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Estimation of the Density and Cumulative Distribution Functions of the Topp-Leone Distribution

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ABSTRACT

In this paper, we estimate the probability density function and the cumulative distribution function of the Topp-Leone distribution. We use the maximum likelihood estimation, uniformly minimum variance unbiased estimation, least squares estimation, weighted least squares estimation, Cramér-von Mises estimation, Anderson-Darling estimation and method of percentile estimation. The consistency of these methods is illustrated in a simulation study. Finally, a real data set is given to assess the performance of proposed methods.

Keywords: Topp-Leone distribution, Maximum likelihood least squares estimation, Weighted least squares estimation, Cramér-von Mises estimation, Anderson-Darling estimation.

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Introduction

We say that the continuous random variable *X* has the Topp-Leone distribution if its probability density function (PDF) is given by $for\ 0 < x < 1$,

$$f(x) = \alpha(2-2x)(2x-x^2)^{\alpha-1}, \qquad \alpha > 0. \quad (1)$$

The corresponding cumulative distribution function (CDF) is

$$F(x) = (2x-x^2)^{\alpha}. \quad (2)$$

The Topp-Leone distribution was developed in 1955 by Topp and Leone [22]. In the last two decades, this distribution has a great attention by several authors. For exemple, the structural properties of this distribution are obtained in [18]. Many reliability measures of the Topp-Leone distribution are studied by Ghitany et al. [10]. Vicari et al. [23] proposed a two-sided generalized version of this distribution. Genç [8] derived the moments of order statistics from this distribution. Genç [9] estimated the probability *P(X>Y)* for this distribution. Bayoud [5] derived the admissible minimax estimates for the shape of this distribution. Akgül [1] estimated system reliability of multicomponent stress-strength model for Topp-Leone distribution. More recently, Long [15] obtained the estimation and prediction for this distribution based on the double Type-I hybrid censored data. Using progressively censored samples from this distribution, the inference of multicomponent stress-strength reliability are derived by Saini et al. [20]. Benkhelifa [6] proposed a new distribution called the alpha power Topp-Leone Weibull distribution.

In many research fields, it is necessary to estimate the PDF and/or CDF. For example, to estimate the Kullback-Leibler divergence, Rényi entropy and Fisher information we estimate the PDF whereas we use the CDF for estimating the quantile function, Bonferroni and Lorenz curves. To estimate the hazard rate function, the probability weighted moments and the mean deviation about mean; we use the PDF and CDF together.

There are several studies on the estimation of PDF and CDF for some distributions. We mention: Pareto distribution by Dixit and Jabbari [7], generalized exponential-Poisson distribution by Bagheri et al. [3], exponentiated Weibull distribution by Alizadeh et al. [2], Weibull extension model by Bagheri et al. [4], Lindley distribution by Maity and Mukherjee [16], inverse Rayleigh distribution by Maleki Jebely et al. [27] and exponentiated Burr XII distribution by Hassan et al. [12].

In this paper, we estimate the PDF and CDF of the Topp-Leone distribution by the following methods: maximum likelihood (ML), uniformly minimum variance unbiased (UMVU), least squares (LS), weighted least squares (WLS), Cramér-von Mises (CvM), Anderson-Darling (AD) and percentile (PC) methods of estimation.

Maximum Likelihood Estimators

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the Topp-Leone distribution. It easy to show that the ML estimator of α, denoted by $\hat{\alpha}_{ML}$ is given by

$$\hat{\alpha}_{ML} = \frac{-n}{\sum_{i=1}^{n} ln(2X_i - X_i^2)}.$$

So, we use the invariance property of ML method to obtain the ML estimators of the PDF (1) and the CDF (2) which are,

$$\hat{f}(x) = \hat{\alpha}_{ML}(2-2x)(2x-x^2)^{\hat{\alpha}_{ML}-1},$$

and

$$\hat{F}(x) = (2x-x^2)^{\hat{\alpha}_{ML}},$$

respectively for $0 < x < 1$.

It is obliged to find the PDF of random variable $\hat{\alpha}_{ML}$ to compute $E([\hat{f}(x)]^r)$ and $E([\hat{F}(x)]^r)$.

Let $Z_i = -ln(2X_i - X_i^2), i = 1, \dots, n$ and $T = \sum_{i=1}^{n} Z_i$. So, it easy to show that Z_1 has an exponential distribution with PDF given by

$$f_{Z_1}(z) = \alpha e^{-\alpha z}, \qquad \text{for } z > 0,$$

and then T has a gamma distribution with the following PDF:

$$F_T(t) = \frac{\alpha^n\ t^{n-1}}{\Gamma(n)} e^{-\alpha t}, \qquad \text{for } t > 0.$$

So we can obtain the PDF of $\hat{\alpha}_{ML} = S = n/T$ which is

$$f_S(s) = \frac{n^n\ \alpha^n}{\Gamma(n) s^{n+1}} e^{-\frac{\alpha n}{s}}, \qquad \text{for } s > 0. \tag{3}$$

In the following Theorem, we give $E([\hat{f}(x)]^r)$ and $E([\hat{F}(x)]^r)$.

Theorem 2.1. We have $E([\hat{f}(x)]^r) = \frac{2(\alpha n)^{\frac{n+r}{2}}(2-2x)^r}{\Gamma(n)(2x-x^2)^r}\left(\frac{-1}{rln(2x-x^2)}\right)^{\frac{r-n}{2}} K_{r-n}\left(2\sqrt{-n\alpha r ln(2x-x^2)}\right)$
and

$$E([\hat{F}(x)]^r) = \frac{2(\alpha n)^{\frac{n}{2}}}{\Gamma(n)}\left(\frac{-1}{rln(2x-x^2)}\right)^{\frac{-n}{2}} K_{-n}\left(2\sqrt{-n\alpha r ln(2x-x^2)}\right),$$

where K_v is the modified Bessel function of the second kind of order v.

Proof. From Equation (3), we can write

$$\begin{aligned} E([\hat{f}(x)]^r) &= \int_0^{+\infty} [s(2-2x)(2x-x^2)^{s-1}]^r e^{-\frac{\alpha n}{s}} ds \\ &= \frac{n^n\ \alpha^n\ (2-2x)^r}{\Gamma(n)(2x-x^2)^r} \int_0^{+\infty} s^{r-n-1}(2x-x^2)^{rs} e^{-\frac{\alpha n}{s}} ds \\ &= \frac{n^n\ \alpha^n\ (2-2x)^r}{\Gamma(n)(2x-x^2)^r} \int_0^{+\infty} s^{r-n-1} e^{rsln(2x-x^2)} e^{-\frac{\alpha n}{s}} ds. \end{aligned}$$

Using the formula (3.471.9) in [11], we get

$$\int_0^{+\infty} s^{r-n-1} e^{rsln(2x-x^2)} e^{-\frac{\alpha n}{s}} ds = 2\left(\frac{-\alpha n}{rln(2x-x^2)}\right)^{\frac{r-n}{2}} K_{r-n}\left(2\sqrt{-n\alpha r ln(2x-x^2)}\right).$$

So

$$E([\hat{f}(x)]^r) = \frac{2(\alpha n)^{\frac{n+r}{2}}\ (2-2x)^r}{\Gamma(n)(2x-x^2)^r}\left(\frac{-1}{rln(2x-x^2)}\right)^{\frac{r-n}{2}} K_{r-n}\left(2\sqrt{-n\alpha r ln(2x-x^2)}\right).$$

We can get $E([\hat{F}(x)]^r)$ in a similar manner, and then the proof of Theorem 2.1 is finished.

Remark. When *r=1*, we observe that the estimators $\hat{f}(x)$ and $\hat{F}(x)$ are biased for *f(x)* and *F(x)*, respectively.

In the following Theorem, we give the MSEs of $\hat{f}(x)$ and $\hat{F}(x)$.

Theorem 2.2. The MSE of $\hat{f}(x)$ is given by

$$\begin{aligned} \text{MSE}(\hat{f}(x)) = &\frac{2(\alpha n)^{\frac{n+2}{2}}\ (2-2x)^2}{\Gamma(n)(2x-x^2)^2}\left(\frac{-1}{2ln(2x-x^2)}\right)^{\frac{2-n}{2}} K_{2-n}\left(2\sqrt{-2n\alpha ln(2x-x^2)}\right) \\ &- \frac{8(\alpha n)^{\frac{n+1}{2}}(1-x)f(x)}{\Gamma(n)(2x-x^2)}\left(\frac{-1}{ln(2x-x^2)}\right)^{\frac{1-n}{2}} K_{1-n}\left(2\sqrt{-n\alpha ln(2x-x^2)}\right) + f^2(x), \end{aligned}$$

while the MSEs of $\hat{F}(x)$ is

$$\mathrm{MSE}\big(\hat{F}(x)\big)=\frac{2(\alpha n)^{\frac{n}{2}}}{\Gamma(n)}\left(\frac{-1}{2ln(2x-x^2)}\right)^{\frac{-n}{2}}K_{-n}\left(2\sqrt{-2n\alpha ln(2x-x^2)}\right)$$
$$-\frac{4(\alpha n)^{\frac{n}{2}}F(x)}{\Gamma(n)}\left(\frac{-1}{ln(2x-x^2)}\right)^{\frac{-n}{2}}K_{-n}\left(2\sqrt{-n\alpha ln(2x-x^2)}\right)+F^2(x).$$

Proof. We know that

$$\mathrm{MSE}\big(\hat{f}(x)\big)=E([\hat{f}(x)]^2)-2f(x)\mathrm{E}\big(\hat{f}(x)\big)+f^2(x),$$

then, we obtain $\mathrm{MSE}\big(\hat{F}(x)\big)$) by setting *r=1* and $E([\hat{f}(x)]^2)$ by *r=2* in Theorem 2.1. Similarly, we can get $\mathrm{MSE}\big(\hat{F}(x)\big)$.

UMVU Estimators

Here, we derive the UMVU estimators of the PDF and the CDF of the Topp-Leone distribution. Also, we give the MSEs of these estimators.

If $X_1, X_2, \dots, X_n$ is a random sample from the Topp-Leone distribution, then $T=-\sum_{i=1}^{n} ln\big(2X_i-X_i{}^2\big)$ is a complete sufficient statistic for α. Let $f^*(x)$ be the UMVU estimator of *f(x)*. Then using Lehmann-Scheffé theorem, we have

$$\mathrm{E}(f^*(x))=\int f_{X_1|\mathrm{T}}(x_1|t)f_{\mathrm{T}}(t)dt=\int f_{X_1,\mathrm{T}}(x_1,t)dt=f_{X_1}(x_1),$$

where $f_{X_1|\mathrm{T}}(x_1|t)=f^*(x)$ is the conditional PDF of X_1 given *T* and $f_{X_1,\mathrm{T}}(x_1,t)$ is the joint PDF of X_1 and *T*. To find $f_{X_1|\mathrm{T}}(x_1|t)$, we need the following Lemma.

Lemma 3.1. The conditional PDF of X_1 given *T=t* is

$$f_{X_1|\mathrm{T}}(x|t)=\frac{(n-1)(2-2x)\big(t+ln(2x-x^2)\big)^{n-2}}{(2x-x^2)t^{n-1}},\qquad for\ -ln(2x-x^2)<t<\infty.$$

Proof. We know that $T=\sum_{i=1}^{n} Z_i$ is a random variable has the PDF given by (3). So, after some elementary algebra, the conditional PDF of Z_1 given *T* is

$$h_{Z_1|\mathrm{T}}(z|t)=\frac{(n-1)(t-z)^{n-2}}{t^{n-1}},\qquad \text{for } 0<z<t.$$

Then

$$f_{X_1|\mathrm{T}}(x|t)=\frac{2-2x}{2x-x^2}h_{Z_1|\mathrm{T}}(-ln(2x-x^2)|t),\qquad for\ -ln(2x-x^2)<t<\infty,$$

and the proof of Lemma is finished.

Theorem 3.1. The UMVU estimators of *f(x)* and *F(x)* are given by

$$\tilde{f}(x)=\frac{(n-1)(2-2x)\big(t+ln(2x-x^2)\big)^{n-2}}{(2x-x^2)t^{n-1}},$$

and

$$\tilde{F}(x)=\left(\frac{t+ln(2x-x^2)}{t}\right)^{n-1},$$

respectively for $-ln(2x-x^2)<t<\infty$.

Proof. From Lemma 3.1, we observe immediately that $\tilde{f}(x)$ is the UMVU estimator of *f(x)*. By integrating $\tilde{f}(x)$, we obtain $\tilde{F}(x)$.

The following Theorem gives the MSEs of $\tilde{f}(x)$ and $\tilde{F}(x)$.

Theorem 3.2. Let *T=t* be given. The MSEs of $\tilde{f}(x)$ and $\tilde{F}(x)$ are respectively given by

$$\mathrm{MSE}\left(\tilde{f}(x)\right)=\frac{A^2}{\Gamma(n)}\sum_{i=1}^{2n-4}\binom{2n-4}{i}b^i a^{i+3}\Gamma(n-i-2,-ab)-f^2(x),$$

and

$$\mathrm{MSE}\left(\tilde{F}(x)\right) = \frac{1}{\Gamma(n)} \sum_{i=1}^{2n-2} \binom{2n-2}{i} b^i a^{i+1} \Gamma(n-i, -ab) - F^2(x),$$

where $A = \frac{(n-1)(2-2x)}{2x-x^2}$, $b = ln(2x - x^2)$ and $\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t} dt$ is the complementary incomplete gamma function.

Proof. We have

$$\tilde{f}(x) = \frac{A(t+b)^{n-2}}{t^{n-1}}$$

and

$$\mathrm{MSE}\left(\hat{f}(x)\right) = E([\hat{f}(x)]^2) - f^2(x).$$

From Equation (3),

$$E([\hat{f}(x)]^2) = \frac{A^2\alpha^n}{\Gamma(n)} \int_{-b}^{+\infty} \left[\frac{(t+b)^{n-2}}{t^{n-1}}\right]^2 t^{n-1} e^{-\alpha t} dt = \frac{A^2\alpha^n}{\Gamma(n)} \int_{-b}^{+\infty} \frac{(t+b)^{2n-4}}{t^{2n-2}} t^{n-1} e^{-\alpha t} dt$$
$$= \frac{A^2\alpha^n}{\Gamma(n)} \int_{-b}^{+\infty} \frac{(t+b)^{2n-4}}{t^{2n-4}} t^{n-3} e^{-\alpha t} dt = \frac{A^2\alpha^n}{\Gamma(n)} \int_{-b}^{+\infty} \left(\frac{t+b}{t}\right)^{2n-4} t^{n-3} e^{-\alpha t} dt$$
$$= \frac{A^2\alpha^n}{\Gamma(n)} \int_{-b}^{+\infty} \left(1+\frac{b}{t}\right)^{2n-4} t^{n-3} e^{-\alpha t} dt.$$

We have

$$\left(1+\frac{b}{t}\right)^{2n-4} = \sum_{i=1}^{2n-4} \binom{2n-4}{i} \left(\frac{b}{t}\right)^i.$$

So

$$E([\hat{f}(x)]^2) = \frac{A^2\alpha^n}{\Gamma(n)} \int_{-b}^{+\infty} \left[\sum_{i=1}^{2n-4} \binom{2n-4}{i} \left(\frac{b}{t}\right)^i\right] t^{n-3} e^{-\alpha t} dt = \frac{A^2\alpha^n}{\Gamma(n)} \sum_{i=1}^{2n-4} \binom{2n-4}{i} b^i \int_{-b}^{+\infty} t^{n-i-3} e^{-\alpha t} dt.$$

The change of variables *u=αt*, yields

$$E([\hat{f}(x)]^2) = \frac{A^2\alpha^n}{\Gamma(n)} \sum_{i=1}^{2n-4} \binom{2n-4}{i} \frac{b^i}{\alpha^{n-i-3}} \int_{-\alpha b}^{+\infty} u^{n-i-3} e^{-u} du = \frac{A^2}{\Gamma(n)} \sum_{i=1}^{2n-4} \binom{2n-4}{i} b^i \alpha^{i+3} \Gamma(n-i-2, -ab).$$

In a similar manner, we can find $\mathrm{MSE}\left(\tilde{F}(x)\right)$.

Least Squares and Weighted Least Squares Estimators

Swain et al. [21] proposed the LS and WLS methods. In this section, we use these methods to estimate α. Let $x_{(1)}, \dots, x_{(n)}$ be the order observations of the random variables following the Topp-Leone distribution with size *n*. The LS estimator of α, denoted by $\tilde{\alpha}_{LS}$, is obtained by minimizing the function

$$\sum_{i=1}^{n} \left(\left(2x_{(i)} - x_{(i)}^2\right)^\alpha - \frac{i}{n+1}\right)^2,$$

while the WLS estimator of α, denoted by $\tilde{\alpha}_{WLS}$, is obtained by minimizing the function

$$\sum_{i=1}^{n} \frac{(n+2)(n+1)^2}{i(n-i+1)} \left(\left(2x_{(i)} - x_{(i)}^2\right)^\alpha - \frac{i}{n+1}\right)^2.$$

So, the LS and WLS estimators of the PDF and the CDF of the Topp-Leone distribution given, for *0 < x < 1*, by

$$\tilde{f}(x) = \tilde{\alpha}(2-2x)(2x-x^2)^{\tilde{\alpha}-1},$$

and

$$\tilde{F}(x) = (2x-x^2)^{\tilde{\alpha}},$$

where $\tilde{f} = \tilde{f}_{LS}$, $\tilde{F} = \tilde{F}_{LS}$ and $\tilde{\alpha} = \tilde{\alpha}_{LS}$ when we use the LS method and $\tilde{f} = \tilde{f}_{WLS}$, $\tilde{F} = \tilde{F}_{WLS}$ and $\tilde{\alpha} = \tilde{\alpha}_{WLS}$ for WLS method. Since it's difficult to find the MSEs of these estimators analytically, we shall calculate them by simulation.

Cramér-von Mises and Anderson Darling estimators

Let $x_{(1)}, \dots, x_{(n)}$ be the order observations of the random variables following the Topp-Leone distribution with size n. The CvM estimator of the parameter α, denoted by $\tilde{\alpha}_{\mathrm{CvM}}$, of the Topp-Leone distribution is obtained by minimizing the function

$$\frac{1}{12n} + \sum_{i=1}^{n} \left(\left(2x_{(i)} - x_{(i)}^2\right)^\alpha - \frac{2i-1}{2n}\right)^2,$$

whereas the AD estimator of α, denoted by $\tilde{\alpha}_{\mathrm{AD}}$, is obtained by minimizing

$$-n+\sum_{i=1}^{n}(2i-1)\left(\alpha ln\left(2x_{(i)}-x_{(i)}^2\right)+ln\left(1-\left(2x_{(i)}-x_{(i)}^2\right)^{\alpha}\right)\right)^2.$$

So, the CvM and AD estimators of the PDF and the CDF of the Topp-Leone distribution given, for 0 < x < 1, by

$$\tilde{f}(x)=\tilde{\alpha}(2-2x)(2x-x^2)^{\tilde{\alpha}-1},$$
and
$$\tilde{F}(x)=(2x-x^2)^{\tilde{\alpha}},$$

where $\tilde{f}=\tilde{f}_{\mathrm{CvM}}$, $\tilde{F}=\tilde{F}_{\mathrm{CvM}}$ and $\tilde{\alpha}=\tilde{\alpha}_{\mathrm{CvM}}$ when we use the CvM method and $\tilde{f}=\tilde{f}_{\mathrm{AD}}$, $\tilde{F}=\tilde{F}_{\mathrm{AD}}$ and $\tilde{\alpha}=\tilde{\alpha}_{\mathrm{AD}}$ for AD method. Since it's difficult to find the MSEs of these estimators analytically, we shall calculate them by simulation.

Estimators Based on Percentiles

The PC method proposed by Kao [13-14]. In this section, the PC estimators of the PDF and the CDF of the Topp-Leone distribution are derived. Let $x_{(1)},\ldots,x_{(n)}$ be the order observations of the random variables following the Topp-Leone distribution with size *n*. The PC estimator of α, denoted by $\tilde{\alpha}_{\mathrm{PC}}$, is obtained by minimizing

$$\sum_{i=1}^{n}\left(ln(p_i)-\alpha ln\left(2x_{(i)}-x_{(i)}^2\right)\right)^2$$

where $p_i=\frac{i}{n+1}$. It easy to show that

$$\tilde{\alpha}_{\mathrm{PC}}=\frac{\sum_{i=1}^{n}ln\left(2x_{(i)}-x_{(i)}^2\right)ln(p_i)}{\sum_{i=1}^{n}ln^2\left(2x_{(i)}-x_{(i)}^2\right)}.$$

So, the PC estimators of the PDF and the CDF are given by

$$\tilde{f}(x)=\tilde{\alpha}_{\mathrm{PC}}(2-2x)(2x-x^2)^{\tilde{\alpha}_{\mathrm{PC}}-1},$$
and
$$\tilde{F}(x)=(2x-x^2)^{\tilde{\alpha}_{\mathrm{PC}}},$$

respectively. Since it's difficult to find the MSEs of these estimators analytically, we shall compute them by simulation.

Simulation study

In this section, we carry out a simulation study to compare ML, UMVU, LS, WLS, PC, CvM and AD estimators of the PDF and the CDF for the Topp-Leone distribution. The random samples are generated from the Topp-Leone distribution for a given parameter *α*=1, 2 and 3 by using the inversion method: $X=\sqrt{1-U^{1/\alpha}}$, where *U* is a standard uniform random variable. The sample sizes of generated random samples are *n*=10, 20, 50 and 100, where the simulation is repeated 1000 times.

Table 1 gives the results of the simulation. We observe that the values of MSE converge to zero when *n* increases for all methods. So we can say that the estimators are consistent. Also, we see that the ML estimates of the PDF and the CDF have the lowest value of MSEs. This means that the ML estimators perform better than the others estimators.

Table 1. MSEs of the PDF and the CDF estimators.

		α=1		α=2		α=3	
n	**Methods**	**PDF**	**CDF**	**PDF**	**CDF**	**PDF**	**CDF**
10	ML	0.0837	0.0112	0.0238	0.1021	0.0511	0.0464
	UMVU	0.1021	0.0920	0.0436	0.1355	0.0518	0.0678
	PC	0.1313	0.1047	0.0982	0.1513	0.1099	0.0897
	LS	0.1967	0.2001	0.1869	0.1901	0.2010	0.1649
	WLS	0.1741	0.1699	0.1107	0.1812	0.1334	0.1121
	CvM	0.1450	0.1218	0.1005	0.1697	0.1101	0.1026
	AD	0.1601	0.1497	0.1087	0.1801	0.1254	0.1059
20	ML	0.0716	0.0053	0.0200	0.0921	0.0341	0.0344
	UMVU	0.0921	0.1152	0.0398	0.1040	0.0403	0.0516
	PC	0.1296	0.1007	0.0772	0.1106	0.0885	0.0662
	LS	0.1941	0.2158	0.1552	0.1681	0.1819	0.1398
	WLS	0.1601	0.1841	0.1079	0.1282	0.1223	0.0997
	CvM	0.1309	0.1181	0.0905	0.1609	0.1004	0.0992
	AD	0.1575	0.1367	0.0992	0.1788	0.1187	0.1003
50	ML	0.0102	0.0026	0.0010	0.0352	0.0028	0.0039
	UMVU	0.0467	0.0788	0.0035	0.0502	0.0177	0.0207
	PC	0.0816	0.0985	0.0225	0.0651	0.0412	0.0388
	LS	0.1511	0.1772	0.1398	0.1436	0.1196	0.1006
	WLS	0.1153	0.1380	0.0996	0.1041	0.0757	0.0967
	CvM	0.1098	0.1101	0.0702	0.1410	0.0866	0.0756
	AD	0.1271	0.1167	0.0803	0.1575	0.0982	0.0811
100	ML	0.0005	0.0016	0.00016	0.0017	0.00018	0.00047
	UMVU	0.0050	0.0042	0.00077	0.0123	0.0092	0.0084
	PC	0.0071	0.0056	0.0087	0.0209	0.0104	0.0196
	LS	0.0930	0.1004	0.0799	0.0874	0.0889	0.0917
	WLS	0.0931	0.1086	0.0599	0.0676	0.0727	0.0729
	CvM	0.0801	0.0998	0.0404	0.0590	0.0363	0.0606
	AD	0.1053	0.1060	0.0503	0.0602	0.0542	0.0699

Application to Real Data

To compare between ML, LS, WLS, CvM, AD and PC estimators for the PDF and the CDF, we use the following data set: 0.2160, 0.0150, 0.4082, 0.0746, 0.0358, 0.0199, 0.0402, 0.0101, 0.0605, 0.0954, 0.1359, 0.0273, 0.0491, 0.3465, 0.0070, 0.6560, 0.1060, 0.0062, 0.4992, 0.0614, 0.5320, 0.0347 and 0.1921. We took this data set from [19]. It represents the times between failures of secondary reactor pumps. In order to obtain data between 0 and 1, a normalization operation is done by dividing the original data set by 10.

To assess the behavior of ML, PC, LS and WLS estimators, we use the following criteria:

- Maximum likelihood criterion, which is given by $\mathrm{MLC} = -2\ln\mathrm{L}$,
- Akaike information criterion, that is defined as $\mathrm{AIC} = -2\ln\mathrm{L} + 2\mathrm{k}$,
- Bayes information criterion that is defined as by $BIC = -2lnL + kln(n)$,
- Akaike information criterion corrected, which is defined by $AICc = AIC + ((2k(k+1))/(n-k-1))$,
- Hannan-Quinn criterion, that is defined as $HQC = -2lnL + 2kln(ln\,(n))$,

where lnL is the estimated value of the maximum log-likelihood, k is the number of parameters and n is the number of observations.

The model which has lowest values of MLC, AIC, BIC, $AICc$ and HQC is selected as the best model to fit the data. The values of the estimates of the parameter and the model selection criteria for the different estimation methods are given in Table 2. It is clear from Tables 2 that the ML estimate has the smallest value for all the model selection criteria. This indicates that the ML estimator is superior than the LS, WLS, CvM, AD and PC estimators.

Plots the fitted PDFs versus the histogram are given in Figure 1(a) while the fitted CDFs plots versus the empirical CDF are given in Figure 1(b). It is easy to see from these plots that the ML method provides the good fit compared to the other methods.

Table 2. Estimations of the parameter and the model selection criteria for the data set.

Methods	Estimate of α	MLC	AIC	BIC	AICc	HQC
ML	0.4891	-37.5653	-35.5653	-34.4298	-35.3749	-35.2798
LS	0.3988	-36.6694	-34.6694	-33.5339	-34.4789	-34.3838
WLS	0.4375	-37.2901	-35.2901	-34.1546	-35.0996	-35.0045
CvM	0.4023	-36.7424	-34.7424	-33.6069	-34.5520	-34.4569
AD	0.4156	-36.9867	-34.9867	-33.8512	-34.7962	-34.7011
PC	0.4979	-37.5580	-35.5580	-34.4226	-35.3676	-35.2725

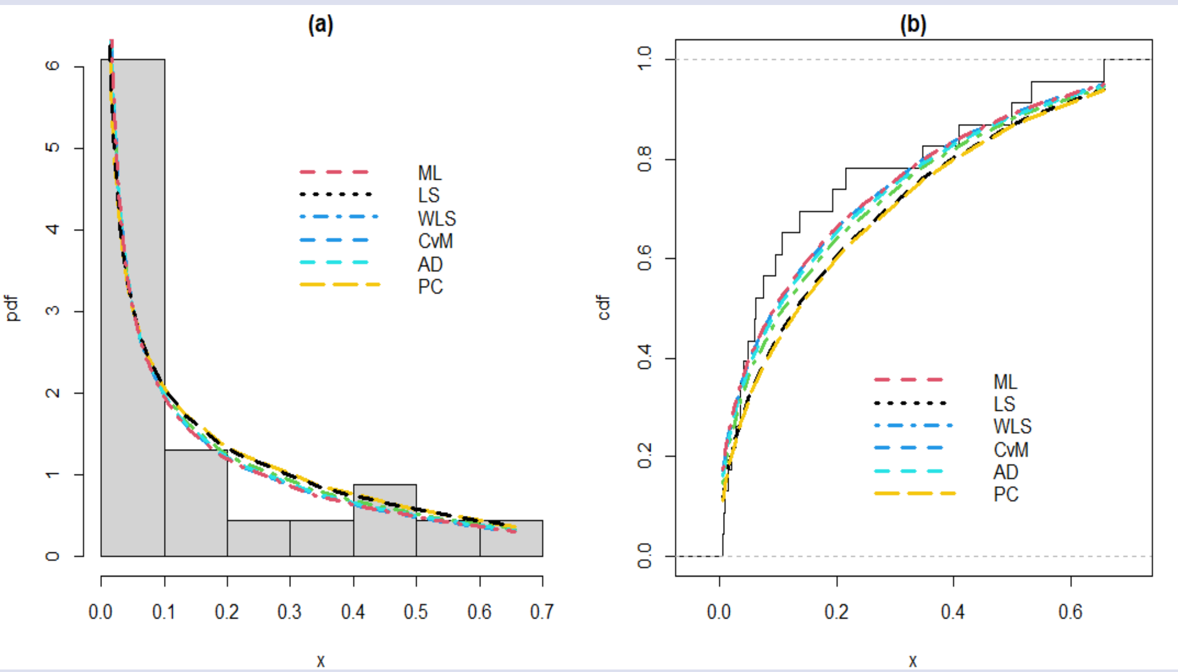

Figure 1. Fitted (a) PDFs versus the histogram and (b) CDFs versus the empirical CDF for the various estimation methods.

Conclusion

In this paper, we have considered seven methods of estimation: ML, UMVU, LS, WLS, CvM, AD and PC method for the PDF and the CDF of the Topp-Leone distribution. We have compared these estimators by a simulation study and a real data application. A simulation study shows that all estimators are consistent and the ML estimator performs better than the other estimators. Also, an application to real data set proves that the ML method provides the good fit than the other methods of estimation.

Conflicts of interest

There are no conflicts of interest in this work.

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