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Common Fixed Point Results for Suzuki Type Contractions on Partial Metric Spaces with an Application

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ABSTRACT

In this article, we prove a common fixed point theorem for Suzuki type contractions on complete partial metric spaces. Moreover, we state some corollaries related to Suzuki type common fixed point theorem. And we give an example where we apply our main theorem on complete partial metric spaces. Finally, to show usability of our results, we give its an application showing existence and uniqueness of a common solution for a class of functional equations in dynamic programming.

Keywords: Partial metric spaces, Fixed point theory, Completeness.

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Introduction

In 2008, Suzuki [1] introduced a useful generalization of Banach fixed point theorem called as Suzuki fixed point theorem as follows:

Let (X,d) be a complete metric space and let T be a self-mapping on X. We consider a nonincreasing function by

$$\theta(r)=\begin{cases} 1, & 0\le r\le \dfrac{\sqrt{5}-1}{2}, \\ \dfrac{1-r}{r^2}, & \dfrac{\sqrt{5}-1}{2}\le r\le \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r}, & \dfrac{1}{\sqrt{2}}\le r<1. \end{cases}$$

Assume that there exists $r\in[0,1)$ such that

$$\theta(r)d(x,Tx)\le d(x,y)\Rightarrow d(Tx,Ty)\le rd(x,y)$$

for all $x,y\in X$. Then, there exists a unique fixed point u of T. Moreover, $\lim_{n\to\infty} T^n x=u$ for all $x\in X$. In view of this generalization, several authors generalized Suzuki's fixed point theorem, see [2-9] and the references therein. In recently, Wangwe and Kumar [10] combined Kannan and Suzuki results and introduced a new fixed point theorem in TVS valued cone metric space.

In 1994, Matthews introduced the concept of partial metric spaces [11]. They are seen as a part of the study of denotational semantics of dataflow networks and play an important role in the creation of models in the computational theory. So, many authors studied on partial metric spaces, and they gave different fixed point theorems on these type metric spaces, such as Kannan's, Caristi's, Nadler's and Suzuki's. For more details, the readers can refer to [12-19].

In this paper, we prove a common fixed point theorem for Suzuki type contractions on complete partial metric spaces. We also state some corollaries related to Suzuki type common fixed point theorem. We also give an example where we apply our main theorem on complete partial metric spaces. Finally, to show usability of our results, we give its an application showing existence and uniqueness of a common solution for a class of functional equations in dynamic programming

Preliminaries

We start by recalling a series of definitions of some fundamental notions related to partial metric spaces. In the following $\mathbb{R}^+$ stands for the set of all non-negative real numbers, i.e., $\mathbb{R}^+=[0,\infty)$.

Definition 1. Let $X\neq\varnothing$. A function $p:X\times X\to\mathbb{R}^+$ is called a partial metric, if it holds the following properties for all $x,y,z\in X$

(p1) $x=y\Leftrightarrow p(x,x)=p(x,y)=p(y,y)$,

(p2) $p(x,x)\le p(x,y)$,

(p3) $p(x,y)=p(y,x)$,

(p4) $p(x,y)\le p(x,z)+p(z,y)-p(z,z)$.

A pair (X,p) is called partial metric space. Shortly, we write PMS for the partial metric space. From (p1) and (p2), we get that if $p(x,y)=0$, then $x=y$. But the opposite may not be true. If we define partial metric as $p(x,y)=\max\{x,y\}$ for all $x,y\in\mathbb{R}^+$, then the pair $(\mathbb{R}^+,p)$ is a PMS. This is a basic for PMS [11].

p induces a T_0 topology τ_p on X having the base

$$\{B_p(a,\varepsilon): a\in X, \varepsilon>0\},$$

where $B_p(a,\varepsilon)=\{b\in X: p(a,b)<p(a,a)+\varepsilon\}$ for all $a\in X$ and $\varepsilon>0$ [9].

Definition 2. Let (X,p) be a PMS.

(1) A sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to a point $x\in X$ if and only if $p(x,x)=\lim\limits_{n\to\infty} p(x,x_n)$.

(2) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called a Cauchy sequence if there exists (and is finite)

$$\lim_{n,m\to\infty} p(x_n,x_m).$$

(3) (X,p) is called complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X converges, with respect to τ_p, to a point $x\in X$ such that $p(x,x)=\lim\limits_{n,m\to\infty} p(x_n,x_m)$ [11].

Lemma 1. Let (X,p) be a PMS and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X Suppose that $x_n\to u$ as $n\to\infty$ in a PMS (X,p) such that $p(u,u)=0$. Then $\lim\limits_{n\to\infty} p(x_n,y)=p(u,y)$ for every $y\in X$ [20].

The function $p^s: X\times X\to[0,\infty)$ given by $p^s(x,y)=2p(x,y)-p(x,x)-p(y,y)$ is a metric on X where p is a partial metric on X [11].

Lemma 2. Let (X,p) be a PMS and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(2) (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore,

$$\lim_{n\to\infty} p^s(x,x_n)=0 \Leftrightarrow p(x,x)=\lim_{n\to\infty} p(x,x_n)=\lim_{n\to\infty} p(x_n,x_m) \text{ [11].}$$

Main Results

Theorem 1. Let (X,p) be a complete PMS, $T,S: X\to X$ be two self-maps and a nonincreasing function $\theta:[0,1)\to\left(\frac{1}{2},1\right]$ be defined by

$$\theta(r)=\begin{cases} 1, & 0\le r\le \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2}, & \frac{\sqrt{5}-1}{2}\le r\le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \frac{1}{\sqrt{2}}\le r<1. \end{cases}$$

If there exists $r\in\left[0,\frac{1}{2}\right)$ such that

$$\theta(r)\min\{p(x,Tx),p(x,Sx)\}\le p(x,y) \tag{1}$$

Implies

$$\max\left\{p(Sx,Sy),p(Tx,Ty),\frac{1}{2}\left[p(Sx,Ty)+p(Sy,Tx)\right]\right\}\le rp(x,y),$$

for all $x,y\in X$, then T and S have a unique common fixed point.

Proof. Firstly, we prove that if u is a fixed point of T (or S), then it is also fixed point of S (or T). We suppose that u is a fixed point of T, that is, $Tu=u$. We want to show that $Su=u$. Taking $x=u$ and $y=Tu$ in (1), we get

$$0 \leq \theta(r)\min\{p(u,Tu),p(u,Su)\} \leq p(u,Tu)$$

implies

$$p(Su,u) \leq \max\left\{p(Su,STu),p(Tu,T^2u),\frac{1}{2}\left[p(Su,T^2u)+p(STu,Tu)\right]\right\}$$
$$\leq rp(u,Tu).$$

Hence, we have $p(Su,u) \leq rp(u,u)$. From property $(p2)$ of PMS, we get

$$p(u,u) \leq p(Su,u) \leq rp(u,u).$$

So, $p(u,u)=0$. Then we get $Su=u$. Similarly, the contrary can be shown easily. Therefore, it is enough to show that T has a fixed point to complete the proof. Putting $y=Sx$ in (1), we have

$$\theta(r)\min\{p(x,Tx),p(x,Sx)\} \leq p(x,Sx)$$

implies

$$\max\left\{p(Sx,S^2x),p(Tx,TSx),\frac{1}{2}\left[p(Sx,TSx)+p(S^2x,Tx)\right]\right\} \leq rp(x,Sx)$$

for every $x \in X$. Hence, we get

$$\frac{1}{2}p(Sx,TSx) \leq \frac{1}{2}\left[p(Sx,TSx)+p(S^2x,Tx)\right] \leq rp(x,Sx). \tag{2}$$

Now, putting $y=Tx$ in (1), we have

$$\theta(r)\min\{p(x,Tx),p(x,Sx)\} \leq p(x,Tx)$$

implies

$$\max\left\{p(Sx,STx),p(Tx,T^2x),\frac{1}{2}\left[p(Sx,T^2x)+p(STx,Tx)\right]\right\} \leq rp(x,Tx)$$

for every $x \in X$. So, we get

$$p(Tx,T^2x) \leq rp(x,Tx) \tag{3}$$

and

$$\frac{1}{2}p(Tx,STx) \leq \frac{1}{2}\left[p(Sx,T^2x)+p(STx,Tx)\right] \leq rp(x,Tx). \tag{4}$$

Let x_0 be an arbitrary element in X. We obtain a sequence $\{x_n\}$ such that $x_{2n+1}=Sx_{2n}$ and $x_{2n+2}=Tx_{2n+1}$ for each $n \in \mathbb{N}\cup\{0\}$. By (4), we get

$$p(x_{2n},x_{2n+1}) = p(Tx_{2n-1},STx_{2n-1}) \leq 2rp(x_{2n-1},Tx_{2n-1}) = 2rp(x_{2n-1},x_{2n}).$$

And also, from (2), we get

$$p(x_{2n+1},x_{2n+2}) = p(Sx_{2n},TSx_{2n}) \leq 2rp(x_{2n},Sx_{2n}) = 2rp(x_{2n},x_{2n+1}).$$

Therefore, for each $n \in \mathbb{N}\cup\{0\}$, we get

$$\begin{aligned} p(x_n,x_{n+1}) &\le 2rp(x_{n-1},x_n) \\ &\le (2r)^2 p(x_{n-2},x_{n-1}) \\ &\vdots \\ &\le (2r)^n p(x_0,x_1). \end{aligned} \tag{5}$$

Taking limit as $n \to \infty$ in inequality (5), we get $\lim_{n\to\infty} p(x_n,x_{n+1}) = 0$ for $r \in \left[0,\frac{1}{2}\right)$. From properties of PMS, we get

$$p(x_n,x_n) \le p(x_n,x_{n+1}) \text{ and } p(x_{n+1},x_{n+1}) \le p(x_n,x_{n+1}).$$

Since $\lim_{n\to\infty} p(x_n,x_{n+1}) = 0$ for $r \in \left[0,\frac{1}{2}\right)$, we get

$$\lim_{n\to\infty} p(x_n,x_n) = 0 \text{ and } \lim_{n\to\infty} p(x_{n+1},x_{n+1}) = 0. \tag{6}$$

$$\begin{aligned} p^s(x_n,x_{n+1}) &= 2p(x_n,x_{n+1}) - p(x_n,x_n) - p(x_{n+1},x_{n+1}) \\ &\le 2p(x_n,x_{n+1}) \\ &\le 2((2r)^n p(x_0,x_1)). \end{aligned} \tag{7}$$

From (7), we have for any $k,n \in \mathbb{N}^+$

$$\begin{aligned} p^s(x_n,x_{n+k}) &\le p^s(x_n,x_{n+1}) + p^s(x_{n+1},x_{n+2}) + \cdots + p^s(x_{n+k-1},x_{n+k}) \\ &\le 2(2r)^n p(x_0,x_1) + 2(2r)^{n+1} p(x_0,x_1) + \cdots + 2(2r)^{n+k-1} p(x_0,x_1) \\ &= 2\left(\sum_{i=n}^{n+k-1} (2r)^i p(x_0,x_1)\right) \\ &\le 2\left(\sum_{i=n}^{\infty} (2r)^i p(x_0,x_1)\right). \end{aligned}$$

Then there exists a positive integer $n_0 \in \mathbb{N}$ such that $p^s(x_n,x_{n+k}) < \varepsilon$ for every $n \ge n_0$, all $k \in \mathbb{N}^+$ and an arbitrary $\varepsilon > 0$. We say that $\{x_n\}$ is a Cauchy sequence in the metric space (X,p^s). Since (X,p) is a complete PMS, (X,p^s) is also complete metric space. Hence, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$ in (X,p^s). So, we get $\lim_{n\to\infty} p^s(x_n,u) = 0$ implies

$$p(u,u) = \lim_{n\to\infty} p(x_n,u) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Since $\{x_n\}$ is a Cauchy sequence in the metric space (X,p^s), we get $\lim_{n,m\to\infty} p^s(x_n,x_m) = 0$. So, we have

$$\lim_{n,m\to\infty} p^s(x_n,x_m) = \lim_{n,m\to\infty} 2p(x_n,x_m) - \lim_{n,m\to\infty} p(x_m,x_m) - \lim_{n,m\to\infty} p(x_n,x_n) = 0.$$

From (6), we get

$$\lim_{n,m\to\infty} p(x_m,x_m) = \lim_{n,m\to\infty} p(x_n,x_n) = 0$$

implies $\lim_{n,m\to\infty} p(x_n,x_m) = 0$ and

$$p(u,u) = \lim_{n\to\infty} p(x_n,u) = \lim_{n,m\to\infty} p(x_n,x_m) = 0.$$

So, we get

$$\lim_{n\to\infty} p(x_{2n+1},u) = \lim_{n\to\infty} p(x_{2n+2},u) = 0$$

implies

$$\lim_{n\to\infty} p(Sx_{2n},u) = \lim_{n\to\infty} p(Tx_{2n+1},u) = 0. \tag{8}$$

We take $x,u \in X$ such that $x \neq u$. As $p(u,u)=0$, we get, $\lim_{n\to\infty} p(x_{2n+1},Tx_{2n+1}) = 0$

and $\lim_{n\to\infty} p(x_{2n+1},x) \neq 0$. Then, there exists some $x_{2n_k+1} \in X$ such that

$$\theta(r)\min\{p(x_{2n_k+1},Tx_{2n_k+1}),p(x_{2n_k+1},Sx_{2n_k+1})\} \le p(x_{2n_k+1},x)$$

implies

$$\max\left\{p(Sx_{2n_k+1},Sx),p(Tx_{2n_k+1},Tx),\frac{1}{2}\left[p(Sx_{2n_k+1},Tx)+p(Sx,Tx_{2n_k+1})\right]\right\} \le rp(x_{2n_k+1},x).$$

Hence, we get

$$p(Tx_{2n_k+1},Tx) \le rp(x_{2n_k+1},x).$$

If we take the limit of both sides as $n\to\infty$, from (8), we get

$$p(u,Tx) = \lim_{n\to\infty} p(Tx_{2n_k+1},Tx) \le r\lim_{n\to\infty} p(x_{2n_k+1},x) = rp(u,x).$$

Then, we have for each $x \neq u$

$$p(u,Tx) \le rp(u,x). \tag{9}$$

To show that the equation

$$p(T^n u,u) \le p(Tu,u) \tag{10}$$

provides for each $n\in\mathbb{N}$, we use induction. For $n=1$, the inequality is obvious. We suppose that the inequality (10) is true for some $m\in\mathbb{N}$. So, we get

$$p(T^m u,u) \le p(Tu,u).$$

For $n=m+1$, if $T^m u = u$, then

$$p(T^{m+1}u,u) = p(T(T^m u),u) = p(Tu,u). \tag{11}$$

If $T^m u \neq u$, then by (9)

$$p(T^{m+1}u,u) \le rp(T^m u,u) \le rp(Tu,u) \le p(Tu,u). \tag{12}$$

So, from (11) and (12), we get

$$p(T^{m+1}u,u) \le p(Tu,u).$$

Then, inequality (10) is satisfied for all $n\in\mathbb{N}$.

Now, we will show that $Tu=u$. We assume that $Tu \neq u$. Since $0 \le r < \frac{1}{2}$, so $\theta(r) \le \frac{1-r}{r^2}$. Formerly, using induction, we prove that

$$p(T^n u,Tu) \le rp(Tu,u) \tag{13}$$

for each $n\in\mathbb{N}$. For $n=1$, it is obvious. Moreover, for $n=2$, from (3) we get inequality (13) is satisfied. We suppose that the inequality (13) is true for some $n>2$. So, we have

$$\begin{aligned} p(Tu,u) &\le p(u,T^n u)+p(T^n u,Tu)-p(T^n u,T^n u) \\ &\le p(u,T^n u)+p(T^n u,Tu) \\ &\le p(u,T^n u)+rp(Tu,u). \end{aligned}$$

So, we get

$(1-r)p(u,Tu) \le p(u,T^n u)$.

Then, from (3), we obtain that

$$\begin{aligned}\theta(r)\min\left\{p(ST^n z,T^n u),p(T^n u,T^{n+1}u)\right\} &\le \theta(r)p(T^n u,T^{n+1}u)\\ &= \frac{1-r}{r^2}p(T^n u,T^{n+1}u)\\ &\le \frac{1-r}{r^n}p(T^n u,T^{n+1}u)\\ &\le \frac{1-r}{r^n}r^n p(u,Tu)\\ &= (1-r)p(u,Tu)\\ &\le p(u,T^n u).\end{aligned}$$

This implies

$$\begin{aligned}p(T^{n+1}u,Tu) &\le \max\left\{p(ST^n u,Su),p(T^{n+1}u,Tu),\frac{1}{2}\left[p(ST^n u,Tu)+p(Su,T^{n+1}u)\right]\right\}\\ &\le rp(u,T^n u).\end{aligned}$$

Then from (10), we get

$$p(T^{n+1}u,Tu) \le rp(T^n u,u) \le rp(Tu,u).$$

Hence, the inequality (13) is satisfied for each $n \in \mathbb{N}$.

Now, $Tu \ne u$ and (13) implies that $T^n u \ne u$. If not,

$$p(T^n u,Tu) \le rp(Tu,u) \Rightarrow p(u,Tu) \le rp(u,Tu) \le p(u,Tu)$$

for $r \in \left[0,\frac{1}{2}\right)$, which is impossible. So, from (9), we have

$$p(u,T^{n+1}u) \le rp(u,T^n u) \le r^2 p(u,T^{n-1}u) \le \cdots \le r^n p(u,Tu).$$

Taking the limit of both sides, we get

$$\lim_{n\to\infty} p(u,T^{n+1}u) = 0 = p(u,u)$$

for $r \in \left[0,\frac{1}{2}\right)$. Then, $T^n u \to u$. Since $p(T^n u,Tu) \le rp(Tu,u)$, we get

$$p(u,Tu) = \lim_{n\to\infty} p(T^n u,Tu) \le \lim_{n\to\infty} rp(Tu,u) = rp(Tu,u).$$

So, we get $p(Tu,u) = 0$, which is a contradiction. So, $Tu = u$. Hence, u is fixed point of T. Therefore, u is also a fixed point of S. As a result, u is a common fixed point of T and S.

Now, to show the uniqueness of this common fixed point, we assume that u and v are common fixed points of T and S where $u \ne v$. Taking $x = u$ and $y = v$ in inequality (1), we have

$$0 = \theta(r)\min\left\{p(u,Tu),p(u,Su)\right\} \le p(u,v)$$

implies

$$\max\left\{p(Su,Sv),p(Tu,Tv),\frac{1}{2}\left[p(Su,Tv)+p(Tu,Sv)\right]\right\}\leq rp(u,v)$$

$$\Rightarrow \max\left\{p(u,v),p(u,v),\frac{1}{2}\left[2p(u,v)\right]\right\}\leq rp(u,v)$$

$$\Rightarrow p(u,v)\leq rp(u,v)<p(u,v).$$

So, $p(u,v)=0$ which is a contradiction. Hence, $u=v$.

In Theorem 1, if we take as $S=T$, then we get the following corollary which is Suzuki type result in partial metric spaces [15].

Corollary 1. Let (X,p) be a complete PMS, $T:X\to X$ be a self-mapping and a nonincreasing function $\theta:[0,1)\to\left(\frac{1}{2},1\right]$ be defined by

$$\theta(r)=\begin{cases}1, & 0\leq r\leq \frac{\sqrt{5}-1}{2},\\ \frac{1-r}{r^2}, & \frac{\sqrt{5}-1}{2}\leq r\leq \frac{1}{\sqrt{2}},\\ \frac{1}{1+r}, & \frac{1}{\sqrt{2}}\leq r<1.\end{cases}$$

If there exists $r\in\left[0,\frac{1}{2}\right)$ such that T satisfies the condition

$$\theta(r)p(x,Tx)\leq p(x,y)$$

implies

$$p(Tx,Ty)\leq rp(x,y)$$

for each $x,y\in X$, then T has a unique fixed point.

Corollary 2. Let (X,p) be a complete PMS, $f,S,T:X\to X$ be three self-maps and a nonincreasing function $\theta:[0,1)\to\left(\frac{1}{2},1\right]$ be defined by

$$\theta(r)=\begin{cases}1, & 0\leq r\leq \frac{\sqrt{5}-1}{2},\\ \frac{1-r}{r^2}, & \frac{\sqrt{5}-1}{2}\leq r\leq \frac{1}{\sqrt{2}},\\ \frac{1}{1+r}, & \frac{1}{\sqrt{2}}\leq r<1.\end{cases}$$

If there exists $r\in\left[0,\frac{1}{2}\right)$ such that

$$\theta(r)\min\left\{p(x,fTx),p(x,fSx)\right\}\leq p(x,y)$$

implies

$$\max\left\{p(fSx,fSy),p(fTx,fTy),\frac{1}{2}\left[p(fSx,fTy)+p(fSy,fTx)\right]\right\}\le rp(x,y), \tag{14}$$

also, if f is one to one, $fS = Sf$ and $fT = Tf$, then f, T and S have a common fixed point.

Proof. If we consider fS and fT as two maps with given contractive condition of Theorem 1, then fS and fT have a common fixed point $u \in X$. Namely, $fSu = fTu = u$. Since f is one to one, we get

$$fSu = fTu = u \Rightarrow Su = Tu.$$

Then, putting $x = u$ and $y = Tu$ in inequality (14)

$$\theta(r)\min\{p(u,fTu),p(u,fSu)\}\le p(u,Tu)$$

implies

$$\max\left\{p(fSu,fSTu),p(fTu,fT^2u),\frac{1}{2}\left[p(fSu,fT^2u)+p(fSTu,fTu)\right]\right\}\le rp(u,Tu)$$

$$\Rightarrow \max\left\{p(fSu,SfTu),p(fTu,TfTu),\frac{1}{2}\left[p(fSu,TfTu)+p(SfTu,fTu)\right]\right\}\le rp(u,Tu)$$

$$\Rightarrow \max\left\{p(u,Su),p(u,Tu),\frac{1}{2}\left[p(u,Tu)+p(Su,u)\right]\right\}\le rp(u,Tu)$$

$$\Rightarrow p(u,Tu)\le rp(u,Tu).$$

Then, $p(u,Tu) = 0$. So, we get $Tu = u$ which implies $Tu = Su = u$ and also $fu = fTu = u$. So, f, T and S have a common fixed point.

Example 1. Let $X = \left\{0,\frac{1}{4},\frac{1}{3},2\right\}\subset\mathbb{R}$. Define $p: X\times X\to\mathbb{R}^+$ by $p(x,y) = \max\{x,y\}$ for all $x,y\in X$. Then, it is obvious that (X,p) is a complete PMS. And define two maps T and S by

$$Tx = \begin{cases} 0 & , \quad x\neq 2 \\ \frac{1}{3} & , \quad x = 2 \end{cases}$$

and

$$Sx = \begin{cases} 0 & , \quad x\neq 2 \\ \frac{1}{4} & , \quad x = 2 \end{cases}$$

for $x\in X$. Moreover, we choose as $r = \frac{1}{6}$. So, we get $\theta(r) = 1$.

Case I If $x,y\in\left\{0,\frac{1}{4},\frac{1}{3}\right\}$, we get

$$\begin{aligned}\theta(r)\min\{p(x,Tx),p(x,Sx)\} &= \min\{p(x,0),p(x,0)\} = \min\{\max\{x,0\},\max\{x,0\}\} \\ &= x\le\max\{x,y\} = p(x,y)\end{aligned}$$

implies

$$\max\left\{p(Sx,Sy),p(Tx,Ty),\frac{1}{2}\left[p(Sx,Ty),p(Sy,Tx)\right]\right\}=\max\left\{p(0,0),p(0,0),\frac{1}{2}[p(0,0)+p(0,0)]\right\}$$

$$=0\le\frac{1}{6}\max\{x,y\}=rp(x,y).$$

Case II If $x\in\left\{0,\frac{1}{4},\frac{1}{3}\right\}$ and $y=2$, we get

$$\begin{aligned}\theta(r)\min\left\{p(x,Tx),p(x,Sx)\right\}&=\min\{p(x,0),p(x,0)\}\\&=\min\{\max\{x,0\},\max\{x,0\}\}\\&=x\le\max\{x,y\}=p(x,y)\end{aligned}$$

implies

$$\max\left\{p(Sx,Sy),p(Tx,Ty),\frac{1}{2}\left[p(Sx,Ty),p(Sy,Tx)\right]\right\}=\max\left\{p\left(0,\frac{1}{4}\right),p\left(0,\frac{1}{3}\right),\frac{1}{2}\left[p\left(0,\frac{1}{3}\right)+p\left(\frac{1}{4},0\right)\right]\right\}$$

$$=\frac{1}{3}\le\frac{1}{6}\max\{x,2\}=rp(x,y).$$

Case III If $y\in\left\{0,\frac{1}{4},\frac{1}{3}\right\}$ and $x=2$, we get

$$\theta(r)\min\left\{p(x,Tx),p(x,Sx)\right\}=\min\left\{p\left(2,\frac{1}{3}\right),p\left(2,\frac{1}{4}\right)\right\}=2\le\max\{2,y\}=p(x,y)$$

implies

$$\max\left\{p(Sx,Sy),p(Tx,Ty),\frac{1}{2}\left[p(Sx,Ty),p(Sy,Tx)\right]\right\}=\max\left\{p\left(\frac{1}{4},0\right),p\left(\frac{1}{3},0\right),\frac{1}{2}\left[p\left(\frac{1}{4},0\right)+p\left(0,\frac{1}{3}\right)\right]\right\}$$

$$=\frac{1}{3}\le\frac{1}{6}\max\{2,y\}=rp(x,y).$$

Case IV If $x=y=2$, we get

$$\begin{aligned}\theta(r)\min\left\{p(x,Tx),p(x,Sx)\right\}&=\min\left\{p\left(2,\frac{1}{3}\right),p\left(2,\frac{1}{4}\right)\right\}\\&=2\le\max\{x,y\}=p(x,y)\end{aligned}$$

implies

$$\max\left\{p(Sx,Sy),p(Tx,Ty),\frac{1}{2}\left[p(Sx,Ty),p(Sy,Tx)\right]\right\}=\max\left\{p\left(\frac{1}{4},\frac{1}{4}\right),p\left(\frac{1}{3},\frac{1}{3}\right),\frac{1}{2}\left[p\left(\frac{1}{4},\frac{1}{3}\right)+p\left(\frac{1}{4},\frac{1}{3}\right)\right]\right\}$$

$$=\frac{1}{3}\le\frac{1}{6}\max\{x,y\}=rp(x,y).$$

Thus, T and S satisfy all the hypotheses of Theorem 1. So, T and S have a unique common fixed point. Moreover, it is $0\in\mathbb{R}$.

Application

Let Y and Z be Banach spaces, $S\subseteq Y$ be a state space, $D\subseteq Z$ be a decision space and $H_i:S\times D\times\mathbb{R}\to\mathbb{R}$ for $i=1,2$. The problem of dynamic programming related to the multistage process reduces to the problem of solving the following functional equation:

$$q_i(x) = \sup_{y\in D}\{H_i(x, y, q_i(x, y))\}, \quad x \in S, i = 1, 2. \tag{15}$$

In this section, using Theorem 1, we show existence and uniqueness of a bounded common solution of the functional equation (15). $B(S)$ is a Banach space which consists of all bounded real functionals on S with the norm $\| h \| = \sup_{x\in S} | h(x) |$ for an arbitrary $h \in B(S)$. $(B(S), \| . \|)$ endowed with the metric d defined by

$$d(h, k) = \sup_{x\in S} | h(x) - k(x) |$$

for $h, k \in B(S)$. Then, the convergence in the space $B(S)$ corresponds to uniform convergence. So, if we take a Cauchy sequence $\{h_n\}$ in $B(S)$, the sequence $\{h_n\}$ converges uniformly to a function h^*. It is bounded. Hence, $h^* \in B(S)$. We define the partial metric such that

$$p(h, k) = d(h, k) + b \tag{16}$$

for all $h, k \in B(S)$ where $b > 0$. Then, $(B(S), p)$ is a complete PMS. Now, we define mappings $A_i : B(S) \to B(S)$ by

$$A_i h(x) = \sup_{y\in D}\{H_i(x, y, h(x, y))\} \tag{17}$$

$$A_i k(x) = \sup_{y\in D}\{H_i(x, y, k(x, y))\} \tag{18}$$

for $i = 1, 2$, $h, k \in B(S)$, $x \in S$. It is clear that if the functions H_i are bounded, then A_i are well defined for $i = 1, 2$.

Theorem 2. Let the functions $H_i : S \times D \times \mathbb{R} \to \mathbb{R}$ be bounded and $A_i : B(S) \to B(S)$ be defined as in (17). If there exists $r \in \left[0, \frac{1}{2}\right)$ such that

$$\theta(r)\min\{| h(t) - A_1 h(t) |, | h(t) - A_2 h(t) |\} \leq | h(t) - k(t) | \tag{19}$$

implies

$$| H_i(x, y, h(t)) - H_j(x, y, k(t)) | \leq r | h(t) - k(t) | \tag{20}$$

for every $(x, y) \in S \times D$, $h, k \in B(S)$ and $t \in S$ and $i, j = 1, 2$, then the functional equations (15) have a unique common bounded solution in $B(S)$ for $i = 1, 2$.

Proof. Let ϵ be an arbitrary positive number and $h, k \in B(S)$. By (17), there exist $y_1, y_2 \in D$ such that

$$A_1 h(x) < H_1(x, y_1, h(x, y_1)) + \epsilon \tag{21}$$

$$A_1 k(x) < H_1(x, y_1, k(x, y_1)) + \epsilon \tag{22}$$

$$A_2 h(x) < H_2(x, y_2, h(x, y_2)) + \epsilon \tag{23}$$

$$A_2 k(x) < H_2(x, y_2, k(x, y_2)) + \epsilon \tag{24}$$

and

$$A_1 h(x) \geq H_1(x, y_1, h(x, y_1)) \tag{25}$$

$$A_1 k(x) \geq H_1(x, y_1, k(x, y_1)) \tag{26}$$

$$A_2 h(x) \geq H_2(x, y_2, h(x, y_2)) \tag{27}$$

$$A_2 k(x) \geq H_2(x, y_2, k(x, y_2)) \tag{28}$$

for $x \in S$. The inequality (19)

$$\theta(r)\min\{|h(x)-A_1h(x)|,|h(x)-A_2h(x)|\}\le|h(x)-k(x)| \tag{29}$$

is satisfied for $r\in\left[0,\frac{1}{2}\right)$. Then, from (23), (24), (27) and (28), we get the functional equations

$$\begin{aligned} A_2h(x)-A_2k(x) &\le H_2(x,y_2,h(x,y_2))-H_2(x,y_2,k(x,y_2))+\epsilon \\ &\le|H_2(x,y_2,h(x,y_2))-H_2(x,y_2,k(x,y_2))|+\epsilon \end{aligned} \tag{30}$$

and

$$\begin{aligned} A_2k(x)-A_2h(x) &\le H_2(x,y_2,k(x,y_2))-H_2(x,y_2,h(x,y_2))+\epsilon \\ &\le|H_2(x,y_2,k(x,y_2))-H_2(x,y_2,h(x,y_2))|+\epsilon \end{aligned} \tag{31}$$

From (30) and (31), we get

$$|A_2h(x)-A_2k(x)|\le|H_2(x,y_2,h(x,y_2))-H_2(x,y_2,k(x,y_2))|+\epsilon$$

Using (20), we can say that

$$|A_2h(x)-A_2k(x)|\le r\,|h(x)-k(x)|+\epsilon. \tag{32}$$

Similarly, using (20), (21), (22), (25) and (26), we get

$$|A_1h(x)-A_1k(x)|\le r\,|h(x)-k(x)|+\epsilon. \tag{33}$$

On the other hand, from (22), (23), (26) and (27), we get

$$|A_2h(x)-A_1k(x)|\le|H_2(x,y_2,h(x,y_2))-H_1(x,y_2,k(x,y_1))|+\epsilon \tag{34}$$

and from (21), (24), (25) and (28), we get

$$|A_2k(x)-A_1h(x)|\le|H_2(x,y_2,k(x,y_2))-H_1(x,y_2,h(x,y_1))|+\epsilon \tag{35}$$

If we sum both sides of the inequalities (34) and (35) and multiply them by $\frac{1}{2}$, then we get

$$\frac{1}{2}\left[|A_2h(x)-A_1k(x)|+|A_2k(x)-A_1h(x)|\right]\le r\,|h(x)-k(x)|+\epsilon. \tag{36}$$

Since the inequalities (32), (33) and (36) hold for any $x\in S$ and $\epsilon>0$, we can ignore ϵ. Then, we get

$$|A_2h(x)-A_2k(x)|\le r\,|h(x)-k(x)|,\ |A_1h(x)-A_1k(x)|\le r\,|h(x)-k(x)|,$$

and

$$\frac{1}{2}\left[|A_2h(x)-A_1k(x)|+|A_2k(x)-A_1h(x)|\right]\le r\,|h(x)-k(x)|.$$

Thus, it is true that

$$\max\left\{|A_2h(x)-A_2k(x)|+|A_1h(x)-A_1k(x)|,\frac{1}{2}[|A_2h(x)-A_1k(x)|+|A_2k(x)-A_1h(x)|]\right\}\le r\,|h(x)-k(x)|. \tag{37}$$

Now, if we take supremum in the inequalities (29) and (37) and add $b>0$ to them, we get from (16)

$$\theta(r)\min\{p(h(x),A_1h(x)),p(h(x),A_2h(x))\}\le p(h(x),k(x))$$

implies

$$\max\left\{p(A_2h(x),A_2k(x)),p(A_1h(x),A_1k(x)),\frac{1}{2}[p(A_2h(x),A_1k(x))+p(A_2k(x),A_1h(x))]\right\}\le rp(h(x),k(x))$$

for $h,k \in B(S)$, $x \in S$ and $r \in \left[0, \frac{1}{2}\right)$. So, from Theorem 1, equations (15) and (17) there exists a unique common fixed point $h^* \in B(S)$. Namely, the system of functional equations (15) has a unique common bounded solution for $i = 1, 2$.

Conflicts of interest

There are no conflicts of interest in this work.

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