Cumhuriyet Sci. J., 44(2) (2023) 315-327
DOI: https://doi.org/10.17776/csj.1173496

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

Parametric Extension of a Certain Family of Summation-Integral Type Operators

Nadire Fulda Odabaşı [1,a,*], İsmet Yüksel [1,b]

[1] *Department of Mathematics, Faculty of Sciences, Gazi University, Ankara, Türkiye.*

**Corresponding author*

Research Article

History
Received: 10/09/2022
Accepted: 24/02/2023

ABSTRACT

In this paper, we introduce a parametric extension of a certain family of summation-integral type operators on the interval [0,∞). Firstly, we obtain test functions and central moments. Secondly, we investigate weighted approximation properties for these operators and estimate the rate of convergence. Then, we give a pointwise approximation for the Peetre K-functional and functions of the Lipschitz class. Moreover, we demonstrate Voronovskaja type theorem for the operators. Finally, the convergence properties of operators to some functions are illustrated by graphics.

Keywords: Rate of convergence, Weighted spaces, Weighted modulus of continuity.

[a] nadirefulda.odabasi@gazi.edu.tr https://orcid.org/0000-0001-5825-1177 [b] iyuksel@gazi.edu.tr https://orcid.org/0000-0002-2631-2382

Introduction

Approximation to real-valued continuous functions with the help of positive linear operators is an important area in the approximation theory. All studies in this area originate from the Weierstrass theory [1]. A sequence of positive linear operators was introduced by Baskakov in 1957 [2], and in 1967, Durrmeyer defined a sequence of positive linear operators [3]. In 1977, Phillips defined an operator [4]. In 2003, Gupta introduced a general sequence of summation integral type operators $G_{n,c}$ [5], and then the approximation properties of the operators $G_{n,c}$ for continuous and unbounded functions specified on the interval [0,∞) were studied by Yüksel and İspir [6,7]. Chen, Tan and Liu studied on generalized Bernstein operators [8]. Later, Cai, Lian and Zhou introduced λ-Bernstein operators with parameter $\lambda \in [-1,1]$ [9]. Recently, Aral proposed Baskakov operators which have a non-negative real parametric generalization [10]. A lot of studies concerning generalizations of a sequence of linear positive operators have been done by many researchers [11-31].

In this paper, motivated by the α-Baskakov operators by Aral [10], we introduce a parametric extension of summation integral type operators named them $G_{n,c}^{\alpha}$ operators and study the approximation properties of these operators.

$G_{n,c}^{\alpha}$ operators

Let us define $G_{n,c}^{\alpha}$ operators on subset of all continuous functions on $[0,\infty)$ for which the following integral exists finitely.

$$G_{n,c}^{\alpha}(f;x) = n\sum_{k=1}^{\infty} P_{n,k}^{\alpha}(x;c)\int_0^{\infty} P_{n+c,k-1}(t;c)\,f(t)dt + P_{n,0}^{\alpha}(x;c)f(0)$$

$$P_{n,k}^{\alpha}(x;c) = \begin{cases} e^{-nx}\dfrac{(nx)^k}{k!}, & if\ c=0 \\ \dfrac{(cx)^{k-1}}{(1+cx)^{\frac{n}{c}+k-1}}\left\{\dfrac{\alpha cx}{(1+cx)}\dfrac{\Gamma\left(\frac{n}{c}+k\right)}{k!\,\Gamma\left(\frac{n}{c}\right)}+(1-\alpha)(cx)\dfrac{\Gamma\left(\frac{n}{c}+k\right)}{k!\,\Gamma\left(\frac{n}{c}\right)}\right\}, & if\ c\in\mathbb{N}\backslash\{0\}\ and\ k\in\{0,1\} \\ \dfrac{(cx)^{k-1}}{(1+cx)^{\frac{n}{c}+k-1}}\left\{\dfrac{\alpha cx}{(1+cx)}\dfrac{\Gamma\left(\frac{n}{c}+k\right)}{k!\,\Gamma\left(\frac{n}{c}\right)}-(1-\alpha)(1+cx)\dfrac{\Gamma\left(\frac{n}{c}+k-2\right)}{(k-2)!\,\Gamma\left(\frac{n}{c}\right)}+(1-\alpha)(cx)\dfrac{\Gamma\left(\frac{n}{c}+k\right)}{k!\,\Gamma\left(\frac{n}{c}\right)}\right\}, & if\ c\in\mathbb{N}\backslash\{0\}\ and\ k\in\mathbb{N}\backslash\{0,1\} \end{cases} \quad (1)$$

with

$$P_{n+c,k-1}(t;c) = \frac{\Gamma\left(\frac{n}{c}+k\right)}{(k-1)!\,\Gamma\left(\frac{n}{c}+1\right)}\frac{(cx)^{k-1}}{(1+cx)^{\frac{n}{c}+k}}. \quad (2)$$

Theorem 2.1 The $G_{n,c}^{\alpha}$operators for $f(x)$ satisfy the following equation for $n>c$.

$$G_{n,c}^{\alpha}(f;x)=\alpha G_{n,c}(f;x)+(1-\alpha)\tilde{G}_{n,c}(f;x),$$

where

$$\tilde{G}_{n,c}(f;x)=n\sum_{k=1}^{\infty}P_{n-c,k}(x;c)\int_{0}^{\infty}g_{n,k}(t;c)f(t)dt+P_{n-c,0}(x;c)f(0)$$

and

$$g_{n,k}(t;c)=\left[\left(1+\frac{k}{\frac{n-c}{c}}\right)P_{n+c,k-1}(t;c)-\left(\frac{k}{\frac{n-c}{c}}\right)P_{n+c,k}(t;c)\right].$$

Proof.

The following identities hold.

$$\left(1+\frac{k}{\frac{n-c}{c}}\right)\frac{\Gamma\left(\frac{n-c}{c}+k\right)}{k!\,\Gamma\left(\frac{n-c}{c}\right)}=\frac{\Gamma\left(\frac{n-c}{c}+k+1\right)}{k!\,\Gamma\left(\frac{n-c}{c}+1\right)}, \tag{3}$$

$$\frac{\Gamma\left(\frac{n-c}{c}+k\right)}{(k-1)!\,\Gamma\left(\frac{n-c}{c}+1\right)}=\left(\frac{k}{\frac{n-c}{c}}\right)\frac{\Gamma\left(\frac{n-c}{c}+k\right)}{k!\,\Gamma\left(\frac{n-c}{c}\right)}. \tag{4}$$

$$G_{n,c}^{\alpha}(f;x)=\alpha n\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\,\Gamma\left(\frac{n}{c}\right)}\frac{(cx)^{k}}{(1+cx)^{\frac{n}{c}+k}}\int_{0}^{\infty}P_{n+c,k-1}(t;c)f(t)dt+\frac{\alpha}{(1+cx)^{\frac{n}{c}}}f(0)$$
$$+(1-\alpha)\left(k_{n,c}^{1}(f;x)-k_{n,c}^{2}(f;x)\right)$$

where

$$k_{n,c}^{1}(f;x)=n\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{n-c}{c}+k+1\right)}{k!\,\Gamma\left(\frac{n-c}{c}+1\right)}\frac{(cx)^{k}}{(1+cx)^{\frac{n-c}{c}+k}}\int_{0}^{\infty}P_{n+c,k-1}(t;c)f(t)dt+\frac{1}{(1+cx)^{\frac{n-c}{c}}}f(0).$$

If we use equation (3), we get

$$k_{n,c}^{1}(f;x)=n\sum_{k=1}^{\infty}\left(1+\frac{k}{\frac{n-c}{c}}\right)\frac{\Gamma\left(\frac{n-c}{c}+k\right)}{k!\,\Gamma\left(\frac{n-c}{c}\right)}\frac{(cx)^{k}}{(1+cx)^{\frac{n-c}{c}+k}}\int_{0}^{\infty}P_{n+c,k-1}(t;c)f(t)dt+\frac{1}{(1+cx)^{\frac{n-c}{c}}}f(0)$$

$$=n\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{n-c}{c}+k\right)}{k!\,\Gamma\left(\frac{n-c}{c}\right)}\frac{(cx)^{k}}{(1+cx)^{\frac{n-c}{c}+k}}\int_{0}^{\infty}\left(1+\frac{k}{\frac{n-c}{c}}\right)P_{n+c,k-1}(t;c)f(t)dt+\frac{1}{(1+cx)^{\frac{n-c}{c}}}f(0).$$

Similarly, we have

$$k_{n,c}^{2}(f;x)=n\sum_{k=2}^{\infty}\frac{\Gamma\left(\frac{n-c}{c}+k-1\right)}{(k-2)!\,\Gamma\left(\frac{n-c}{c}+1\right)}\frac{(cx)^{k-1}}{(1+cx)^{\frac{n-c}{c}+k-1}}\int_{0}^{\infty}P_{n+c,k-1}(t;c)f(t)dt.$$

When we write $k+1$ instead of k in $k_{n,c}^{2}(f;x)$, we obtain

$$k_{n,c}^{2}(f;x)=n\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{n-c}{c}+k\right)}{(k-1)!\,\Gamma\left(\frac{n-c}{c}+1\right)}\frac{(cx)^{k}}{(1+cx)^{\frac{n-c}{c}+k}}\int_{0}^{\infty}P_{n+c,k}(t;c)f(t)dt.$$

Again, if we use equation (4), we get

$$k_{n,c}^2(f;x) = n\sum_{k=1}^{\infty}\left(\frac{k}{\frac{n-c}{c}}\right)\frac{\Gamma\left(\frac{n-c}{c}+k\right)}{k!\,\Gamma\left(\frac{n-c}{c}\right)}\frac{(cx)^k}{(1+cx)^{\frac{n-c}{c}+k}}\int_0^{\infty} P_{n+c,k}(t;c)\,f(t)dt$$

$$= n\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{n-c}{c}+k\right)}{k!\,\Gamma\left(\frac{n-c}{c}\right)}\frac{(cx)^k}{(1+cx)^{\frac{n-c}{c}+k}}\int_0^{\infty}\left(\frac{k}{\frac{n-c}{c}}\right)P_{n+c,k}(t;c)\,f(t)dt\,.$$

Finally, substracting $k_{n,c}^2(f;x)$ from $k_{n,c}^1(f;x)$, we have

$$k_{n,c}^1(f;x) - k_{n,c}^2(f;x) = n\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{n-c}{c}+k\right)}{k!\Gamma\left(\frac{n-c}{c}\right)}\frac{(cx)^k}{(1+cx)^{\frac{n-c}{c}+k}}\int_0^{\infty}\left[\left(1+\frac{k}{\frac{n-c}{c}}\right)P_{n+c,k-1}(t;c) - \left(\frac{k}{\frac{n-c}{c}}\right)P_{n+c,k}(t;c)\right]f(t)dt + \frac{1}{(1+cx)^{\frac{n-c}{c}}}f(0)$$

$$= n\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{n-c}{c}+k\right)}{k!\Gamma\left(\frac{n-c}{c}\right)}\frac{(cx)^k}{(1+cx)^{\frac{n-c}{c}+k}}\int_0^{\infty} g_{n,k}(t;c)\,f(t)dt + \frac{1}{(1+cx)^{\frac{n-c}{c}}}f(0)$$

$$= n\sum_{k=1}^{\infty}P_{n-c,k}(x;c)\int_0^{\infty} g_{n,k}(t;c)\,f(t)dt + P_{n-c,0}(x;c)f(0)$$

$$= \tilde{G}_{n,c}(f;x)$$

and proof of the Theorem 2.1 is completed.

According to Theorem 2.1, $G_{n,c}^{\alpha}$ operators are positive provided that $\alpha \in [0,1]$. Here, the operators reduce to $G_{n,c}$ operators [7] for $\alpha = 1$.

Lemma 2.1 Let $\sum_{k=0}^{\infty} k^m P_{n,k}^{\alpha}(x;c)$ be series of $P_{n,k}^{\alpha}$ base functions defined in equation (1). Then, we have the following equalities for $m \in \{0,1,2,3,4\}$.

(i) $\sum_{k=0}^{\infty} k^0 P_{n,k}^{\alpha}(x;c) = 1,$

(ii) $\sum_{k=0}^{\infty} k^1 P_{n,k}^{\alpha}(x;c) = nx - 2(1-\alpha)cx,$

(iii) $\sum_{k=0}^{\infty} k^2 P_{n,k}^{\alpha}(x;c) = nx - 4(1-\alpha)cx - 4nc(1-\alpha)x^2 + n(n+c)x^2,$

(iv) $\sum_{k=0}^{\infty} k^3 P_{n,k}^{\alpha}(x;c) = nx - 8(1-\alpha)cx - 18nc(1-\alpha)x^2 + 3n(n+c)x^2$
$-6nc(n+c)(1-\alpha)x^3 + n(n+c)(n+2c)x^3,$

(v) $\sum_{k=0}^{\infty} k^4 P_{n,k}^{\alpha}(x;c) = nx - 16c(1-\alpha)x - 64nc(1-\alpha)x^2$
$+7n(n+c)x^2 - 48nc(n+c)(1-\alpha)x^3$
$+6n(n+c)(n+2c)x^3 - 8nc(n+c)(n+2c)(1-\alpha)x^4$
$+n(n+c)(n+2c)(n+3c)x^4.$

Proof.

(i) $\sum_{k=0}^{\infty} k^0 P_{n,k}^{\alpha}(x;c) = \alpha\sum_{k=0}^{\infty}\frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}}\frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)} - (1-\alpha)\sum_{k=2}^{\infty}\frac{(cx)^{k-1}}{(1+cx)^{\frac{n}{c}+k-2}}\frac{\Gamma\left(\frac{n}{c}+k-2\right)}{(k-2)!\Gamma\left(\frac{n}{c}\right)}$
$+(1-\alpha)(1+cx)\sum_{k=0}^{\infty}\frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}}\frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)}.$

When we write $k+2$ instead of k in the second term of the above equation, we get

$$\sum_{k=0}^{\infty} k^0 P_{n,k}^{\alpha}(x;c) = \alpha\sum_{k=0}^{\infty}\frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}}\frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)} - (1-\alpha)\sum_{k=0}^{\infty}\frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k}}\frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)}$$
$$+(1-\alpha)(1+cx)\sum_{k=0}^{\infty}\frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}}\frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)}$$
$$= \alpha - (1-\alpha)cx + (1-\alpha)(1+cx)$$
$$= 1.$$

(ii) $\sum_{k=0}^{\infty} k^1 P_{n,k}^{\alpha}(x;c) = \sum_{k=1}^{\infty}\frac{\alpha(cx)^k}{(1+cx)^{\frac{n}{c}+k}}\frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)}k - (1-\alpha)\sum_{k=2}^{\infty}\frac{(cx)^{k-1}}{(1+cx)^{\frac{n}{c}+k-2}}\frac{\Gamma\left(\frac{n}{c}+k-2\right)}{(k-2)!\Gamma\left(\frac{n}{c}\right)}k$
$(1-\alpha)(1+cx)\sum_{k=1}^{\infty}\frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}}\frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)}k\,.$

When we write $k+1$ instead of k in first and third terms of the above equation, we get

$$\sum_{k=0}^{\infty} k^1 P_{n,k}^{\alpha}(x;c) = \alpha\sum_{k=0}^{\infty}\frac{(cx)^{k+1}}{(1+cx)^{\frac{n}{c}+k+1}}\frac{\Gamma\left(\frac{n}{c}+k+1\right)}{(k+1)k!\Gamma\left(\frac{n}{c}\right)}(k+1) - (1-\alpha)\sum_{k=2}^{\infty}\frac{(cx)^{k-1}}{(1+cx)^{\frac{n}{c}+k-2}}\frac{\Gamma\left(\frac{n}{c}+k-2\right)}{(k-2)!\left(\frac{n}{c}\right)}\,k$$
$$+(1-\alpha)(1+cx)\sum_{k=0}^{\infty}\frac{(cx)^{k+1}}{(1+cx)^{\frac{n}{c}+k+1}}\frac{\Gamma\left(\frac{n}{c}+k+1\right)}{(k+1)k!\Gamma\left(\frac{n}{c}\right)}(k+1).$$

When we write $k+2$ instead of k in the second term of the above equation, we get

$$\sum_{k=0}^{\infty} k^1 P_{n,k}^{\alpha}(x;c) = \alpha \sum_{k=0}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k+1}} \frac{\Gamma\left(\frac{n}{c}+k+1\right)}{k!\Gamma\left(\frac{n}{c}\right)} - (1-\alpha)\sum_{k=0}^{\infty} \frac{(cx)^{k+1}}{(1+cx)^{\frac{n}{c}+k}} \frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)}(k+2)$$
$$+(1-\alpha)(1+cx)\sum_{k=0}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k+1}} \frac{\Gamma\left(\frac{n}{c}+k+1\right)}{k!\Gamma\left(\frac{n}{c}\right)}$$
$$= \alpha \sum_{k=0}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k+1}} \frac{\Gamma\left(\frac{n}{c}+k+1\right)}{k!\Gamma\left(\frac{n}{c}\right)} - 2(1-\alpha)\sum_{k=0}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k}} \frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)}$$
$$-(1-\alpha)\sum_{k=1}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k}} \frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)} k + (1-\alpha)(1+cx)\sum_{k=0}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k+1}} \frac{\Gamma\left(\frac{n}{c}+k+1\right)}{k!\Gamma\left(\frac{n}{c}\right)}.$$

When we write $k+1$ instead of k in the third term of the above equation, we yield

$$\sum_{k=0}^{\infty} k^1 P_{n,k}^{\alpha}(x;c) = \alpha \sum_{k=0}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k+1}} \frac{\Gamma\left(\frac{n}{c}+k+1\right)}{k!\Gamma\left(\frac{n}{c}\right)} - 2(1-\alpha)\sum_{k=0}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k}} \frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)}$$
$$-(1-\alpha)\sum_{k=0}^{\infty} \frac{(cx)^{k+1} cx}{(1+cx)^{\frac{n}{c}+k+1}} \frac{\Gamma\left(\frac{n}{c}+k+1\right)}{(k+1)k!\Gamma\left(\frac{n}{c}\right)}(k+1)$$
$$+(1-\alpha)(1+cx)\sum_{k=0}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k+1}} \frac{\Gamma\left(\frac{n}{c}+k+1\right)}{k!\Gamma\left(\frac{n}{c}\right)}.$$

When we multiply and divide first, third and fourth terms of the above equation with n/c, we get

$$\sum_{k=0}^{\infty} k^1 P_{n,k}^{\alpha}(x;c) = \alpha \sum_{k=0}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k+1}} \frac{\Gamma\left(\frac{n}{c}+k+1\right)}{k!\Gamma\left(\frac{n}{c}+1\right)}\frac{n}{c} - 2(1-\alpha)\sum_{k=0}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k}} \frac{\Gamma\left(\frac{n}{c}+k\right)}{k!\Gamma\left(\frac{n}{c}\right)}$$
$$-(1-\alpha)\sum_{k=0}^{\infty} \frac{(cx)^k (cx)^2}{(1+cx)^{\frac{n}{c}+k+1}} \frac{\Gamma\left(\frac{n}{c}+k+1\right)}{k!\Gamma\left(\frac{n}{c}+1\right)}\frac{n}{c} + (1-\alpha)(1+cx)\sum_{k=0}^{\infty} \frac{(cx)^k cx}{(1+cx)^{\frac{n}{c}+k+1}} \frac{\Gamma\left(\frac{n}{c}+k+1\right)}{k!\Gamma\left(\frac{n}{c}+1\right)}\frac{n}{c}$$
$$= \alpha nx - 2(1-\alpha)cx - (1-\alpha)ncx^2 + (1-\alpha)(1+cx)nx$$
$$= nx - 2(1-\alpha)cx.$$

We have proof (i) and (ii). Finally, making same process in (i) and (ii), we can obtain (iii), (iv) and (v) easily.

Lemma 2.2 Let $e_i(t) = t^i$, $n \neq jc$ and $j \leq i$ for every $i, j \in \{0,1,2,3,4\}$. We have

(i) $G_{n,c}^{\alpha}(e_0;x) = 1,$

(ii) $G_{n,c}^{\alpha}(e_1;x) = \frac{n+2(\alpha-1)c}{n-c}x,$

(iii) $G_{n,c}^{\alpha}(e_2;x) = \frac{n(n+c)+4(\alpha-1)nc}{(n-c)(n-2c)}x^2 + \frac{2n+6(\alpha-1)c}{(n-c)(n-2c)}x,$

(iv) $G_{n,c}^{\alpha}(e_3;x) = \frac{n(n+c)(n+2c)+6(\alpha-1)n(n+c)c}{(n-c)(n-2c)(n-3c)}x^3 + \frac{6n(n+c)+30(\alpha-1)nc}{(n-c)(n-2c)(n-3c)}x^2 + \frac{6n+24(\alpha-1)c}{(n-c)(n-2c)(n-3c)}x,$

(v) $G_{n,c}^{\alpha}(e_4;x) = \frac{n(n+c)(n+2c)(n+3c)+8(\alpha-1)n(n+c)(n+2c)c}{(n-c)(n-2c)(n-3c)(n-4c)}x^4 + \frac{12n(n+c)(n+2c)+84(\alpha-1)n(n+c)c}{(n-c)(n-2c)(n-3c)(n-4c)}x^3$
$+\frac{36n(n+c)+216(\alpha-1)nc}{(n-c)(n-2c)(n-3c)(n-4c)}x^2 + \frac{24n+120(\alpha-1)c}{(n-c)(n-2c)(n-3c)(n-4c)}x.$

Proof.
The following identities are derived from Beta functions and equation (2).

$$\int_0^{\infty} P_{n+c,k-1}(t;c)\, t^i dt = \frac{\Gamma\left(\frac{n}{c}+k\right)}{(k-1)!\,\Gamma\left(\frac{n}{c}+1\right)} \frac{B\left(k+i, \frac{n}{c}-i\right)}{c^{i+1}}. \tag{5}$$

From equation (5) and Lemma 2.1, we have

(i) $G_{n,c}^{\alpha}(e_0;x) = n\sum_{k=1}^{\infty} P_{n,k}^{\alpha}(x;c)\frac{1}{n} + P_{n,0}^{\alpha}(x;c)$
$= \sum_{k=0}^{\infty} P_{n,k}^{\alpha}(x;c)$
$= 1,$

(ii) $G_{n,c}^{\alpha}(e_1;x) = n\sum_{k=1}^{\infty} P_{n,k}^{\alpha}(x;c)\frac{k}{n(n-c)}$

$$= \frac{1}{n-c}\sum_{k=0}^{\infty} P_{n,k}^{\alpha}(x;c)k$$
$$= \frac{n+2(\alpha-1)c}{n-c}x,$$

(iii) $G_{n,c}^{\alpha}(e_2;x) = n\sum_{k=1}^{\infty} P_{n,k}^{\alpha}(x;c)\frac{k(k+1)}{n(n-c)(n-2c)}$

$$= \sum_{k=0}^{\infty} P_{n,k}^{\alpha}(x;c)\frac{k^2}{(n-c)(n-2c)} + \sum_{k=0}^{\infty} P_{n,k}^{\alpha}(x;c)\frac{k}{(n-c)(n-2c)}$$
$$= \frac{1}{(n-c)(n-2c)}\sum_{k=0}^{\infty} P_{n,k}^{\alpha}(x;c)k^2 + \frac{1}{(n-c)(n-2c)}\sum_{k=0}^{\infty} P_{n,k}^{\alpha}(x;c)k$$
$$= \frac{n(n+c)+4(\alpha-1)nc}{(n-c)(n-2c)}x^2 + \frac{2n+6(\alpha-1)c}{(n-c)(n-2c)}x.$$

We have proof (i), (ii) and (iii). Finally, if we make same process in (i), (ii) and (iii), we obtain (iv) and (v) easily.

Lemma 2.3 Let $\eta_i(t) = (t-x)^i$, $i \in \{1,2,4\}$ and $n \neq jc,\ j \leq i$ for every $j \in \{0,1,2,3,4\}$. We have

(i) $G_{n,c}^{\alpha}(\eta_1;x) = \frac{c+2(\alpha-1)c}{(n-c)}x,$

(ii) $G_{n,c}^{\alpha}(\eta_2;x) = \frac{2(n+c)c+8(\alpha-1)c^2}{(n-c)(n-2c)}x^2 + \frac{2n+6(\alpha-1)c}{(n-c)(n-2c)}x,$

(iii) $G_{n,c}^{\alpha}(\eta_4;x) = \frac{12n^2c^2+156nc^3+24c^4+8(\alpha-1)c(24nc^2+24c^3)}{(n-c)(n-2c)(n-3c)(n-4c)}x^4$

$$+ \frac{24n^2c+264nc^2+12(\alpha-1)c(26nc+36c^2)}{(n-c)(n-2c)(n-3c)(n-4c)}x^3$$
$$+ \frac{12n^2+132nc+24(\alpha-1)c(5n+16c)}{(n-c)(n-2c)(n-3c)(n-4c)}x^2$$
$$+ \frac{24n+120(\alpha-1)c}{(n-c)(n-2c)(n-3c)(n-4c)}x$$

Weighted Approximation

$B_\rho[0,\infty)$ is the space of all functions that are defined on the unbounded interval $[0,\infty)$ satisfying the inequality

$|f(x)| \leq M_f\rho(x),$

where M_f is a positive constant only depending on function f and $\rho(x) = 1 + \varphi(x)^2$. Here, $\varphi(x)$ is monotone increasing continuous function on the real axis. Also, let us define the spaces

$C_\rho[0,\infty) = B_\rho[0,\infty) \cap C[0,\infty)$

and

$$C_\rho^*[0,\infty) = \left\{f \in C_\rho[0,\infty): \lim_{x\to\infty}\frac{|f(x)|}{\rho(x)} < \infty\right\}$$

and endow them with the norm

$$\|f\|_\rho = \sup\left\{\frac{|f(x)|}{\rho(x)}: x \in [0,\infty)\right\}.$$

Theorem 3.1[32,33]
a) There exists a sequence of linear positive operators $A_n: C_\rho[0,\infty) \to B_\rho[0,\infty)$ such that

$$\lim_{n\to\infty}\|A_n(\varphi^\nu) - \varphi^\nu\|_\rho = 0, \quad \nu = 0,1,2, \tag{6}$$

and there exists a function $f^* \in C_\rho[0,\infty)\backslash C_\rho^*[0,\infty)$ with
$\lim_{n\to\infty}\|A_n(f^*) - f^*\|_\rho \geq 1.$

b) If conditions (6) are satisfied by a sequence of linear positive operators $A_n: C_\rho[0,\infty) \to B_\rho[0,\infty)$, then for every $f \in C_\rho^*[0,\infty)$, we have

$$\lim_{n\to\infty} \|A_n(f) - f\|_\rho = 0\,.$$

Choosing $\rho(x) = 1 + x^2$, we obtain the following theorems.
Theorem 3.2 For every $f \in C_\rho^*[0,\infty)$, we have

$$\lim_{n\to\infty} \left\|G_{n,c}^\alpha(f) - f\right\|_\rho = 0.$$

Proof.
From Theorem 3.1, verifying the following three conditions is sufficient.

$$\lim_{n\to\infty} \left\|G_{n,c}^\alpha(e_i) - e_i\right\|_\rho = 0, \qquad i = 0,1,2. \tag{7}$$

Since $G_{n,c}^\alpha(e_0;x) = 1$, condition (7) holds for $i = 0$. From Lemma 2.2, we have the following inequality,

$$\begin{aligned}\left\|G_{n,c}^\alpha(e_1) - e_1\right\|_\rho &\le \sup_{x\ge0} \frac{\left|\frac{n+2(\alpha-1)c}{n-c}x - x\right|}{1+x^2}\\ &\le \sup_{x\ge0} \frac{\left|\frac{(2\alpha-1)c}{n-c}x\right|}{1+x^2}\\ &\le \left|\frac{(2\alpha-1)c}{n-c}\right|.\end{aligned}$$

Hence, for $\lim_{n\to\infty} \left\|G_{n,c}^\alpha(e_1) - e_1\right\|_\rho = 0$ which implies that the condition in (7) holds for $i = 1$.
Similarly, we can write the following inequality.

$$\begin{aligned}\left\|G_{n,c}^\alpha(e_2) - e_2\right\|_\rho &\le \sup_{x\ge0} \frac{\left|\frac{n(n+c)+4(\alpha-1)nc}{(n-c)(n-2c)}x^2 + \frac{2n+6(\alpha-1)c}{(n-c)(n-2c)}x - x^2\right|}{1+x^2}\\ &\le \sup_{x\ge0} \frac{\left|\left(\frac{4n\alpha c-2c^2}{(n-c)(n-2c)}\right)x^2 + \frac{2n+6(\alpha-1)c}{(n-c)(n-2c)}x\right|}{1+x^2}\\ &\le \left|\frac{4n\alpha c-2c^2+2n+6|\alpha-1|c}{(n-c)(n-2c)}\right|.\end{aligned}$$

Then, $\lim_{n\to\infty} \left\|G_{n,c}^\alpha(e_2) - e_2\right\|_\rho = 0$ which implies that the condition in (7) holds for $i = 2$.
Therefore, the proof is completed.

Rate of Convergence

We want to find the rate of convergence for the $G_{n,c}^\alpha$ operators in this section. As we know, if f isn't continuous uniformly on $[0,\infty)$, then $\omega(f,\delta)$ which is the usual first modulus of continuity doesn't tend to zero as $\delta \to 0$. For every $f \in C_\rho^*[0,\infty)$, a weighted modulus of continuity would be liked to define $\Omega(f,\delta)$ that tends to zero as $\delta \to 0$.
Let

$$\Omega(f,\delta) = \sup_{x\ge0,|h|\le\delta} \frac{|f(x+h) - f(x)|}{(1+x^2)(1+h^2)}, \qquad \text{for } f \in C_\rho^*[0,\infty). \tag{8}$$

$\Omega(f,\delta)$ is called the weighted modulus of continuity of the function $f \in C_\rho^*[0,\infty)$ [34].
We will show that the weighted modulus of continuity defined in equation (8) has similar properties to the first modulus of continuity.
Lemma 4.1 [34] Let $f \in C_\rho^*[0,\infty)$. Then, we have
(i) $\Omega(f,\delta)$ is a monotonically increasing function of δ,
(ii) for every $f \in C_\rho^*[0,\infty)$, $\lim_{\delta\to0^+} \Omega(f,\delta) = 0$,
(iii) for every $m \in \mathbb{N}$, $\Omega(f,m\delta) \le m\Omega(f,\delta)$,
(iv) for $\lambda \in \mathbb{R}^+$, $\Omega(f,\lambda\delta) \le (\lambda+1)\Omega(f,\delta)$.

Theorem 4.1 For $f \in C_\rho^*[0,\infty)$, $M > 0$ and $\bar{\rho}(x) = 1 + x^5$, we get

$$\left\| G_{n,c}^{\alpha}(f) - f \right\|_{\bar{\rho}} \leq M\Omega\left(f, \frac{1}{\sqrt{n}}\right).$$

.

Proof.

From the definition $\Omega(f,\delta)$ and Lemma 4.1(iv), we can write

$$|f(t) - f(x)| \leq (1 + (t-x)^2)(1 + x^2)\left(1 + \frac{|t-x|}{\delta}\right)\Omega(f,\delta).$$

Then, we obtain

$$\begin{aligned} \left|G_{n,c}^{\alpha}(f(t) - f(x); x)\right| &\leq G_{n,c}^{\alpha}(|f(t) - f(x)|; x) \\ &\leq \Omega(f,\delta)(1 + x^2) G_{n,c}^{\alpha}\left((1 + (t-x)^2)\left(1 + \frac{|t-x|}{\delta}\right); x\right) \end{aligned}$$

$$= \Omega(f,\delta)(1 + x^2)\left[G_{n,c}^{\alpha}(1 + (t-x)^2; x) + G_{n,c}^{\alpha}\left(\frac{|t-x|}{\delta}; x\right) + G_{n,c}^{\alpha}\left(\frac{|t-x|^3}{\delta}; x\right)\right]. \quad (9)$$

Applying the Cauchy-Schwarz inequality to the second and third term of (9), we obtain

$$G_{n,c}^{\alpha}\left(\frac{|t-x|}{\delta}; x\right) \leq \sqrt{G_{n,c}^{\alpha}\left(\frac{|t-x|^2}{\delta^2}; x\right)}\sqrt{G_{n,c}^{\alpha}(1; x)}$$

and

$$G_{n,c}^{\alpha}\left(\frac{|t-x|^3}{\delta}; x\right) \leq \sqrt{G_{n,c}^{\alpha}((t-x)^4; x)}\sqrt{G_{n,c}^{\alpha}\left(\frac{|t-x|^2}{\delta^2}; x\right)}. \quad (10)$$

Due to (9) and (10) we get

$$G_{n,c}^{\alpha}(|f(t) - f(x)|; x) \leq \Omega(f,\delta)(1 + x^2)\left[G_{n,c}^{\alpha}(1 + (t-x)^2; x) + \frac{1}{\delta}\sqrt{G_{n,c}^{\alpha}((t-x)^2; x)} + \frac{1}{\delta}\sqrt{G_{n,c}^{\alpha}((t-x)^4; x)}\sqrt{G_{n,c}^{\alpha}((t-x)^2; x)}\right].$$

There exist positive constants $M_1$and M_2 such that

$$G_{n,c}^{\alpha}(1 + (t-x)^2; x) \leq M_1(1 + x^2), \quad (11)$$

$$\sqrt{G_{n,c}^{\alpha}((t-x)^4; x)} \leq M_2(1 + x^2). \quad (12)$$

Notice that from Lemma 2.3, we have

$$G_{n,c}^{\alpha}((t-x)^2; x) = \frac{2(n+c)c + 8(\alpha-1)c^2}{(n-c)(n-2c)}x^2 + \frac{2n + 6(\alpha-1)c}{(n-c)(n-2c)}x.$$

Hence, we get

$$\begin{aligned} \sqrt{G_{n,c}^{\alpha}\left(\frac{|t-x|^2}{\delta^2}; x\right)} &\leq \frac{1}{\delta} O\left(\frac{1}{\sqrt{n}}\right)\sqrt{(x + x^2)} \\ &\leq M_3 \frac{(1+x)}{\delta\sqrt{n}}. \end{aligned} \quad (13)$$

Then, from (12) and (13) we yield

$$\sqrt{G_{n,c}^{\alpha}((t-x)^4; x)}\sqrt{G_{n,c}^{\alpha}\left(\frac{|t-x|^2}{\delta^2}; x\right)} \leq M_5 \frac{1+x^3}{\delta\sqrt{n}}, \quad (14)$$

where $M_5 = M_2 M_3 M_4$ and $M_4 = \sup_{x\geq 0} \frac{(1+x^2)(1+x)}{1+x^3}$.
Lastly, from (11)−(14) and choosing $M_8 = M_1 M_6 + M_3 M_7 + M_5$ where $M_6 = \sup_{x\geq 0} \frac{1+x^2}{1+x^3}$, $M_7 = \sup_{x\geq 0} \frac{1+x}{1+x^3}$; choosing $M = M_8 M_9$ where $M_9 = \sup_{x\geq 0} \frac{(1+x^2)(1+x^3)}{1+x^5}$ and choosing $\delta = \frac{1}{\sqrt{n}}$ then combining the estimate between (9) and (14), we end up with the result of

$$\left|G_{n,c}^{\alpha}(f(t) - f(x); x)\right| \leq M(1 + x^5)\Omega\left(f, \frac{1}{\sqrt{n}}\right).$$

Therefore, we reach the result of the theorem.

Pointwise approximation properties by $G_{n,c}^{\alpha}$

The usual modulus of continuity of $f \in \hat{C}_B[0, \infty)$ is given by

$$\omega(f, \delta) = \sup_{|x-y|\leq\delta} |f(x) - f(y)|, \tag{15}$$

where $\hat{C}_B[0, \infty)$ space of uniformly continuous and bounded functions on $[0, \infty)$.
Theorem 5.1 Let $f \in \hat{C}_B[0, \infty)$, then the following inequality holds

$$\left|G_{n,c}^{\alpha}(f; x) - f(x)\right| \leq 2\omega\left(f, \sqrt{\Theta_n(x)}\right),$$

where $\Theta_n(x) = G_{n,c}^{\alpha}(\eta_2; x)$.

Proof.

From Lemma 2.3 and Shisha Mond Theorem [35] which states that if $G_{n,c}^{\alpha}$ is a linear positive operator, then for any bounded function f we obtain

$$\begin{aligned}\left|G_{n,c}^{\alpha}(f; x) - f(x)\right| &\leq \omega(f, \delta)\left(1 + \frac{1}{\delta}\sqrt{G_{n,c}^{\alpha}(\eta_2; x)}\right)\\ &= \omega(f, \delta)\left(1 + \frac{1}{\delta}\sqrt{\Theta_n(x)}\right)\\ &\leq 2\omega\left(f, \sqrt{\Theta_n(x)}\right),\end{aligned}$$

where $\delta = \sqrt{\Theta_n(x)}$.

Now, we mention the rate of convergence of $G_{n,c}^{\alpha}$ whereby Peetre $\mathcal{K}$-functional [36].

Let us define the space $\hat{C}_B^2[0, \infty) = \{f \in \hat{C}_B[0, \infty): f', f'' \in \hat{C}_B[0, \infty)\}$ and endow it with the norm $\|f\|_{\hat{C}_B[0,\infty)} = \|f\|_{\hat{C}_B[0,\infty)} + \|f'\|_{\hat{C}_B[0,\infty)} + \|f''\|_{\hat{C}_B[0,\infty)}$ and $\hat{C}_B[0, \infty)$ with the norm $\|f\|_{\hat{C}_B[0,\infty)} = \sup\{|f(x)|: x \in [0, \infty)\}$.

For $g \in \hat{C}_B^2[0, \infty)$, $f \in \hat{C}_B[0, \infty)$ and $\delta > 0$, the Peetre $\mathcal{K}$-functional is defined by

$$\mathcal{K}(f, \delta) = \inf_{g\in\hat{C}_B^2[0,\infty)} \{\|f - g\|_{\hat{C}_B[0,\infty)} + \delta\|g''\|_{\hat{C}_B[0,\infty)}\}.$$

There exists an absolute constant $M \in \mathbb{R}^+$, such that

$$\mathcal{K}(f, \delta) \leq M\omega_2(f, \sqrt{\delta}),$$

where $\omega_2(f, \sqrt{\delta})$ is the second order modulus of continuity of $f \in \hat{C}_B[0, \infty)$. $\omega_2(f, \sqrt{\delta})$ defined as

$$\omega_2(f, \sqrt{\delta}) \leq \sup_{0<h\leq\sqrt{\delta}} \sup_{x,x+h\in[0,\infty)} |f(x + 2h) - 2f(x + h) + f(x)|.$$

Theorem 5.2 Let $f \in \hat{C}_B[0,\infty)$ and $x \in [0,\infty)$.Then, there exists a constant $M \in \mathbb{R}^+$ such that

$$\left|G_{n,c}^{\alpha}(f;x) - f(x)\right| \le 4M\omega_2\left(f,\sqrt{\delta_n(x)}\right) + \omega\left(f,\frac{2(\alpha-1)c+c}{n-c}x\right),$$

where $\delta_n(x) = \frac{2(n+c)c+8(\alpha-1)c^2}{(n-c)(n-2c)}x^2 + \frac{(2\alpha-1)^2c^2}{(n-c)(n-c)}x^2 + \frac{2n+6(\alpha-1)c}{(n-c)(n-2c)}x, n \ne c$ and $n \ne 2c$.

Proof.

Let $f \in \hat{C}_B[0,\infty)$ and $t \in [0,\infty)$. Then by Taylor's expansion we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du. \tag{16}$$

When we apply $G_{n,c}^{\alpha}$ to both sides of the above equation (16) , we get $G_{n,c}^{\alpha}(t-x;x) \ne 0$. So, we must define an operator as follows

$$\breve{G}_{n,c}^{\alpha}(f;x) = G_{n,c}^{\alpha}(f;x) - f\left(\frac{n+2(\alpha-1)c}{n-c}x\right) + f(x). \tag{17}$$

Now, applying $\breve{G}_{n,c}^{\alpha}$ to both side of the equation (16), we get

$$\breve{G}_{n,c}^{\alpha}(g;x) - g(x) = \breve{G}_{n,c}^{\alpha}\left(\int_x^t (t-u)g''(u)du\,;x\right).$$

From the definition of equation (17), we yield

$$\breve{G}_{n,c}^{\alpha}(g;x) - g(x) = G_{n,c}^{\alpha}\left(\int_x^t (t-u)g''(u)du\,;x\right) - \int_x^{\frac{n+2(\alpha-1)c}{n-c}x}\left(\frac{n+2(\alpha-1)c}{n-c}x-u\right)g''(u)du + \int_x^x (x-u)g''(u)du.$$

Using linearity and positivity properties of $G_{n,c}^{\alpha}$, we write the inequality

$$\begin{aligned}\left|\breve{G}_{n,c}^{\alpha}(g;x) - g(x)\right| &\le \left|G_{n,c}^{\alpha}\left(\int_x^t (t-u)g''(u)du\,;x\right)\right| + \left|\int_x^{\frac{n+2(\alpha-1)c}{n-c}x}\left(\frac{n+2(\alpha-1)c}{n-c}x-u\right)g''(u)du\right| \\ &\le G_{n,c}^{\alpha}\left(\int_x^t |t-u||g''(u)|du\,;x\right) + \int_x^{\frac{n+2(\alpha-1)c}{n-c}x}\left|\frac{n+2(\alpha-1)c}{n-c}x-u\right||g''(u)|du.\end{aligned}$$

Then, we get

$$\left|\breve{G}_{n,c}^{\alpha}(g;x) - g(x)\right| \le \|g''\|_{\hat{C}_B[0,\infty)}\ G_{n,c}^{\alpha}((t-x)^2;x) + \|g''\|_{\hat{C}_B[0,\infty)}\ \left(\frac{n+2(\alpha-1)c}{n-c}x-x\right)^2.$$

From Lemma 2.3, we write

$$\begin{aligned}\left|\breve{G}_{n,c}^{\alpha}(g;x) - g(x)\right| &\le \|g''\|_{\hat{C}_B[0,\infty)}\ \left(\frac{2(n+c)c+8(\alpha-1)c^2}{(n-c)(n-2c)}x^2 + \frac{2n+6(\alpha-1)c}{(n-c)(n-2c)}x\right) \\ &\quad + \|g''\|_{\hat{C}_B[0,\infty)}\ \left(\frac{(2\alpha-1)^2c^2}{(n-c)(n-c)}x^2\right) \\ &= \|g''\|_{\hat{C}_B[0,\infty)}\ \left(\frac{2(n+c)c+8(\alpha-1)c^2}{(n-c)(n-2c)}x^2 + \frac{(2\alpha-1)^2c^2}{(n-c)(n-c)}x^2 + \frac{2n+6(\alpha-1)c}{(n-c)(n-2c)}x\right) \\ &= \|g''\|_{\hat{C}_B[0,\infty)}\ \delta_n(x),\end{aligned} \tag{18}$$

where $\delta_n(x) = \frac{2(n+c)c+8(\alpha-1)c^2}{(n-c)(n-2c)}x^2 + \frac{(2\alpha-1)^2c^2}{(n-c)(n-c)}x^2 + \frac{2n+6(\alpha-1)c}{(n-c)(n-2c)}x.$

From equation (17) for $f \in \hat{C}_B[0,\infty)$, we get

$$\begin{aligned}\left|\breve{G}_{n,c}^{\alpha}(f;x)\right| &\le \left|G_{n,c}^{\alpha}(f;x)\right| + \left|f\left(\frac{n+2(\alpha-1)c}{n-c}x\right)\right| + |f(x)| \\ &\le G_{n,c}^{\alpha}(|f|;x) + \|f\|_{\hat{C}_B[0,\infty)} + \|f\|_{\hat{C}_B[0,\infty)} \\ &\le 3\|f\|_{\hat{C}_B[0,\infty)}.\end{aligned} \tag{19}$$

From equation (17), we obtain

$$\left|G_{n,c}^{\alpha}(f;x)-f(x)\right| \leq \left|\breve{G}_{n,c}^{\alpha}((f-g);x)-(f-g)(x)\right| + \left|\breve{G}_{n,c}^{\alpha}(g;x)-g(x)\right| + \left|f\left(\frac{n+2(\alpha-1)c}{n-c}x\right)-f(x)\right|. \quad (20)$$

From (15) and (17)–(20), we yield

$$\left|G_{n,c}^{\alpha}(f;x)-f(x)\right| \leq 4\|f-g\|_{\hat{C}_B[0,\infty)} + \|g''\|_{\hat{C}_B[0,\infty)}\,\delta_n(x) + \omega\left(f,\frac{2(\alpha-1)c+c}{n-c}x\right).$$

Taking infimum on the right side of the above inequality overall $g \in \hat{C}_B^2[0,\infty)$, we get

$$\left|G_{n,c}^{\alpha}(f;x)-f(x)\right| \leq 4\mathcal{K}\left(f,\delta_n(x)\right) + \omega\left(f,\frac{2(\alpha-1)c+c}{n-c}x\right).$$

Consequently, we get

$$\left|G_{n,c}^{\alpha}(f;x)-f(x)\right| \leq 4M\omega_2\left(f,\sqrt{\delta_n(x)}\right) + \omega\left(f,\frac{2(\alpha-1)c+c}{n-c}x\right).$$

Theorem 5.3 Let $f \in \mathrm{Lip}_M(\sigma)$ with $M > 0$ and $0 < \sigma \leq 1$. Then the operators $G_{n,c}^{\alpha}$ satisfy

$$\left|G_{n,c}^{\alpha}(f;x)-f(x)\right| \leq M\left(\Theta_n(x)\right)^{\sigma/2},$$

where $\mathrm{Lip}_M(\sigma) \coloneqq \left\{f \in \hat{C}_B[0,\infty) \ : |f(t)-f(x)| \leq M|t-x|^{\sigma},\ \ t,x \in [0,\infty)\right\}$.

Proof.

If f satisfies the Lipschitz conditions, we have

$$|f(t)-f(x)| \leq M|t-x|^{\sigma}. \quad (21)$$

By the linearity and positivity properties of $G_{n,c}^{\alpha}$ and inequality (21), we get

Thanks to Lemma 2.3 and using Hölder inequality with $p=\frac{2}{\sigma}$, $q=\frac{2}{2-\sigma}$, we have

$$\begin{aligned}\left|G_{n,c}^{\alpha}(f;x)-f(x)\right| &\leq MG_{n,c}^{\alpha}((t-x)^2;x)^{\sigma/2}\\ &= M\left(\frac{2(n+c)c+8(\alpha-1)c^2}{(n-c)(n-2c)}x^2+\frac{2n+6(\alpha-1)c}{(n-c)(n-2c)}x\right)^{\sigma/2}\\ &= M\left(\Theta_n(x)\right)^{\sigma/2} \text{for } n \neq c \text{ and } n \neq 2c.\end{aligned}$$

Voronovskaja Type Theorem

Theorem 6.1 Let $f \in \hat{C}_B^2[0,\infty)$. Then the following equality holds

$$\lim_{n\to\infty} n\left(G_{n,c}^{\alpha}(f;x)-f(x)\right) = (2\alpha-1)(cx)f'(x) + (cx^2+x)f''(x).$$

.

Proof.

For any $x \geq 0$, using Lemma 2.3 we have

$$\lim_{n\to\infty} nG_{n,c}^{\alpha}(n_1;x) = (2\alpha-1)(cx), \quad (22)$$

$$\lim_{n\to\infty} nG_{n,c}^{\alpha}(n_2;x) = 2cx^2+2x, \quad (23)$$

$$\lim_{n\to\infty} n^2 G_{n,c}^{\alpha}(n_4; x) = 12c^2x^4 + 24cx^3 + 12x^2. \tag{24}$$

By using Taylor's formula, we write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \mu(t,x)(t-x)^2, \tag{25}$$

where $\mu(t;x)$ is Peano form of the remainder term and $\lim_{t\to x}\mu(t,x) = 0$.
When we apply $G_{n,c}^{\alpha}$ to (25) and use linearity of $G_{n,c}^{\alpha}$, we have

$$G_{n,c}^{\alpha}(f;x) - f(x) = f'(x)G_{n,c}^{\alpha}((t-x);x) + \frac{1}{2}f''(x)G_{n,c}^{\alpha}((t-x)^2;x) + G_{n,c}^{\alpha}(\mu(t,x)(t-x)^2;x). \tag{26}$$

When we multiply (26) by n, take the limit n goes to infinity and use (22) and (23), we achieve

$$\lim_{n\to\infty} n\left(G_{n,c}^{\alpha}(f;x) - f(x)\right) = (2\alpha-1)(cx)f'(x) + (cx^2+x)f''(x) + \lim_{n\to\infty} nG_{n,c}^{\alpha}(\mu(t,x)(t-x)^2;x). \tag{27}$$

If we use Cauchy-Schwarz inequality for the remainder term in equation (27), we obtain

$$nG_{n,c}^{\alpha}(\mu(t,x)(t-x)^2;x) \le \sqrt{G_{n,c}^{\alpha}(\mu^2(t,x);x)}\sqrt{n^2G_{n,c}^{\alpha}((t-x)^4;x)}.$$

.

Let $\psi(t,x) = \mu^2(t,x)$. Since $\psi(.,x)$ is continuous at $t \in [0,\infty)$ and $\lim_{t\to x}\mu(t,x) = 0$, we observe that

$$\lim_{t\to x}\mu^2(t,x) = \lim_{t\to x}\psi(t,x) = \psi(x,x) = 0. \tag{28}$$

When we use (24) and (28), we have

$$\lim_{n\to\infty} nG_{n,c}^{\alpha}(\mu(t,x)(t-x)^2;x) = 0. \tag{29}$$

Finally, from (27) and (29) we yield

$$\lim_{n\to\infty} n\left(G_{n,c}^{\alpha}(f;x) - f(x)\right) = (2\alpha-1)(cx)f'(x) + (cx^2+x)f''(x).$$

Therefore, we complete the proof.

Graphical Analysis

In this section, the convergence of the operators $G_{5,c}^{\alpha}$ with function $f(x) = x^2$ is illustrated in Figure 1 for different values of $\alpha = 1, \alpha = {}^{1}/_{10}$ and $c = 1$ on the interval $[0,\infty)$. Also, the convergence of the operators $G_{5,c}^{\alpha}$ with the function $f(x) = x^3$ is illustrated in Figure 2 for different values of $\alpha = 1$, $\alpha = {}^{1}/_{10}, \alpha = {}^{1}/_{1000}$ and $c = 1$ on the interval $[0,\infty)$.

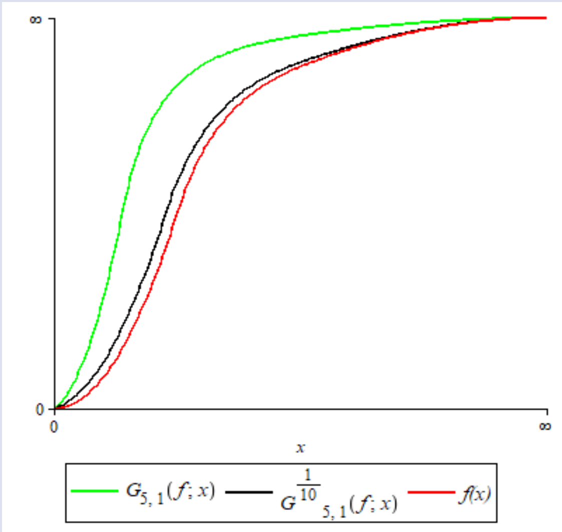

Figure 1. $n = 5$, $c = 1$ and $f(x) = x^2$

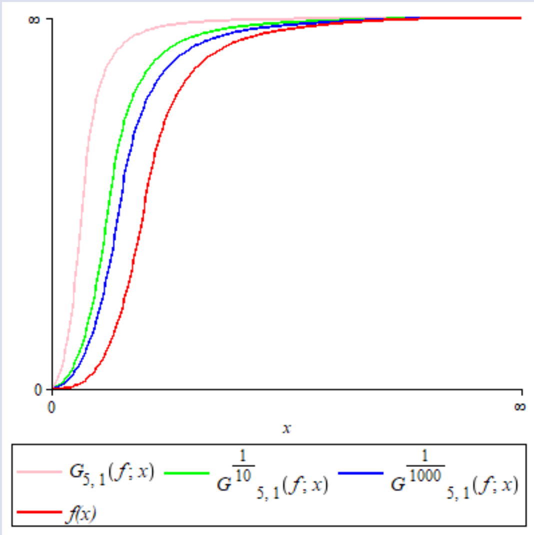

Figure 2. $n = 5$, $c = 1$ and $f(x) = x^3$

Acknowledgment

The authors would like to sincerely thank the editor and the reviewers for the helpful comments and suggestions.

Conflict of interests

The authors report no conflict of interest.

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