N}_2(v)$ are linearly dependent, $\lambda_{\bar{N}}(u,v) = 0$. Therefore, the parameter curves $L_{\bar{N}_1}(u)$ and $L_{\bar{N}_2}(v)$ intersect.

If $\Delta_{\bar{N}}(u,v)$, $\delta^1_{\bar{N}}(u,v)$, $\delta^2_{\bar{N}}(u,v)$ are calculated, we obtain

$$\Delta_{\bar{N}}(u,v) = (\cos(u+v))^2,$$

$$\delta^1_{\bar{N}}(u,v) = 144(v^2\sin(u+v) - u^2)^2,$$

$$\delta^2_{\bar{N}}(u,v) = 144(v^2 - u^2\sin(u+v))^2.$$

For the $\lambda_{\bar{N}}(u,v) = 0$, the solution of the $(u,v) = \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, $\Delta_{\bar{N}}(u,v) = \delta^1_{\bar{N}}(u,v) = \delta^2_{\bar{N}}(u,v) = 0$. Therefore, the parameter curves $L_{\bar{N}_1}(u)$ and $L_{\bar{N}_2}(v)$ intersect. As a result, $\bar{\phi}_{\bar{N}_1}(u,s)$ and $\bar{\phi}_{\bar{N}_2}(v,t)$ ruled surfaces intersect.

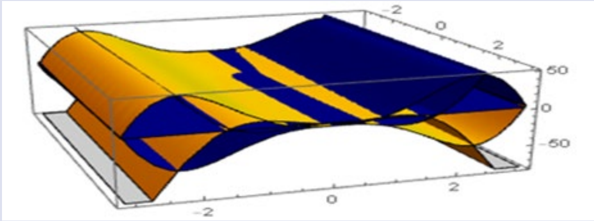

Figure 2. The intersection of $\bar{\phi}_{\bar{N}_1}$ and $\bar{\phi}_{\bar{N}_2}$ ruled surfaces.

The dual binormal spherical indicatrix curves of the $\bar{\alpha}_1(u)$ and $\bar{\alpha}_2(v)$ are

$\bar{B}_1(u) = (0,0,1) + \varepsilon(4u^3\sin(u) + 12u^2\cos(u), -4u^3\cos(u) + 12u^2\sin(u), 0)$,

$\bar{B}_2(v) = (0,0,-1) + \varepsilon(4v^3\cos(v) - 12v^2\sin(v), -4v^3\sin(v) - 12v^2\cos(v), 0)$.

The ruled surface corresponding to the $\bar{B}_1(u)$ is

$$\bar{\phi}_{\bar{B}_1}(u,s) = (4u^3\cos(u) - 12u^2\sin(u), 4u^3\sin(u) + 12u^2\cos(u), 0) + s(0,0,1)$$

where the base curve is $C_{\bar{B}_1}(u) = (4u^3\cos(u) - 12u^2\sin(u), 4u^3\sin(u) + 12u^2\cos(u), 0)$.

The ruled surface corresponding to the $\bar{B}_2(v)$ is

$$\bar{\phi}_{\bar{B}_2}(v,t) = (-4v^3\sin(v) - 12v^2\cos(v), -4v^3\cos(v) + 12v^2\sin(v), 0) + t(0,0,-1)$$

where the base curve is $C_{\bar{B}_2}(v) = (-4v^3\sin(v) - 12v^2\cos(v), -4v^3\cos(v) + 12v^2\sin(v), 0)$.

Let's examine the intersection of two ruled surfaces correspondig to two different binormal spherical indicatrix curves on DS^2. Let $\bar{\phi}_{\bar{B}_1}(u,s) = \bar{\phi}_{\bar{B}_2}(v,t)$, the following equation can be written by

$$(4u^3\cos(u) - 12u^2\sin(u) + 4v^3\sin(v) + 12v^2\cos(v), 4u^3\sin(u) + 12u^2\cos(u) + 4v^3\cos(v) - 12v^2\sin(v), 0) = -s(0,0,1) + t(0,0,-1).$$

Since $C_{\bar{B}_1}(u) - C_{\bar{B}_2}(v)$, $\bar{B}_1(u)$ and $\bar{B}_2(v)$ are linearly dependent, $\lambda_{\bar{B}}(u,v) = 0$. Hence, the parameter curves $L_{\bar{B}_1}(u)$ and $L_{\bar{B}_2}(v)$ intersect.

If $\Delta_{\bar{B}}(u,v)$, $\delta_{\bar{B}}^1(u,v)$, $\delta_{\bar{B}}^2(u,v)$ are calculated, we obtain

$$\Delta_{\bar{B}}(u,v) = 0,$$

$$\begin{aligned}\delta_{\bar{B}}^1(u,v) = 16\{&[3(v^2\sin(v) - u^2\cos(u)) \\ &- (v^3\cos(v) + u^3\sin(u))]^2 \\ &+ [3(v^2\cos(v) - u^2\sin(u)) \\ &+ (v^3\sin(v) + u^3\cos(u))]^2\},\end{aligned}$$

$$\begin{aligned}\delta_{\bar{B}}^2(u,v) = 16\{&[3(u^2\cos(u) - v^2\sin(v)) \\ &+ (u^3\sin(u) + v^3\cos(v))]^2 \\ &+ [3(u^2\sin(u) - v^2\cos(v)) \\ &- (u^3\cos(u) + v^3\sin(v))]^2\}.\end{aligned}$$

For the $\lambda_{\bar{B}}(u,v) = 0$, the solution of the $(u,v) = (0,0)$, $\Delta_{\bar{B}}(u,v) = \delta_{\bar{B}}^1(u,v) = \delta_{\bar{B}}^2(u,v) = 0$. Therefore, the parameter curves $L_{\bar{B}_1}(u)$ and $L_{\bar{B}_2}(v)$ intersect. Then, $\bar{\phi}_{\bar{B}_1}(u,s)$ and $\bar{\phi}_{\bar{B}_2}(v,t)$ ruled surfaces intersect.

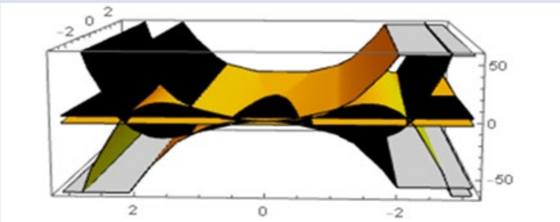

Figure 3. The intersection of $\bar{\phi}_{\bar{B}_1}$ and $\bar{\phi}_{\bar{B}_2}$ ruled surfaces.

The dual pole spherical indicatrix curves of the $\bar{\alpha}_1(u)$ and $\bar{\alpha}_2(v)$ are

$$\begin{aligned}\bar{C}_1(u) = (0,0,1) + \varepsilon(&12u^2\cos(u) \\ &- 24u\sin(u), 12u^2\sin(u) \\ &+ 24u\cos(u), 0),\end{aligned}$$

$$\bar{C}_2(v) = (0,0,-1) + \varepsilon(-12v^2\sin(v) - 24v\cos(v), -12v^2\cos(v) + 24v\sin(v), 0).$$

The ruled surface corresponding to the $\bar{C}_1(u)$ is

$$\begin{aligned}\bar{\phi}_{\bar{C}_1}(u,s) = (&-12u^2\sin(u) - 24u\cos(u), 12u^2\cos(u) \\ &- 24u\sin(u), 0) + s(0,0,1)\end{aligned}$$

where the base curve is $C_{\bar{C}_1}(u) = (-12u^2\sin(u) - 24u\cos(u), 12u^2\cos(u) - 24u\sin(u), 0)$.

The ruled surface corresponding to the $\bar{C}_2(v)$ is

$$\begin{aligned}\bar{\phi}_{\bar{C}_2}(v,t) = (&-12v^2\cos(v) + 24v\sin(v), 12v^2\sin(v) \\ &+ 24v\cos(v), 0) + t(0,0,-1)\end{aligned}$$

where the base curve is $C_{\bar{C}_2}(v) = (-12v^2\cos(v) + 24v\sin(v), 12v^2\sin(v) + 24v\cos(v), 0)$.

Let's examine the intersection of two ruled surfaces correspondig to two different pole spherical indicatrix curves on DS^2. Let $\bar{\phi}_{\bar{C}_1}(u,s) = \bar{\phi}_{\bar{C}_2}(v,t)$, the following equation can be written by

$$\begin{aligned}(-12u^2\sin(u) &- 24u\cos(u) \\ &+ 12v^2\cos(v) \\ &- 24v\sin(v), 12u^2\cos(u) \\ &- 24u\sin(u) - 12v^2\sin(v) \\ &- 24v\cos(v), 0) \\ &= -s(0,0,1) + t(0,0,-1).\end{aligned}$$

Since $C_{\bar{C}_1}(u) - C_{\bar{C}_2}(v)$, $\bar{C}_1(u)$ and $\bar{C}_2(v)$ are linearly dependent, $\lambda_{\bar{C}}(u,v) = 0$. Hence, the parameter curves $L_{\bar{C}_1}(u)$ and $L_{\bar{C}_2}(v)$ intersect.

If $\Delta_{\bar{C}}(u,v)$, $\delta_{\bar{C}}^1(u,v)$, $\delta_{\bar{C}}^2(u,v)$ are calculated, we obtain

$$\Delta_{\bar{C}}(u,v) = 0,$$

$$\begin{aligned}\delta_{\bar{C}}^1(u,v) = 144\{&[(v^2\sin(v) - u^2\cos(u)) \\ &+ 2(u\sin(u) + v\cos(v))]^2 \\ &+ [(v^2\cos(v) - u^2\sin(u)) \\ &- (u\cos(u) + v\sin(v))]^2\},\end{aligned}$$

$$\begin{aligned}\delta_{\bar{C}}^2(u,v) = 144\{&[(u^2\cos(u) - v^2\sin(v)) \\ &- 2(u\sin(u) + v\cos(v))]^2 \\ &+ [(u^2\sin(u) - v^2\cos(v)) \\ &+ 2(u\cos(u) + v\sin(v))]^2\}.\end{aligned}$$

For the $\lambda_{\bar{C}}(u,v) = 0$, the solution of the $(u,v) = (0,0)$, $\Delta_{\bar{C}}(u,v) = \delta_{\bar{C}}^1(u,v) = \delta_{\bar{C}}^2(u,v) = 0$. Therefore, the parameter curves $L_{\bar{C}_1}(u)$ and $L_{\bar{C}_2}(v)$ intersect. Then, $\bar{\phi}_{\bar{C}_1}(u,s)$ and $\bar{\phi}_{\bar{C}_2}(v,t)$ ruled surfaces intersect.

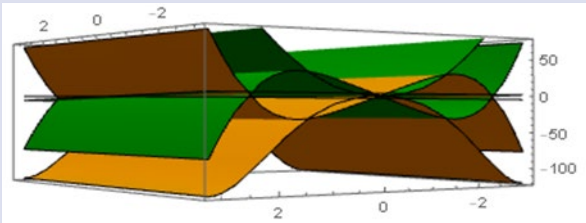

Figure 4. The intersection of $\bar{\phi}_{\bar{C}_1}$ and $\bar{\phi}_{\bar{C}_2}$ ruled surfaces.

Conclusion

E. Study mapping plays an important role to establish correspondence between dual space and Euclidean space. Considering the E. Study mapping, the curve on DS^2 corresponds to the ruled surfaces in $\mathbb{R}^3$. Also, the intersection curve of two ruled surfaces in $\mathbb{R}^3$ is represented by bivariate functions. In this study, two different ruled surfaces in $\mathbb{R}^3$ are examined with E. Study mapping to two different dual spherical indicatrix curves taken on DS^2. Later, the intersection curve for the corresponding ruled surfaces is presented by some main theorems. Additionally, these theorems are verified by giving examples. These results could have important applications in geometry, computer modeling systems, and solid modeling

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Conflict of interests

There are no conflicts of interest in this work.

References

[1] Carmo M. Do., Differential Geometry of Curves and Surfaces, Prentice Hall, Englewood Cliffs, New Jersey, (1976) 1-114.

[2] Karaca E., Çalışkan M., Ruled Surfaces and Tangent Bundle of Unit 2-Sphere of Natural Lift Curves, *Gazi Univ. J. Sci.,* 33(5) (2020) 751-759.

[3] Altınkaya A., Çalışkan M., On the Curvatures of the Ruled Surfaces of b-Lift Curves, *Cumhuriyet Sci. J.,* 42(4) (2021) 873-877.

[4] Izumiya S., Takeuchi N., New Special Curves and Developable Surfaces, *Turk. J. Math.*, 28(2) (2004) 153-163.

[5] Uyar Düldül B., Çalışkan M., On the Geodesic Torsion of a Tangential Intersection Curve of Two Surfaces in $\mathbb{R}^3$, *Acta Math. Univ. Comen.,* 82(2) (2013) 177-189.

[6] Ye X., Maekawa T., Differential Geometry of Intersection Curves of Two Surfaces, *Comput. Aided Geom. Des.,* 16(8) (1999) 767-788.

[7] Heo H-S., Kim M-S., Elber G., The Intersection of Two Ruled Surfaces, *Comput. Aided Des.*, 31(1) (1999) 33-50.

[8] Fischer I.S., Dual-Number Methods in Kinematics, Statics and Dynamics. Boca Raton, London, New York, Washington DC: CRC Press, (1999).

[9] Yaylı Y., Saraçoğlu S., Different Approaches to Ruled Surfaces, *SDU J. Sci.*, 7(1) (2012) 56-68.

[10] Karaca E., Çalışkan M., Dual Spherical Curves of Natural Lift Curve and Tangent Bundles of Unit 2-Sphere, *J. Sci.and Arts,* 3(48) (2019) 561-574.

[11] Arslan Güven İ., Kaya Nurhan S., Karacan M-K., Ruled Weingarten Surfaces Related to Dual Spherical Curves, *Gen. Math. Not.,* 24(2) (2014) 10-17.